N. ANTONOV
M. VYGODSKY, V. NIKITIN
A. SANKIN

PROBLEMS IN ELEMENTARY MATHEMATICS for Home Study

Translated from the Russian
by
LEONID LEVANT

MIR PUBLISHERS
MOSCOW

Н. П. АНТОНОВ
М. Я. ВЫГОДСКИЙ
В. В. НИКИТИН
А. И. САНКИН

СБОРНИК ЗАДАЧ ПО ЭЛЕМЕНТАРНОЙ МАТЕМАТИКЕ

ИЗДАТЕЛЬСТВО
«НАУКА»
МОСКВА

510
H6M

GOVT.
8091

На английском языке

CONTENTS

Formulas for Reference 6

PART ONE

ARITHMETIC AND ALGEBRA

	Problems	Answers and Solutions
Chapter I. Arithmetic Calculations	11	89
Chapter II. Algebraic Transformations	14	90
Chapter III. Algebraic Equations	22	112
Chapter IV. Logarithmic and Exponential Equations	29	142
Chapter V. Progressions	32	159
Chapter VI. Combinatorics and Newton's Binomial Theorem	36	169
Chapter VII. Algebraic and Arithmetic Problems .	39	177

PART TWO

GEOMETRY AND TRIGONOMETRY

Chapter VIII. Plane Geometry	55	208
Chapter IX. Polyhedrons	62	246
Chapter X. Solids of Revolution	76	327
Chapter XI. Trigonometric Transformations . . .	81	363
Chapter XII. Trigonometric Equations	84	372
Chapter XIII. Inverse Trigonometric Functions . .	87	395

FORMULAS FOR REFERENCE

I. ARITHMETIC AND ALGEBRA

Proportions

1. In the proportion $\frac{a}{b}=\frac{c}{d}$ a and d are the extremes, b and c are the means. The principal property of the proportion:

$$a\cdot d=b\cdot c$$

2. Interchanging the terms:

$$\text{(a)}\quad \frac{a}{b}=\frac{c}{d};\ \text{(b)}\quad \frac{d}{b}=\frac{c}{a};\ \text{(c)}\quad \frac{a}{c}=\frac{b}{d};\ \text{(d)}\quad \frac{d}{c}=\frac{b}{a}$$

3. Derived proportions: if $\frac{a}{b}=\frac{c}{d}$, then the following proportions hold true:

$$\text{(a)}\quad \frac{a\pm b}{a}=\frac{c\pm d}{c};\ \text{(b)}\quad \frac{a\pm c}{b\pm d}=\frac{a}{b}=\frac{c}{d}$$

Involution

1. $(a\cdot b\cdot c)^n=a^n\cdot b^n\cdot c^n$, that is $a^n\cdot b^n\cdot c^n=(a\cdot b\cdot c)^n$

2. $\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}$, that is $\frac{a^n}{b^n}=\left(\frac{a}{b}\right)^n$; 3. $a^m\cdot a^n=a^{m+n}$

4. $a^m : a^n=a^{m-n}$; 5. $1 : a^n=a^0 : a^n=a^{-n}$

6. $(a^m)^n=a^{mn}$

Evolution *

1. $\sqrt[m]{a\cdot b\cdot c}=\sqrt[m]{a}\cdot\sqrt[m]{b}\cdot\sqrt[m]{c}$, that is $\sqrt[m]{a}\cdot\sqrt[m]{b}\cdot\sqrt[m]{c}=\sqrt[m]{a\cdot b\cdot c}$

2. $\sqrt[m]{\frac{a}{b}}=\frac{\sqrt[m]{a}}{\sqrt[m]{b}}$, that is $\frac{\sqrt[m]{a}}{\sqrt[m]{b}}=\sqrt[m]{\frac{a}{b}}$

---

* The roots are supposed to be arithmetic, cf. p 90.

3. $a^{\frac{n}{m}} = \sqrt[m]{a^n}$, that is $\sqrt[m]{a^n} = a^{\frac{n}{m}}$
4. $(\sqrt[m]{a^n})^p = \sqrt[m]{a^{np}}$
5. $\sqrt[m]{a^n} = \sqrt[mp]{a^{np}}$, that is $\sqrt[mp]{a^{np}} = \sqrt[m]{a^n}$

Quadratic Equations

1. The equation of the form $x^2+px+q=0$ is solved by using the formula

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4}-q}$$

2. The equation of the form $ax^2+bx+c=0$ is solved by using the formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

3. The equation of the form $ax^2+2kx+c=0$ is solved by using the formula

$$x_{1,2} = \frac{-k \pm \sqrt{k^2-ac}}{a}$$

4. If x_1 and x_2 are the roots of the equation $x^2+px+q=0$, then $x_1+x_2=-p$ and $x_1x_2=q$
5. $x^2+px+q=(x-x_1)(x-x_2)$, where x_1 and x_2 are the roots of the equation $x^2+px+q=0$
6. $ax^2+bx+c=a(x-x_1)(x-x_2)$, where x_1 and x_2 are the roots of the equation $ax^2+bx+c=0$

Progressions (see page 32)

Logarithms*

1. Symbolically, $\log_a N = x$ is equivalent to $a^x = N$, hence we have the identity $a^{\log_a N} = N$

2. $\log_a a = 1$; 3. $\log_a 1 = 0$; 4. $\log_a (N \cdot M) = \log_a N + \log_a M$

5. $\log_a \frac{N}{M} = \log_a N - \log_a M$; 6. $\log_a (N^m) = m \log_a N$

7. $\log_a \sqrt[m]{N} = \frac{1}{m} \log_a N$

8. For the modulus which makes possible conversion from a system of logarithms to the base b to that of logarithms to the base a see page 142.

---

* The numbers a (the logarithmic base) and N are assumed to be positive, a being not equal to unity.

Combinatorics

1. $A_m^n = m(m-1)(m-2)\ldots(m-n+1)$;   2. $P_m = 1\cdot 2\cdot 3\ldots m = m!$

3. $C_m^n = \frac{A_m^n}{P_n} = \frac{m(m-1)(m-2)\ldots(m-n+1)}{1\cdot 2\cdot 3\ldots n}$;   4. $C_m^n = C_m^{m-n}$

Newton's Binomial

1. $(x+a)^m = x^m + C_m^1 a x^{m-1} + C_m^2 a^2 x^{m-2} + \ldots + C_m^{m-2} a^{m-2} x^2 + C_m^{m-1} a^{m-1} x + a^m$
2. General term of expansion:

$$T_{k+1} = C_m^k a^k x^{m-k}$$

3. $1 + C_m^1 + C_m^2 + \ldots + C_m^{m-2} + C_m^{m-1} + 1 = 2^m$
4. $1 - C_m^1 + C_m^2 - C_m^3 + \ldots \pm 1 = 0$

II. GEOMETRY AND TRIGONOMETRY

The Circumference of a Circle and the Length of Its Arc

$C = 2\pi R$; $l = \frac{\pi R a}{180} = R\alpha$ (a is the degree measure of the arc and α, its radian measure)

Areas

Triangle: $S = \frac{ah}{2}$ (a is the base and h, the altitude); $S = \sqrt{p(p-a)(p-b)(p-c)}$ (p is the semiperimeter and a, b and c, the sides); $S = \frac{ab\sin C}{2}$

For an equilateral triangle $S = \frac{a^2\sqrt{3}}{4}$ (a is the side)

Parallelogram: $S = bh$ (b is the base and h, the altitude)

Rhombus: $S = \frac{d_1 d_2}{2}$ (d_1 and d_2 are the diagonals)

Trapezoid: $S = \frac{a+b}{2}h$ (a and b are the bases and h, the altitude); $S = mh$ (m is the median).

Regular polygon: $S = \frac{Pa}{2}$ (P is the perimeter and a, the apothem)

Circle: $S = \pi R^2$

Circular sector: $S = \frac{Rl}{2} = \frac{R^2\alpha}{2} = \frac{\pi R^2 a}{360}$ (a is the degree measure of the sector arc; α, its radian measure and l, the length of the arc)

Surfaces

Prism: $S_{lat} = Pl$ (P is the perimeter of a right section and l, the lateral edge)

Regular pyramid: $S_{lat} = \frac{Pa}{2}$ (P is the perimeter of the base and a, the slant height)

Frustum of a regular pyramid: $S_{lat} = \frac{P_1+P_2}{2}a$ (P_1 and P_2 are the perimeters of the bases and a, the slant height)
Cylinder: $S_{lat} = 2\pi RH$
Cone: $S_{lat} = \pi Rl$ (l is the generator)
Frustum of a cone: $S_{lat} = \pi(R_1+R_2)l$
Sphere: $S = 4\pi R^2$

Volumes

Prism: $V = SH$ (S is the area of the base and H, the altitude)

Pyramid: $V = \frac{SH}{3}$

Frustum of a pyramid: $V = \frac{H}{3}(S_1+S_2+\sqrt{S_1S_2})$

Cylinder: $V = \pi R^2 H$

Cone: $V = \frac{\pi R^2 H}{3}$

Frustum of a cone: $V = \frac{\pi H}{3}(R_1^2+R_2^2+R_1R_2)$

Sphere: $V = \frac{4}{3}\pi R^3$

Conversion of the Degree Measure of an Angle to its Radian Measure and Vice Versa

$\alpha = \frac{\pi \cdot a^\circ}{180^\circ}$; $a^\circ = \alpha\frac{180^\circ}{\pi}$ (α is the radian measure of the angle and a, its degree measure)

Addition Formulas

1. $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
2. $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
3. $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

Double-Angle and Half-Angle Formulas

1. $\sin 2\alpha = 2\sin\alpha\cos\alpha$; 2. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

3. $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$; 4. $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$

5. $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$; 6. $\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$

7. $\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$; 8. $\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$

Reducing Trigonometric Expressions to Forms Convenient for Taking Logarithms

1. $\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

2. $\sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

3. $\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

4. $\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

5. $\tan\alpha\pm\tan\beta=\frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}$

6. $1+\cos\alpha=2\cos^2\frac{\alpha}{2}$; 7. $1-\cos\alpha=2\sin^2\frac{\alpha}{2}$

Some Important Relations

1. $\sin\alpha\sin\beta=\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$

2. $\cos\alpha\cos\beta=\frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}$

3. $\sin\alpha\cos\beta=\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$

Basic Relations Between the Elements of a Right-Angled Triangle

(a and b are the legs; c, the hypotenuse; A and B, the acute angles; C, the right angle)

1. $a=c\sin A=c\cos B$; 2. $b=c\sin B=c\cos A$
3. $a=b\tan A=b\cot B$; 4. $b=a\tan B=a\cot A$

Basic Relations Between the Elements of an Arbitrary Triangle

(a, b and c are the sides; A, B and C, the angles)

1. $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ (law of sines)

2. $a^2=b^2+c^2-2bc\cos A$ (law of cosines)

3. $\frac{a+b}{a-b}=\frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}$ (law of tangents)

Relations Between the Values of Inverse Trigonometric Functions

(arcsin x, arccos x and arctan x are the principal values of the corresponding inverse trigonometric functions)

1. $\mathrm{Arcsin}\,x=\pi k+(-1)^k\arcsin x$
2. $\mathrm{Arccos}\,x=2\pi k\pm\arccos x$
3. $\mathrm{Arctan}\,x=\pi k+\arctan x$; k is any integer (positive or negative).

PROBLEMS

PART ONE

ARITHMETIC AND ALGEBRA

CHAPTER I

ARITHMETIC CALCULATIONS

1. $\dfrac{\left(152\frac{3}{4}-148\frac{3}{8}\right)\cdot 0.3}{0.2}$; **2.** $\dfrac{172\frac{5}{6}-170\frac{1}{3}+3\frac{5}{12}}{0.8\cdot 0.25}$

3. $\dfrac{215\frac{9}{16}-208\frac{3}{4}+\frac{1}{2}}{0.0001:0.005}$; **4.** $\left(\dfrac{0.012}{5}+\dfrac{0.04104}{5.4}\right)\cdot 4560-42\frac{1}{3}$

5. $\dfrac{\left(85\frac{7}{30}-83\frac{5}{18}\right):2\frac{2}{3}}{0.04}$; **6.** $\dfrac{\left(140\frac{7}{30}-138\frac{5}{12}\right):18\frac{1}{6}}{0.002}$

7. $\dfrac{\left(95\frac{7}{30}-93\frac{5}{18}\right)\cdot 2\frac{1}{4}+0.373}{0.2}$; **8.** $\dfrac{\left(49\frac{5}{24}-46\frac{7}{20}\right)\cdot 2\frac{1}{3}+0.6}{0.2}$

9. $\dfrac{\left(12\frac{1}{6}-6\frac{1}{27}-5\frac{1}{4}\right)\cdot 13.5+0.111}{0.02}$; **10.** $\dfrac{\left(1\frac{1}{12}+2\frac{5}{32}+\frac{1}{24}\right)\cdot 9\frac{3}{5}+2.13}{0.4}$

11. $\dfrac{\left(6\frac{3}{5}-3\frac{3}{14}\right)\cdot 5\frac{5}{6}}{(21-1.25):2.5}$; **12.** $\dfrac{2\frac{5}{8}-\frac{2}{3}\cdot 2\frac{5}{14}}{\left(3\frac{1}{12}+4.375\right):19\frac{8}{9}}$

13. $\dfrac{0.134+0.05}{18\frac{1}{6}-1\frac{11}{14}-\frac{2}{15}\cdot 2\frac{6}{7}}$; **14.** $\dfrac{\left(58\frac{4}{15}-56\frac{7}{24}\right):0.8+2\frac{1}{9}\cdot 0.225}{8\frac{3}{4}\cdot\frac{3}{5}}$

15. $\dfrac{\left(68\frac{7}{30}-66\frac{5}{18}\right):6\frac{1}{9}+\left(\frac{7}{40}+\frac{3}{32}\right)\cdot 4.5}{0.04}$

16. $\dfrac{(2.1-1.965):(1.2\cdot 0.045)}{0.00325:0.013}-\dfrac{1:0.25}{1.6\cdot 0.625}$

17. $\dfrac{\left[\left(40\frac{7}{30}-38\frac{5}{12}\right):10.9+\left(\frac{7}{8}-\frac{7}{30}\right)\cdot 1\frac{9}{11}\right]\cdot 4.2}{0.008}$

18. $\left[\dfrac{\left(2.4+1\frac{5}{7}\right)\cdot 4.375}{\frac{2}{3}-\frac{1}{6}}-\dfrac{\left(2.75-1\frac{5}{6}\right)\cdot 21}{8\frac{3}{20}-0.45}\right]:\dfrac{67}{200}$

19. $\left[\dfrac{\left(6-4\frac{1}{2}\right):0.03}{\left(3\frac{1}{20}-2.65\right)\cdot 4+\frac{2}{5}}-\dfrac{\left(0.3-\frac{3}{20}\right)\cdot 1\frac{1}{2}}{\left(1.88+2\frac{3}{25}\right)\cdot\frac{1}{80}}\right]:2\dfrac{1}{20}$

20. $26:\left[\dfrac{3:(0.2-0.1)}{2.5\cdot(0.8+1.2)}+\dfrac{(34.06-33.81)\cdot 4}{6.84:(28.57-25.15)}\right]+\dfrac{2}{3}:\dfrac{4}{21}$

21. $\dfrac{3:\frac{2}{5}-0.09:\left(0.15:2\frac{1}{2}\right)}{0.32\cdot 6+0.03-(5.3-3.88)+0.67}$

22. $1\dfrac{7}{20}:2.7+2.7:1.35+\left(0.4:2\dfrac{1}{2}\right)\cdot\left(4.2-1\dfrac{3}{40}\right)$

23. $\left(10:2\dfrac{2}{3}+7.5:10\right)\cdot\left(\dfrac{3}{40}-\dfrac{7}{30}\cdot 0.25+\dfrac{157}{360}\right)$

24. $\left(\dfrac{0.216}{0.15}+\dfrac{2}{3}:\dfrac{4}{15}\right)+\left(\dfrac{196}{225}-\dfrac{7.7}{24\frac{3}{4}}\right)+0.695:1.39$

25. $1.7:\dfrac{\left(4.5\cdot 1\frac{2}{3}+3.75\right)\cdot\frac{7}{135}}{\frac{5}{9}}-\left(0.5+\dfrac{1}{3}-\dfrac{5}{12}\right)$

26. $\dfrac{1}{3}:\dfrac{2}{3}+0.228:\left[\left(1.5291-\dfrac{14.53662}{3-0.095}\cdot 0.305\right):0.12\right]$

27. $\left\{\dfrac{8.8077}{20-[28.2:(13.333\cdot 0.3+0.0001)]\cdot 2.004}+4.9\right\}\cdot\dfrac{5}{32}$

28. $\dfrac{\left[\left(6.2:0.31-\frac{5}{6}\cdot 0.9\right)\cdot 0.2+0.15\right]:0.02}{\left(2+1\frac{4}{11}\cdot 0.22:0.1\right)\cdot\frac{1}{33}}$

29. $6:\dfrac{1}{3}-0.8:\dfrac{1.5}{\frac{3}{2}\cdot 0.4\cdot\frac{50}{1:\frac{1}{2}}}+\dfrac{1}{4}+\dfrac{1+\frac{1}{2}\cdot\frac{1}{0.25}}{6-\frac{46}{1+2.2\cdot 10}}$

30. $$\frac{\left(1.75:\frac{2}{3}-1.75\cdot 1\frac{1}{8}\right):\frac{7}{12}}{\left(\frac{17}{80}-0.0325\right):400}:(6.79:0.7+0.3)$$

31. $$\frac{4.5:\left[47.375-\left(26\frac{1}{3}-18\cdot 0.75\right)\cdot 2.4:0.88\right]}{17.81:1.37-23\frac{2}{3}:1\frac{5}{6}}$$

32. Find the number, 3.6 per cent of which amount to

$$\frac{3+4.2:0.1}{\left(1:0.3-2\frac{1}{3}\right)\cdot 0.3125}$$

33. Compute

$$\left(46\frac{2}{25}:12+41\frac{23}{35}:260\frac{5}{14}+800:12\frac{28}{31}\right)\cdot\frac{0.8\cdot 7.2\cdot 4.5\cdot 1.3}{6.5\cdot 2.7\cdot 1.92}$$

34. Compute

$$\left[15:\frac{(0.6+0.425-0.005):0.01}{30\frac{5}{9}+3\frac{4}{9}}\right]\left(0.645:0.3-1\frac{107}{180}\right)\times$$
$$\times\left(4:6.25-\frac{1}{5}+\frac{1}{7}\cdot 1.96\right)$$

35. Compute

$$\left[\left(7\frac{2}{3}-6\frac{8}{15}\cdot\frac{5}{14}\right):\left(8\frac{3}{4}\cdot\frac{2}{7}-1\frac{1}{6}\right)+\frac{7}{18}:\frac{14}{27}\right]\times$$
$$\times\left(\frac{5}{6}-0.75\right)\ \frac{20.4\cdot 4.8\cdot 6.5}{22.1\cdot 1.2}$$

36. Compute

$$\frac{2.045\cdot 0.033+10.518395-0.464774:0.0562}{0.003092:0.0001-5.188}$$

37. Compute

$$\left(7\frac{1}{9}-2\frac{14}{15}\right):\left(2\frac{2}{3}+1\frac{3}{5}\right)-\left(\frac{3}{4}-\frac{1}{20}\right)\ \left(\frac{5}{7}-\frac{5}{14}\right)$$

38. Compute

$$\left(41\frac{23}{84}-40\frac{49}{60}\right)\left\{\left[4-3\frac{1}{2}\left(2\frac{1}{7}-1\frac{1}{5}\right)\right]:0.16\right\}$$

39. Compute

$$\frac{45\frac{10}{63}-44\frac{25}{84}}{\left(2\frac{1}{3}-1\frac{1}{9}\right):4-\frac{3}{4}}:31$$

40. Compute

$$\frac{0.8:\left(\frac{4}{5}\cdot 1.25\right)}{0.64-\frac{1}{25}}+\frac{\left(1.08-\frac{2}{25}\right):\frac{4}{7}}{\left(6\frac{5}{9}-3\frac{1}{4}\right)\cdot 2\frac{2}{17}}+(1.2\cdot 0.5):\frac{4}{5}$$

41. Compute

$$\left[41\frac{29}{72}-\left(18\frac{7}{8}-5\frac{1}{4}\right)\left(10\frac{1}{2}-7\frac{2}{3}\right)\right]:22\frac{7}{18}$$

42. Compute

$$\left[\frac{\left(6-4\frac{1}{2}\right):0.003}{\left[\left(3\frac{1}{20}-2.65\right)4\right]:\frac{1}{5}}-\frac{\left(0.3-\frac{3}{20}\right)\cdot 1\frac{1}{2}}{\left(1.88+2\frac{3}{25}\right)\cdot\frac{1}{8}}\right]:62\frac{1}{20}+17.81:0.0137$$

43. Compute x, if

$$5\frac{4}{7}:\left\{x:1.3+8.4\cdot\frac{6}{7}\cdot\left[6-\frac{(2.3+5:6.25)\cdot 7}{8\cdot 0.0125+6.9}\right]\right\}=1\frac{1}{14}$$

44. Compute x, if

$$\frac{\left[\left(4.625-\frac{13}{18}\cdot\frac{9}{26}\right):x+(2.5:1.25):6.75\right]:1\frac{53}{68}}{\left(\frac{1}{2}-0.375\right):0.125+\left(\frac{5}{6}-\frac{7}{12}\right):(0.358-1.4796:13.7)}=\frac{17}{27}$$

45. Find x, if

$$\frac{(2.7-0.8)\cdot 2\frac{1}{3}}{(5.2-1.4):\frac{3}{7}}+x+8\frac{9}{11}-\frac{(1.6+154.66:70.3):1.9}{\left(2\frac{2}{5}-1.3\right):4.3}=2.625$$

CHAPTER II

ALGEBRAIC TRANSFORMATIONS

Simplify the following expressions:

46. $(a^2-b^2-c^2+2bc):\frac{a+b-c}{a+b+c}$

Evaluate the result at $a=8.6$; $b=\sqrt{3}$; $c=3\frac{1}{3}$

47. $\frac{a^2-1}{n^2+an}\cdot\left(\frac{1}{1-\frac{1}{n}}-1\right)\cdot\frac{a-an^3-n^4+n}{1-a^2}$

48. $\frac{x}{ax-2a^2}-\frac{2}{x^2+x-2ax-2a}\cdot\left(1+\frac{3x+x^2}{3+x}\right)$

49. $\frac{2a}{a^2-4x^2}+\frac{1}{2x^2+6x-ax-3a}\cdot\left(x+\frac{3x-6}{x-2}\right)$

50. $\left(\frac{2a+10}{3a-1}+\frac{130-a}{1-3a}+\frac{30}{a}-3\right)\cdot\frac{3a^3+8a^2-3a}{1-\frac{1}{4}a^2}$

51. $\frac{a^2-b^2}{a-b}-\frac{a^3-b^3}{a^2-b^2}$; **52.** $\frac{x}{x^2+y^2}-\frac{y(x-y)^2}{x^4-y^4}$

53. $\frac{2}{3}\left[\frac{1}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}+\frac{1}{1+\left(\frac{2x-1}{\sqrt{3}}\right)^2}\right]$

54. $\left[\frac{a-1}{a^2-2a+1}+\frac{2(a-1)}{a^2-4}-\frac{4(a+1)}{a^2+a-2}+\frac{a}{a^2-3a+2}\right]\times$

$\times\frac{36a^3-144a-36a^2+144}{a^3+27}$

55. $\left[\frac{3(x+2)}{2(x^3+x^2+x+1)}+\frac{2x^2-x-10}{2(x^3-x^2+x-1)}\right]:$

$:\left[\frac{5}{x^2+1}+\frac{3}{2(x+1)}-\frac{3}{2(x-1)}\right]$

56. $\left(\frac{x-y}{2y-x}-\frac{x^2+y^2+y-2}{x^2-xy-2y^2}\right):\frac{4x^4+4x^2y+y^2-4}{x^2+y+xy+x}$

57. $\frac{a^2+a-2}{a^{n+1}-3a^n}\cdot\left[\frac{(a+2)^2-a^2}{4a^2-4}-\frac{3}{a^2-a}\right]$

58. $\frac{2a^2(b+c)^{2n}-\frac{1}{2}}{an^2-a^3-2a^2-a}:\frac{2a(b+c)^n-1}{a^2c-a(nc-c)}$

59. $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}$

60. $\frac{1+(a+x)^{-1}}{1-(a+x)^{-1}}\cdot\left[1-\frac{1-(a^2+x^2)}{2ax}\right]$

Evaluate the result at $x=\frac{1}{a-1}$.

61. $\left[\frac{2+ba^{-1}}{a+2b}-6b(4b^2-a^2)^{-1}\right]:\left(2a^nb+3a^{n+1}-\frac{6a^{n+2}}{2a-b}\right)^{-1}$

62*. $\frac{\left[1-\left(\frac{a}{b}\right)^{-2}\right]a^2}{(\sqrt{a}-\sqrt{b})^2+2\sqrt{ab}}$

---

* Prior to solving subsequent problems read the notes on pp. 90 to 92.

63. $\frac{b}{a-b}\sqrt[3]{(a^2-2ab+b^2)(a^2-b^2)(a+b)}\cdot\frac{a^3-b^3}{\sqrt[3]{(a+b)^2}}$

64. $\sqrt[6]{8x(7+4\sqrt{3})}\sqrt[3]{2\sqrt{6x}-4\sqrt{2x}}$

65. $\frac{a}{2}\sqrt[4]{(a+1)(a^2-1)(1+2a+a^2)}\cdot\left(\frac{a^2+3a+2}{\sqrt{a-1}}\right)^{-1}$

66. $\sqrt{\frac{(1+a)\sqrt[3]{1+a}}{3a}}\sqrt[3]{\frac{\sqrt{3}}{9+18a^{-1}+9a^{-2}}}$

67. $ab\sqrt[n]{a^{1-n}b^{-n}-a^{-n}b^{1-n}}\sqrt[n]{(a-b)^{-1}}$

68. $\left(\frac{15}{\sqrt{6}+1}+\frac{4}{\sqrt{6}-2}-\frac{12}{3-\sqrt{6}}\right)(\sqrt{6}+11)$

69. $\left(\frac{1}{\sqrt{a}-\sqrt{a-b}}+\frac{1}{\sqrt{a}+\sqrt{a+b}}\right):\left(1+\sqrt{\frac{a+b}{a-b}}\right)$

70. $\left(\frac{1}{b-\sqrt{a}}+\frac{1}{b+\sqrt{a}}\right):\frac{\sqrt[-2]{\frac{1}{9}a^{-2}b^{-1}}}{a^{-2}-a^{-1}b^{-2}}$

71. $\frac{\sqrt{\frac{1+a}{1-a}}+\sqrt{\frac{1-a}{1+a}}}{\sqrt{\frac{1+a}{1-a}}-\sqrt{\frac{1-a}{1+a}}}-\frac{1}{a}$

72. Evaluate the expression

$$\frac{xy-\sqrt{x^2-1}\sqrt{y^2-1}}{xy+\sqrt{x^2-1}\sqrt{y^2-1}}$$

at $x=\frac{1}{2}\left(a+\frac{1}{a}\right)$, $y=\frac{1}{2}\left(b+\frac{1}{b}\right)$ $(a\geqslant 1,\ b\geqslant 1)$.

73. Evaluate the expression

$$\frac{\sqrt{a+bx}+\sqrt{a-bx}}{\sqrt{a+bx}-\sqrt{a-bx}}$$

at $x=\frac{2am}{b(1+m^2)}$, $|m|<1$.

Simplify the following expressions:

74. $\frac{(m+x)^{\frac{1}{2}}+(m-x)^{\frac{1}{2}}}{(m+x)^{\frac{1}{2}}-(m-x)^{\frac{1}{2}}}$

if $x=\frac{2mn}{n^2+1}$ and $m>0,\ 0<n<1$.

75. $$\left[\frac{(1-x^2)^{-\frac{1}{2}}+1}{2}\right]^{-\frac{1}{2}}+\left[\frac{(1-x^2)^{-\frac{1}{2}}-1}{2}\right]^{-\frac{1}{2}}$$

if $x=2k^{\frac{1}{2}}(1+k)^{-1}$ and $k>1$.

76. $$\left(\frac{1}{2}-\frac{1}{4a^{-1}}-\frac{2^{-2}}{a}\right)\left[(a-1)\sqrt[3]{(a+1)^{-3}}-\frac{(a+1)^{\frac{3}{2}}}{\sqrt{(a^2-1)(a-1)}}\right]$$

77. $$\left(2\sqrt{x^4-a^2x^2}-\frac{2a^2}{\sqrt{1-a^2x^{-2}}}\right)\cdot\frac{(x^2a^{-2}-4+4a^2x^{-2})^{-\frac{1}{2}}}{2ax\,(x^2-a^2)^{-\frac{1}{2}}}$$

78. $$\frac{a\left(\frac{\sqrt{a}+\sqrt{b}}{2b\sqrt{a}}\right)^{-1}+b\left(\frac{\sqrt{a}+\sqrt{b}}{2a\sqrt{b}}\right)^{-1}}{\left(\frac{a+\sqrt{ab}}{2ab}\right)^{-1}+\left(\frac{b+\sqrt{ab}}{2ab}\right)^{-1}}$$

79. $$\left(\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a+x}}-\frac{\sqrt{a+x}}{\sqrt{a}+\sqrt{x}}\right)^{-2}-\left(\frac{\sqrt{a}-\sqrt{x}}{\sqrt{a+x}}-\frac{\sqrt{a+x}}{\sqrt{a}-\sqrt{x}}\right)^{-2}$$

80. $$\frac{1}{2}\left(\sqrt{x^2+a}+\frac{x^2}{\sqrt{x^2+a}}\right)+\frac{a}{2}\cdot\frac{1+\frac{x}{\sqrt{x^2+a}}}{x+\sqrt{x^2+a}}$$

81. $$2x+\sqrt{x^2-1}\left(1+\frac{x^2}{x^2-1}\right)-\frac{1+\frac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}$$

82. Compute
$$\left[a^{-\frac{3}{2}}b\,(ab^{-2})^{-\frac{1}{2}}\,(a^{-1})^{-\frac{2}{3}}\right]^3$$
at $a=\frac{\sqrt{2}}{2},\ b=\frac{1}{\sqrt[3]{2}}$.

83. Evaluate the expression
$$(a+1)^{-1}+(b+1)^{-1}$$
at $a=(2+\sqrt{3})^{-1}$ and $b=(2-\sqrt{3})^{-1}$.

Simplify the following expressions:

84. $\dfrac{x+\sqrt{x^2-4x}}{x-\sqrt{x^2-4x}}-\dfrac{x-\sqrt{x^2-4x}}{x+\sqrt{x^2-4x}}$

85. $\dfrac{n+2+\sqrt{n^2-4}}{n+2-\sqrt{n^2-4}}+\dfrac{n+2-\sqrt{n^2-4}}{n+2+\sqrt{n^2-4}}$

86. $\sqrt{\dfrac{x}{x-a^2}}:\left(\dfrac{\sqrt{x}-\sqrt{x-a^2}}{\sqrt{x}+\sqrt{x-a^2}}-\dfrac{\sqrt{x}+\sqrt{x-a^2}}{\sqrt{x}-\sqrt{x-a^2}}\right)$

87. $\dfrac{x^{\frac{1}{2}}+1}{x+x^{\frac{1}{2}}+1}:\dfrac{1}{x^{1.5}-1}$; 88. $(2^{\frac{3}{2}}+27y^{\frac{3}{5}}):\left[\left(\frac{1}{2}\right)^{-\frac{1}{2}}+3y^{\frac{1}{5}}\right]$

89. Prove the identity

$$a^{\frac{1}{2}}-\frac{a-a^{-2}}{a^{\frac{1}{2}}-a^{-\frac{1}{2}}}+\frac{1-a^{-2}}{a^{\frac{1}{2}}+a^{-\frac{1}{2}}}+\frac{2}{a^{\frac{3}{2}}}=0$$

90. Compute

$$\frac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{(a^2-ab)^{\frac{2}{3}}}:\frac{a^{-\frac{2}{3}}\sqrt[3]{a-b}}{a\sqrt{a}-b\sqrt{b}}$$

at $a=1.2$ and $b=\frac{3}{5}$.

Simplify the following expressions:

91. $[(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}+5b^{\frac{1}{2}})-(a^{\frac{1}{2}}+2b^{\frac{1}{2}})(a^{\frac{1}{2}}-2b^{\frac{1}{2}})]:(2a+3a^{\frac{1}{2}}b^{\frac{1}{2}})$

Evaluate the result at $a=54$ and $b=6$.

92. $\dfrac{\left[(a+b)^{-\frac{1}{2}}+(a-b)^{-\frac{1}{2}}\right]^{-1}+\left[(a+b)^{-\frac{1}{2}}-(a-b)^{-\frac{1}{2}}\right]^{-1}}{\left[(a+b)^{-\frac{1}{2}}+(a-b)^{-\frac{1}{2}}\right]^{-1}-\left[(a+b)^{-\frac{1}{2}}-(a-b)^{-\frac{1}{2}}\right]^{-1}}$

93. $a^2(1-a^2)^{-\frac{1}{2}}-\dfrac{1}{1+\left[a(1-a^2)^{-\frac{1}{2}}\right]^2}\cdot\dfrac{(1-a^2)^{\frac{1}{2}}+a^2(1-a^2)^{-\frac{1}{2}}}{1-a^2}$

94. $$\frac{x^{\frac{5}{2}}-x^{-\frac{1}{2}}}{(x+1)(x^2+1)}-\left(x-\frac{x^3}{1+x^2}\right)^{-\frac{1}{2}}\cdot\frac{x^2\sqrt{(1+x^2)^{-1}}-\sqrt{1+x^2}}{1+x^2}$$

95. $$(R^2-x^2)^{\frac{1}{2}}-x^2(R^2-x^2)^{-\frac{1}{2}}+R^2\frac{(R^2-x^2)^{\frac{1}{2}}+x^2(R^2-x^2)^{-\frac{1}{2}}}{(R^2-x^2)\left[1+\left(\frac{\sqrt{R^2-x^2}}{x}\right)^{-2}\right]}$$

96. $$(p^{\frac{1}{2}}+q^{\frac{1}{2}})^{-2}(p^{-1}+q^{-1})+\frac{2}{(p^{\frac{1}{2}}+q^{\frac{1}{2}})^3}(p^{-\frac{1}{2}}+q^{-\frac{1}{2}})$$

97. $$\left[\frac{(a+\sqrt[3]{a^2x}):(x+\sqrt[3]{ax^2})-1}{\sqrt[3]{a}-\sqrt[3]{x}}-\frac{1}{\sqrt[3]{x}}\right]^6$$

98. $$\left[\frac{(\sqrt{a}+1)^2-\frac{a-\sqrt{ax}}{\sqrt{a}-\sqrt{x}}}{(\sqrt{a}+1)^3-a\sqrt{a}+2}\right]^{-3};$$ **99.** $$\left[\frac{4a-9a^{-1}}{2a^{\frac{1}{2}}-3a^{-\frac{1}{2}}}+\frac{a-4+3a^{-1}}{a^{\frac{1}{2}}-a^{-\frac{1}{2}}}\right]^2$$

100. $$\left[(a-b)\sqrt{\frac{a+b}{a-b}}+a-b\right]\left[(a-b)\left(\sqrt{\frac{a+b}{a-b}}-1\right)\right]$$

101. $$\left(\sqrt{ab}-\frac{ab}{a+\sqrt{ab}}\right):\frac{\sqrt[4]{ab}-\sqrt{b}}{a-b}$$

102. $$(a+b^{\frac{3}{2}}:\sqrt{a})^{\frac{2}{3}}\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}}+\frac{\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)^{-\frac{2}{3}}$$

103. $$\left[\frac{1}{x^{\frac{1}{2}}-4x^{-\frac{1}{2}}}+\frac{2\sqrt[3]{x}}{x\sqrt[3]{x}-4\sqrt[3]{x}}\right]^{-2}-\sqrt{x^2+8x+16}$$

104. $$x^3\left[\frac{(\sqrt[4]{x}+\sqrt[4]{y})^2+(\sqrt[4]{x}-\sqrt[4]{y})^2}{x+\sqrt{xy}}\right]^5\sqrt[3]{x\sqrt{x}}$$

105. $$\left(\frac{\sqrt[4]{ax^3}-\sqrt[4]{a^3x}}{\sqrt{a}-\sqrt{x}}+\frac{1+\sqrt{ax}}{\sqrt[4]{ax}}\right)^{-2}\sqrt{1+2\sqrt{\frac{a}{x}}+\frac{a}{x}}$$

106. $$\frac{(a-b^2)\sqrt{3}-b\sqrt{3}\sqrt[3]{-8b^3}}{\sqrt{2(a-b^2)^2+(2b\sqrt{2a})^2}}\cdot\frac{\sqrt{2a}-\sqrt{2c}}{\sqrt{\frac{3}{a}}-\sqrt{\frac{3}{c}}}$$

107. $$\left\{\sqrt{1+\left[(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{1}{2}}x^{-\frac{1}{3}}\right]_2}\right\}^{-6}-\frac{1}{a^2}\sqrt{(a^2-x^2)^2+4a^2x^2}$$

108. $\left[(\sqrt[4]{x}-\sqrt[4]{a})^{-1}+(\sqrt[4]{x}+\sqrt[4]{a})^{-1}\right]^{-2}:\dfrac{x-a}{4\sqrt{x}+4\sqrt{a}}$

109. $\left[\dfrac{\sqrt[6]{a^2x}+\sqrt{x}}{\sqrt[3]{x}+\sqrt[3]{a}}+\sqrt[6]{x}\right]^3+4(x+1)+\left(\sqrt[3]{x\sqrt{x}}+1\right)^2$

110. $\left[\dfrac{3x^{-\frac{1}{3}}}{x^{\frac{2}{3}}-2x^{-\frac{1}{3}}}-\dfrac{x^{\frac{1}{3}}}{x^{\frac{4}{3}}-x^{\frac{1}{3}}}\right]^{-1}-\left(\dfrac{1-2x}{3x-2}\right)^{-1}$

111. $\sqrt[-\frac{1}{2}]{\sqrt{a}}\left[\sqrt{a^2+a\sqrt{a^2-b^2}}-\sqrt{a^2-a\sqrt{a^2-b^2}}\right]^2$

112. $\left[\dfrac{(\sqrt[3]{x}-\sqrt[3]{a})^3+2x+a}{(\sqrt[3]{x}-\sqrt[3]{a})^3-x-2a}\right]^3+\sqrt{(a^3+3a^2x+3ax^2+x^3)^{\frac{2}{3}}:a}$

113. $\left[\dfrac{(\sqrt{a}+\sqrt{b})^2-(2\sqrt{b})^2}{a-b}-(a^{\frac{1}{2}}-b^{\frac{1}{2}})(a^{\frac{1}{2}}+b^{\frac{1}{2}})^{-1}\right]:\dfrac{(4b)^{\frac{3}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$

114. $\left(\dfrac{a-4b}{a+(ab)^{\frac{1}{2}}-6b}-\dfrac{a-9b}{a+6(ab)^{\frac{1}{2}}+9b}\right)\cdot\dfrac{b^{-\frac{1}{2}}}{a^{\frac{1}{2}}-3b^{\frac{1}{2}}}$

115. $\dfrac{\left(\dfrac{a-b}{\sqrt{a}+\sqrt{b}}\right)^3+2a\sqrt{a}+b\sqrt{b}}{3a^2+3b\sqrt{ab}}+\dfrac{\sqrt{ab}-a}{a\sqrt{a}-b\sqrt{a}}$

116. $\dfrac{(\sqrt{a}-\sqrt{b})^3+2a^2:\sqrt{a}+b\sqrt{b}}{a\sqrt{a}+b\sqrt{b}}+\dfrac{3\sqrt{ab}-3b}{a-b}$

117. $\left[\dfrac{1}{(a^{\frac{1}{2}}+b^{\frac{1}{2}})^{-2}}-\left(\dfrac{\sqrt{a}-\sqrt{b}}{a^{\frac{3}{2}}-b^{\frac{3}{2}}}\right)^{-1}\right](ab)^{-\frac{1}{2}}$

118. $\left[\dfrac{\dfrac{1}{a}-a}{\left(\sqrt[3]{a}+\sqrt[3]{\dfrac{1}{a}+1}\right)\left(\sqrt[3]{a}+\sqrt[3]{\dfrac{1}{a}-1}\right)}+\sqrt[3]{a}\right]^{-3}$

119. $\left[\dfrac{a\sqrt[3]{a}+\sqrt[3]{a^2}}{a+\sqrt[3]{a}}-\sqrt[3]{x}\right]\left[(\sqrt[3]{a}-\sqrt[3]{x})^2+3(\sqrt[3]{a}+\sqrt[3]{x})^2\right]$

120. $\left[\left(\dfrac{a^2-b\sqrt{a}}{\sqrt{a}-\sqrt[3]{b}}+a\sqrt[3]{b}\right):(a+\sqrt[6]{a^3b^2})-\sqrt[3]{b}\right]^2$

121. $$\left[\frac{a^2\sqrt[4]{x}+x\sqrt{a}}{a\sqrt[4]{x}+\sqrt{ax}}-\sqrt{a^2+x+2a\sqrt{x}}\right]^4$$

122. $$\left[\frac{x\sqrt{x}-x}{\left(\frac{\sqrt[4]{x^3}-1}{\sqrt[4]{x}-1}-\sqrt{x}\right)\left(\frac{\sqrt[4]{x^3}+1}{\sqrt[4]{x}+1}-\sqrt{x}\right)}\right]^3$$

123. $$\sqrt{a}\left[\frac{a+\sqrt[4]{a^3b^2}+b\sqrt[4]{ab^2}+b^2}{(\sqrt[4]{a}+\sqrt{b})^2}-b\right]^{-1}+\frac{1}{a^{-\frac{1}{4}}b^{\frac{1}{2}}-1}$$

124. $$\frac{\frac{a+x}{\sqrt[3]{a^2}-\sqrt[3]{x^2}}+\frac{\sqrt[3]{ax^2}-\sqrt[3]{a^2x}}{\sqrt[3]{a^2}-2\sqrt[3]{ax}+\sqrt[3]{x^2}}}{\sqrt[6]{a}-\sqrt[6]{x}}-\sqrt[6]{x}$$

125. $$\frac{1}{a^{\frac{1}{4}}+a^{\frac{1}{8}}+1}+\frac{1}{a^{\frac{1}{4}}-a^{\frac{1}{8}}+1}-\frac{2a^{\frac{1}{4}}-2}{a^{\frac{1}{2}}-a^{\frac{1}{4}}+2}$$

126. $$\frac{\sqrt{\sqrt{2}-1}\,\sqrt[4]{3+2\sqrt{2}}+\sqrt[3]{(x+12)\sqrt{x}-6x-8}}{\frac{x-\sqrt{x}}{\sqrt{x}-1}-\sqrt{\sqrt{2}+1}\,\sqrt[4]{3-2\sqrt{2}}}$$

127. $$\frac{\sqrt{a^3b}\,\sqrt[3]{a^4}+\sqrt{a^4b^3}:\sqrt[6]{a}}{(b^2-ab-2a^2)\sqrt{ab}}-$$
$$-a^{-\frac{2}{3}}\left(\frac{3a^2}{3b-6a+2ab-b^2}:\frac{a+b}{3a-ab}-\frac{ab}{a+b}\right)$$

128. $$\left[\frac{10x^2+3ax}{4x^2-a^2}+\frac{bx-x^2-ax+ab}{2x+a}:(b-x)-2\right]\times$$
$$\times\left[\frac{(a+2x)^{-\frac{1}{2}}+(2x-a)^{\frac{1}{2}}}{(4x^2-a^2)^{-\frac{1}{2}}+1}\right]^2$$

129. $$\left[\frac{x+4}{2x^2-2x-4}+\frac{x+2}{2(x^2+3x+2)}\right]\sqrt{2}\,x-$$
$$-\left(\sqrt{2}+\sqrt{x}-\frac{x+6}{\sqrt{x}+\sqrt{2}}\right):(x^{\frac{1}{2}}-2^{\frac{1}{2}})^2$$

130. $$\frac{(1-x^2)^{-\frac{1}{2}}+1}{(1+x)^{-\frac{1}{2}}+(1-x)^{\frac{1}{2}}}:\frac{\sqrt{1-x}}{x-2}+$$
$$+(x+1)\left(\frac{1}{x+1}+\frac{4}{x^2-4x}-\frac{5}{x^2-3x-4}\right)$$

131. $$\frac{a^2\sqrt{ab^{-1}}\sqrt[3]{b^2\sqrt{ab}}-2\sqrt{a^3b}\sqrt[6]{ab^5}}{(a^2-ab-2b^2)\sqrt[3]{a^5b}}-$$
$$-\frac{a-3}{a+2b}\left[\frac{a+2b}{a^2+ab-3a-3b}-(a-1)(a^2-4a+3)^{-1}\right]$$

132. $$\frac{\sqrt{a\sqrt{ab}}-(ab)^{\frac{3}{4}}:\sqrt{a}}{(a^2-b^2)\,a^{-1}}\cdot\left(\sqrt[4]{\frac{a}{b}}+\sqrt[4]{\frac{b}{a}}\right)+$$
$$+\frac{b}{a}\cdot\left(\frac{2a+2b}{a-4b}+\frac{a+3b}{2a+2b}-\frac{a^2+21ab}{2a^2-6ab-8b^2}\right)$$

133. $$\left[\frac{\left(\sqrt[3]{ab^2\sqrt{b}}-\sqrt[3]{ab}\sqrt{a}\right)^2}{ab\sqrt[6]{ab}}+4\right]:\frac{a\sqrt{b}+b\sqrt{a}}{\sqrt{a}-\sqrt{b}}+$$
$$+\frac{b^2-4a^2}{4a}\cdot\left(\frac{1}{b^2+3ab+2a^2}-\frac{3}{2a^2+ab-b^2}\right)$$

134. $$\frac{4\,(2ab)^{\frac{3}{4}}(a+2b)^{-1}}{\sqrt{a}-\sqrt{2b}}:\frac{\sqrt{2b\sqrt{2ab}}+\sqrt[4]{2a^3b}}{\sqrt{2ab}}-$$
$$-6\left(\frac{a}{6a-48b}-\frac{2b}{3a-6b}-\frac{8b^2}{a^2-10ab+16b^2}\right)$$

CHAPTER III

ALGEBRAIC EQUATIONS

Solve the following equations:

135. $\dfrac{6b+7a}{6b}-\dfrac{3ay}{2b^2}=1-\dfrac{ay}{b^2-ab}$; **136.** $\dfrac{ax-b}{a+b}+\dfrac{bx+a}{a-b}=\dfrac{a^2+b^2}{a^2-b^2}$

137. $$\frac{x-a-b}{c}+\frac{x-b-c}{a}+\frac{x-c-a}{b}=3$$

138. $$\frac{c+3z}{4c^2+6cd}-\frac{c-2z}{9d^2-6cd}=\frac{2c+z}{4c^2-9d^2}$$

139. $$\frac{x-1}{n-1}+\frac{2n^2(1-x)}{n^4-1}=\frac{2x-1}{1-n^4}-\frac{1-x}{1+n}$$

140. $$\frac{3ab+1}{a}x=\frac{3ab}{a+1}+\frac{(2a+1)\,x}{a\,(a+1)^2}+\frac{a^2}{(a+1)^3}$$

141. $$\frac{3abc}{a+b}+\frac{a^2b^2}{(a+b)^3}+\frac{(2a+b)\,b^2x}{a\,(a+b)^2}=3cx+\frac{bx}{a}$$

142. $$\frac{x+m}{a+b}-\frac{ax}{(a+b)^2}=\frac{am}{a^2-b^2}-\frac{b^2x}{a^3-ab^2+a^2b-b^3}$$

143. $\frac{m}{z}+\frac{z}{m}+\frac{m(z-m)}{z(z+m)}-\frac{z(z+m)}{m(z-m)}=\frac{mz}{m^2-z^2}-2$

144. $\frac{a^2+x}{b^2-x}-\frac{a^2-x}{b^2+x}=\frac{4abx+2a^2-2b^2}{b^4-x^2}$

145. $\frac{an}{a-x}+\frac{(a+n)(anx+nx^2+x^3)}{x^3+nx^2-a^2x-a^2n}=\frac{ax}{n+x}+\frac{nx^2}{x^2-a^2}$

146. $\left(\frac{a+1}{ax+1}+\frac{x+1}{x+a^{-1}}-1\right):\left[\frac{a+1}{(x+a^{-1})a}-\frac{a(x+1)}{ax+1}+1\right]=\frac{x}{2}$

147. $\frac{a+x}{a^2+ax+x^2}-\frac{a-x}{ax-x^2-a^2}=\frac{3a}{x(a^4+a^2x^2+x^4)}$

148. $a(\sqrt{x}-a)-b(\sqrt{x}-b)+a+b=\sqrt{x}$

149. $\frac{1}{a}+\frac{1}{a+x}+\frac{1}{a+2x}=0$; **150.** $\frac{2x}{x+b}-\frac{x}{b-x}=\frac{b^2}{4(x^2-b^2)}$

151. $1-\frac{2a}{x-a}=\frac{b^2-a^2}{a^2+x^2-2ax}$; **152.** $\frac{x^2}{ab-2b^2}=\frac{a-b}{ac^2-2bc^2}+\frac{x}{bc}$

153. $\frac{x}{x+a}+\frac{2x}{x-a}=\frac{5a^2}{4(x^2-a^2)}$; **154.** $\frac{x^2+1}{n^2x-2n}-\frac{1}{2-nx}=\frac{x}{n}$

155. $\frac{a-x^2}{(a-x)^2}-\frac{1}{a}=\frac{a-1}{a^3-ax(2a-x)}$; **156.** $1-\frac{2b}{x-a}=\frac{a^2-b^2}{a^2+x^2-2ax}$

157. $\frac{1}{2n+nx}-\frac{1}{2x-x^2}=\frac{2(n+3)}{x^3-4x}$; **158.** $\frac{a+x-2n}{2a-n}-\frac{a-2n}{x}=1$

159. $\frac{a}{nx-x}-\frac{a-1}{x^2-2nx^2+n^2x^2}=1$; **160.** $\frac{\left(\frac{a-x}{x}\right)^2-\left(\frac{a}{a+b}\right)^2}{x^2+a^2-2ax}=\frac{5}{9x^2}$

161. $\frac{x+x^2}{1-x^2}:\frac{1-a^2}{(1+ax)^2-(a+x)^2}=\frac{ab}{(b-a)^2}$

162. Factor the following expression into linear factors:

$$11x-3x^2+70$$

163. Factor the expression $\frac{a}{b}-\frac{b}{a}$ into two factors, whose sum is

$$\frac{a}{b}+\frac{b}{a}$$

164. Factor the following expression:

$$15x^3+x^2-2x$$

165. Factor the following expression:

$$x^3+2x^4+4x^2+2+x$$

165a. Solve the equation

$$(1+x^2)^2=4x(1-x^2)$$

166. Write a quadratic equation, whose roots are

$$\frac{a}{b} \text{ and } \frac{b}{a}$$

167. Set up a quadratic equation, whose roots are

$$\frac{1}{10-\sqrt{72}} \text{ and } \frac{1}{10+6\sqrt{2}}$$

168. Write a quadratic equation, whose roots are

$$\frac{a}{\sqrt{a}\pm\sqrt{a-b}}$$

169. The roots x_1 and x_2 of the quadratic equation

$$x^2+px+12=0$$

possess the following property: $x_1-x_2=1$. Find the coefficient p.

170. In the equation

$$5x^2-kx+1=0$$

determine k such that the difference of the roots be equal to unity.

171. The roots x_1 and x_2 of the equation

$$x^2-3ax+a^2=0$$

are such that $x_1^2+x_2^2=1.75$. Determine a.

172. In the quadratic equation

$$x^2+px+q=0$$

determine the coefficients such that the roots be equal to p and q.

173. The roots of the quadratic equation

$$ax^2+bx+c=0$$

are x_1 and x_2. Set up a new quadratic equation, whose roots are $\frac{x_1}{x_2}$ and $\frac{x_2}{x_1}$.

174. Given a quadratic equation

$$ax^2+bx+c=0$$

Set up a new quadratic equation, whose roots are:
(1) twice as large as the roots of the given equation;
(2) reciprocal to the roots of the given equation.

175. Set up a quadratic equation, whose roots are equal to the cubes of the roots of the equation

$$ax^2 + bx + c = 0$$

176. Set up a biquadratic equation, the sum of the squared roots of which is 50, the product of the roots being equal to 144.

177. Find all the roots of the equation

$$4x^4 - 24x^3 + 57x^2 + 18x - 45 = 0$$

if one of them is $3 + i\sqrt{6}$.

178. Determine the constant term of the equation

$$6x^3 - 7x^2 - 16x + m = 0$$

if it is known that one of its roots is equal to 2. Find the remaining two roots.

179. Knowing that 2 and 3 are the roots of the equation

$$2x^3 + mx^2 - 13x + n = 0$$

determine m and n and find the third root of the equation.

180. At what numerical values of a does the equation

$$x^2 + 2ax\sqrt{a^2 - 3} + 4 = 0$$

have equal roots?

180a. In what interval must the number m vary so that both roots of the equation

$$x^2 - 2mx + m^2 - 1 = 0$$

lie between -2 and 4?

Solve the following equations:

181. $\sqrt{y+2} - \sqrt{y-6} = 2$; **182.** $\sqrt{22-x} - \sqrt{10-x} = 2$

183. $\sqrt{3x+1} - \sqrt{x-1} = 2$; **184.** $\sqrt{x+3} + \sqrt{3x-2} = 7$

185. $\sqrt{x+1} + \sqrt{2x+3} = 1$; **186.** $\sqrt{3x-2} = 2\sqrt{x+2} - 2$

187. $\sqrt{2x+1} + \sqrt{x-3} = 2\sqrt{x}$; **188.** $\sqrt{1 + x\sqrt{x^2+24}} = x + 1$

189. $\frac{3+x}{3x} = \sqrt{\frac{1}{9} + \frac{1}{x}\sqrt{\frac{4}{9} + \frac{2}{x^2}}}$

190. $\sqrt{\frac{x-5}{x+2}} + \sqrt{\frac{x-4}{x+3}} = \frac{7}{x+2}\sqrt{\frac{x+2}{x+}}$

191. $\frac{\sqrt{x^2-16}}{\sqrt{x-3}} + \sqrt{x+3} = \frac{7}{\sqrt{x-3}}$

192. $\dfrac{4}{x+\sqrt{x^2+x}}-\dfrac{1}{x-\sqrt{x^2+x}}=\dfrac{3}{x}$

193. $\dfrac{2}{2+\sqrt{4-x^2}}-\dfrac{1}{2-\sqrt{4-x^2}}=\dfrac{1}{x}$

194. $\sqrt{2\sqrt{7}+\sqrt{x}}-\sqrt{2\sqrt{7}-\sqrt{x}}=\sqrt[4]{28}$

195. $\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\dfrac{3}{2}\sqrt{\dfrac{x}{x+\sqrt{x}}}$

196. $\dfrac{\sqrt{27+x}+\sqrt{27-x}}{\sqrt{27+x}-\sqrt{27-x}}=\dfrac{27}{x}$; **197.** $x=a-\sqrt{a^2-x\sqrt{x^2+a^2}}$

198. $\dfrac{\sqrt{1+a^{-2}x^2}-xa^{-1}}{\sqrt{1+a^{-2}x^2}+xa^{-1}}=\dfrac{1}{4}$; **199.** $\dfrac{\sqrt{1+a^2x^2}-ax}{\sqrt{1+a^2x^2}+ax}=\dfrac{1}{c^2}$

200. $\dfrac{x+c+\sqrt{x^2-c^2}}{x+c-\sqrt{x^2-c^2}}=\dfrac{9(x+c)}{8c}$

201. $\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$

202. $2\sqrt{a+x}+\sqrt{a-x}=\sqrt{a-x+\sqrt{x(a+x)}}$

203. $\sqrt{a^2-x}+\sqrt{b^2-x}=a+b$

204. $\sqrt{a-x}+\sqrt{b+x}=\sqrt{a+b}$

205. $\sqrt{x+a}=a-\sqrt{x}$; **206.** $\dfrac{\sqrt{a+x}}{a}+\dfrac{\sqrt{a+x}}{x}=\sqrt{x}$

207. $\sqrt{x}+\sqrt[4]{x}=12$; **208.** $(x-1)^{\frac{1}{2}}+6(x-1)^{\frac{1}{4}}=16$

209. $\sqrt[3]{2+\sqrt{10+2x}}=-\sqrt[3]{\sqrt{15-2x}-9}$

210. $\sqrt[3]{x}+\sqrt[3]{2x-3}=\sqrt[3]{12(x-1)}$

211. $\sqrt[3]{a-x}+\sqrt[3]{b-x}=\sqrt[3]{a+b-2x}$

212. $\sqrt[3]{x}+2\sqrt[3]{x^2}=3$; **213.** $2\sqrt[3]{z^2}-3\sqrt[3]{z}=20$

214. $\sqrt{a+x}-\sqrt[3]{a+x}=0$; **215.** $\sqrt{\dfrac{2x+2}{x+2}}-\sqrt{\dfrac{x+2}{2x+2}}=\dfrac{7}{12}$

216. $x^2+11+\sqrt{x^2+11}=42$; **217.** $\dfrac{x\sqrt[3]{x}-1}{\sqrt[3]{x^2}-1}-\dfrac{\sqrt[3]{x^2}-1}{\sqrt[3]{x}+1}=4$

218. $\dfrac{x-4}{\sqrt{x}+2}=x-8$; **219.** $\dfrac{(a-x)\sqrt{a-x}+(x-b)\sqrt{x-b}}{\sqrt{a-x}+\sqrt{x-b}}=a-b$

220. $\dfrac{2-x}{2-\sqrt{x}}=\sqrt{\dfrac{2-x}{2}}$; **221.** $\dfrac{x-1}{1+\sqrt{x}}=4-\dfrac{1-\sqrt{x}}{2}$

222. $\sqrt{x^2-3x+5}+x^2=3x+7$

223. $\sqrt{3x^2+5x-8}-\sqrt{3x^2+5x+1}=1$

224. $\sqrt{y^2+4y+8}+\sqrt{y^2+4y+4}=\sqrt{2(y^2+4y+6)}$

Solve the following systems of equations:

225. $\begin{cases} x^2+y^2=2(xy+2) \\ x+y=6 \end{cases}$ **226.** $\begin{cases} x+xy+y=11 \\ x^2y+xy^2=30 \end{cases}$

227. $\begin{cases} x+y^2=7 \\ xy^2=12 \end{cases}$ **228.** $\begin{cases} x^2-y=23 \\ x^2y=50 \end{cases}$

229. $\begin{cases} (x^2-y^2)\,xy=180 \\ x^2-xy-y^2=-11 \end{cases}$ **230.** $\begin{cases} 3x^2-2xy+5y^2-35=0 \\ 5x^2-10y^2-5=0 \end{cases}$

231. $\begin{cases} x^2+y^2=\dfrac{5}{2}\,xy \\ x-y=\dfrac{1}{4}\,xy \end{cases}$ **232.** $\begin{cases} x^2+xy+y^2=13 \\ x+y=4 \end{cases}$

233. $\begin{cases} x^2-xy+y^2=7 \\ x-y=1 \end{cases}$ **234.** $\begin{cases} \dfrac{x}{y}+\dfrac{y}{x}=\dfrac{25}{12} \\ x^2-y^2=7 \end{cases}$

235. $\begin{cases} \left(\dfrac{x}{a}\right)^m\cdot\left(\dfrac{y}{b}\right)^n=c \\ \left(\dfrac{x}{b}\right)^n\cdot\left(\dfrac{y}{a}\right)^m=d \end{cases}$

Give positive solutions only, assuming that $a>0$, $b>0$, $c>0$, $d>0$ and $m\neq n$.

236. $\begin{cases} x^2-xy+y^2=7 \\ x^3+y^3=35 \end{cases}$ **237.** $\begin{cases} x^3+y^3=7 \\ xy(x+y)=-2 \end{cases}$

Give real solutions only.

238. $\begin{cases} xy(x+y)=30 \\ x^3+y^3=35 \end{cases}$ **239.** $\begin{cases} \dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=5\dfrac{1}{5} \\ xy=6 \end{cases}$

240. $$\begin{cases} x+y+z=1 \\ ax+by+cz=d \\ a^2x+b^2y+c^2z=d^2 \end{cases}$$

241. $$\begin{cases} x+2y+3z+4u=30 \\ 2x-3y+5z-2u=3 \\ 3x+4y-2z-u=1 \\ 4x-y+6z-3u=8 \end{cases}$$

242. $$\begin{cases} x+y+z=4 \\ x+2y+3z=5 \\ x^2+y^2+z^2=14 \end{cases}$$

243. $$\begin{cases} \sqrt{4x+y-3z+7}=2 \\ \sqrt[3]{2y+5x+z+25.5}=3 \\ \sqrt{y+z}-\sqrt{6x}=0 \end{cases}$$

244. $$\begin{cases} x+y+z=13 \\ x^2+y^2+z^2=61 \\ xy+xz=2yz \end{cases}$$

245. $$\begin{cases} x^2+y^2=z^2 \\ xy+yz+zx=47 \\ (z-x)(z-y)=2 \end{cases}$$

246. $$\begin{cases} a^3+a^2x+ay+z=0 \\ b^3+b^2x+by+z=0 \\ c^3+c^2x+cy+z=0 \end{cases}$$

247. $$\begin{cases} \dfrac{12}{\sqrt{x-1}}+\dfrac{5}{\sqrt{y+\frac{1}{4}}}=5 \\ \dfrac{8}{\sqrt{x-1}}+\dfrac{10}{\sqrt{y+\frac{1}{4}}}=6 \end{cases}$$

248. $$\begin{cases} x+y-2\sqrt{xy}=4 \\ x+y=10 \end{cases}$$

249. $$\begin{cases} \sqrt{\dfrac{3x}{x+y}}-2+\sqrt{\dfrac{x+y}{3x}}=0 \\ xy-54=x+y \end{cases}$$

250. $$\begin{cases} \frac{1}{4}\sqrt[3]{x^2+y^2}-\frac{1}{2}\sqrt[3]{17}=0 \\ \sqrt{x+y}+\sqrt{x-y}=6 \end{cases}$$

251. $$\begin{cases} \sqrt{x+y}+\sqrt{x-y}=4\sqrt{a} \\ \sqrt{x^2+y^2}-\sqrt{x^2-y^2}=(\sqrt{41}-3)a \end{cases}$$

252. $$\begin{cases} \sqrt{x^2+y^2}-\sqrt{x^2-y^2}=y \\ x^4-y^4=144a^4 \end{cases}$$

253. $$\begin{cases} x^2+xy+y^2=84 \\ x+\sqrt{xy}+y=14 \end{cases}$$

253a. Find all the values of m for which the system of equations

$$\begin{cases} x-y=m(1+xy) \\ 2+x+y+xy=0 \end{cases}$$

has real solutions.

CHAPTER IV

LOGARITHMIC AND EXPONENTIAL EQUATIONS

Determine x without using logarithmic tables:

254[*]. $x+10\cdot 100^{\frac{1}{2}\log 9-\log 2}$; **255.** $x=100^{\frac{1}{2}-\log\sqrt[4]{4}}$

256. $x=\sqrt{10^{2+\frac{1}{2}\log 16}}$; **257.** $x=49^{1-\log_7 2}+5^{-\log_5 4}$

Solve the following equations:

258. $\log_4\log_3\log_2 x=0$

259. $\log_a\{1+\log_b[1+\log_c(1+\log_p x)]\}=0$

260. $\log_4\{2\log_3[1+\log_2(1+3\log_2 x)]\}=\frac{1}{2}$

261. $\log_2(x+14)+\log_2(x+2)=6$

262. $\log_a y+\log_a(y+5)+\log_a 0.02=0$

263. $\frac{\log(35-x^3)}{\log(5-x)}=3$

264. $1+\log x=\frac{1}{3}\log\left[b-\frac{(3a-b)(a^2+ab)^{-1}}{b^{-2}}\right]-$
$-\frac{4}{3}\log b+\frac{1}{3}\log(a^3-ab^2)$

265. $\log\left[x-a(1-a)^{-\frac{1}{2}}\right]-\frac{1}{2}\log\left(1+\frac{1}{a}\right)-$
$-\log\sqrt{\frac{a^3+a}{a+1}-a^2}=0$

266. $\log_x\sqrt{5}+\log_x(5x)-2.25=(\log_x\sqrt{5})^2$

267. $\log_{16}x+\log_4 x+\log_2 x=7$

268. $\log_a x-\log_{a^2}x+\log_{a^4}x=\frac{3}{4}$; **269.** $\left(\frac{3}{7}\right)^{3x-7}=\left(\frac{7}{3}\right)^{7x-3}$

270. $7\cdot 3^{x+1}-5^{x+2}=3^{x+4}-5^{x+3}$; **271.** $0.125\cdot 4^{2x-3}=\left(\frac{\sqrt{2}}{8}\right)^{-x}$

---

[*] Throughout this book, the symbol log stands for the logarithm to the base 10.

272. $0.5^{x^2}\cdot 2^{2x+2} = 64^{-1}$; **273.** $32^{\frac{x+5}{x-7}} = 0.25\cdot 128^{\frac{x+17}{x-3}}$

274. $\left(\frac{4}{9}\right)^x \left(\frac{27}{8}\right)^{x-1} = \frac{\log 4}{\log 8}$; **275.** $\left[2\left(2^{\sqrt{x}+3}\right)^{\frac{1}{2\sqrt{x}}}\right]^{\frac{2}{\sqrt{x}-1}} = 4$

276. $2\left(2^{\sqrt{x}+3}\right)^{2^{-1}x^{-\frac{1}{2}}} - \sqrt[\sqrt{x}-1]{4^2} = 0$

277. $\sqrt[x^2-1]{a^3}\,\sqrt[2x-2]{a}\,\sqrt[4]{a^{-1}} = 1$; **278.** $3\log_{xa^2}x + \frac{1}{2}\log_{\frac{x}{\sqrt{a}}} x = 2$

279. $\log_4(x+12)\cdot\log_x 2 = 1$; **280.** $\log_x(5x^2)\cdot\log_5^2 x = 1$

281. $1 + a + a^2 + a^3 + \ldots + a^{x-1} + a^x = (1+a)(1+a^2)(1+a^4)(1+a^8)$

282. $5^2\cdot 5^4\cdot 5^6\cdot\ldots\cdot 5^{2x} = 0.04^{-28}$; **283.** $4^{x-2} - 17\cdot 2^{x-4} + 1 = 0$

284. $2\cdot 4^{2x} - 17\cdot 4^x + 8 = 0$; **285.** $3\sqrt[x]{81} - 10\sqrt[x]{9} + 3 = 0$

286. $x^{\frac{\log x+7}{4}} = 10^{\log x+1}$

287. $\log\left(4^{-1}\cdot 2^{\sqrt{x}} - 1\right) - 1 = \log\left(\sqrt{2^{\sqrt{x}-2}} + 2\right) - 2\log 2$

288. $2(\log 2 - 1) + \log\left(5^{\sqrt{x}} + 1\right) = \log\left(5^{1-\sqrt{x}} + 5\right)$

289. $5^{\log x} - 3^{\log x - 1} = 3^{\log x + 1} - 5^{\log x - 1}$

290. $x^{2\log^3 x - 1.5\log x} = \sqrt{10}$; **291.** $\log\left(64\sqrt[24]{2^{x^2-40x}}\right) = 0$

292. $\log_2(9 - 2^x) = 3 - x$

293. $\log 2 + \log\left(4^{x-2} + 9\right) = 1 + \log\left(2^{x-2} + 1\right)$

294. $2\log 2 + \left(1 + \frac{1}{2x}\right)\log 3 - \log\left(\sqrt[x]{3} + 27\right) = 0$

295. $\log\left(3^{\sqrt{4x+1}} - 2^{4-\sqrt{4x+1}}\right) - 2 = \frac{1}{4}\log 16 - \sqrt{x+0.25}\log 4$

296. $\dfrac{2\log 2 + \log(x-3)}{\log(7x+1) + \log(x-6) + \log 3} = \dfrac{1}{2}$

297. $\log_5 120 + (x-3) - 2\log_5\left(1 - 5^{x-3}\right) = -\log_5\left(0.2 - 5^{x-4}\right)$

Solve the following systems of equations:

298. $\begin{cases} 8^{2x+1} = 32\cdot 2^{4y-1} \\ 5\cdot 5^{x-y} = \sqrt{25^{2y+1}} \end{cases}$ **299.** $\begin{cases} \log_3 x + \log_3 y = 0 \\ x + y = 3.(3) \end{cases}$

300. $$\begin{cases}\log_a x+\log_a y=2\\ \log_b x-\log_b y=4\end{cases}$$

301. $$\begin{cases}\log(x^2+y^2)-1=\log 13\\ \log(x+y)-\log(x-y)=3\log 2\end{cases}$$

302. $$\begin{cases}\log_{xy}(x-y)=1\\ \log_{xy}(x+y)=0\end{cases}$$

303. $$\begin{cases}\log_a\left(1+\frac{x}{y}\right)=2-\log_a y\\ \log_b x+\log_b y=4\end{cases}$$

304. $$\begin{cases}\log_a x+\log_a y+\log_a 4=2+\log_a 9\\ x+y-5a=0\end{cases}$$

305. $$\begin{cases}xy=a^2\\ \log^2 x+\log^2 y=2.5\log^2(a^2)\end{cases}$$

306. $$\begin{cases}3^x\cdot 2^y=576\\ \log_{\sqrt{2}}(y-x)=4\end{cases}$$

307. $$\begin{cases}\log x+\log y=\log a\\ 2(\log x-\log y)=\log b\end{cases}$$

308. $$\begin{cases}\log_a x+\log_{a^2}y=\frac{3}{2}\\ \log_{b^2}x+\log_b y=\frac{3}{2}\end{cases}$$

309. $$\begin{cases}\log_a x+\log_{a^2}y=\frac{3}{2}\\ \log_{b^2}x-\log_{b^2}y=1\end{cases}$$

310. $$\begin{cases}\log_v u+\log_u v=2\\ u^2+v=12\end{cases}$$

311. $$\begin{cases}x^2+xy+y^2=a^2\\ \log_{\frac{x}{\sqrt{a}}}\sqrt{a}+\log_{\frac{y}{\sqrt{b}}}\sqrt{b}=\frac{a}{\sqrt{3}}\end{cases}$$

312. $$\begin{cases}\log_4 x-\log_2 y=0\\ x^2-5y^2+4=0\end{cases}$$

313. $$\begin{cases}\log_2 x+\log_4 y+\log_4 z=2\\ \log_3 y+\log_9 z+\log_9 x=2\\ \log_4 z+\log_{16}x+\log_{16}y=2\end{cases}$$

314. $$\begin{cases}\sqrt[x-y]{x+y}=2\sqrt{3}\\ (x+y)\,2^{y-x}=3\end{cases}$$

315. $$\begin{cases}\sqrt[10]{2^x}\sqrt{\sqrt[5]{2^y}}=\sqrt[x]{128}\\ \log(x+y)=\log 40-\log(x-y)\end{cases}$$

316. $$\begin{cases}\sqrt[y]{4^x}=32\sqrt[x]{8^y}\\ \sqrt[y]{3^x}=3\sqrt[y]{9^{1-y}}\end{cases}$$

317. $$\begin{cases}9^{-1}\sqrt[y]{9^x}-27\sqrt[x]{27^y}=0\\ \log(x-1)-\log(1-y)=0\end{cases}$$

318. $$\begin{cases}\frac{1}{2}\log x+\frac{1}{2}\log y-\log\left(4-\sqrt{x}\right)=0\\ \left(25^{\sqrt{x}}\right)^{\sqrt{y}}-125\cdot 5^{\sqrt{y}}=0\end{cases}$$

319. $$\begin{cases}\log_x ay=p\\ \log_y bx=q\end{cases}$$

CHAPTER V

PROGRESSIONS

Notation and formulas

a_1 = first term of arithmetic progression
a_n = nth term of arithmetic progression
d = common difference of arithmetic progression
u_1 = first term of geometric progression
u_n = nth term of geometric progression
q = common ratio of geometric progression
S_n = sum of the first n terms of a progression
S = sum of infinitely decreasing geometric progression

Formulas for arithmetic progression

$$a_n = a_1 + d(n-1) \tag{1}$$

$$S_n = \frac{(a_1 + a_n)n}{2} \tag{2}$$

$$S_n = \frac{[2a_1 + d(n-1)]n}{2} \tag{3}$$

Formulas for geometric progression

$$u_n = u_1 q^{n-1} \tag{4}$$

$$S_n = \frac{u_n q - u_1}{q-1} \; (q > 1) \text{ or } S_n = \frac{u_1 - u_n q}{1-q} \; (q < 1) \tag{5}$$

$$S_n = \frac{u_1(q^n - 1)}{q-1} \; (q > 1) \text{ or } S_n = \frac{u_1(1-q^n)}{1-q} \; (q < 1) \tag{6}$$

$$S = \frac{u_1}{1-q} \tag{7}$$

ARITHMETIC PROGRESSION

320. How many terms of the arithmetic progression

$$5;\ 9;\ 13;\ 17;\ \ldots$$

is it necessary to take for their sum to equal 10,877?

321. Find an arithmetic progression, if the sum of its first four terms is equal to 26, and the product of the same terms equals 880.

322. In an arithmetic progression $a_p = q$; $a_q = p$. Express a_n in terms of n, p, and q.

323. Find the sum of all two-digit natural numbers.

324. Find four successive odd numbers, if the sum of their squares exceeds by 48 the sum of the squares of the even numbers contained between them.

325. An arithmetic progression consists of 20 terms. The sum of the terms occupying even places is equal to 250, and that of the terms occupying odd places equals 220. Find the two medium terms of the progression.

326. Given a sequence of expressions: $(a+x)^2$; (a^2+x^2); $(a-x)^2$; . . . . Prove that they form an arithmetic progression, and find the sum of its first n terms.

327. Denoting the sums of the first n_1, first n_2, and first n_3 terms of an arithmetic progression by S_1, S_2, and S_3, respectively, show that

$$\frac{S_1}{n_1}(n_2-n_3)+\frac{S_2}{n_2}(n_3-n_1)+\frac{S_3}{n_3}(n_1-n_2)=0$$

328. Write an arithmetic progression whose first term is 1, the sum of the first five terms being equal to $\frac{1}{4}$ of that of the next five terms.

329. Find an arithmetic progression in which the sum of any number of terms is always three times the squared number of these terms.

330. Find the sum of all two-digit numbers which, when divided by 4, yield unity as a remainder.

GEOMETRIC PROGRESSION

331. Insert three geometric means between the numbers 1 and 256.

332. Find the three numbers forming a geometric progression, if it is known that the sum of the first and third terms is equal to 52, and the square of the second term is 100.

333. Write first several terms of a geometric progression in which the difference between the third and first terms is equal to 9, and that between the fifth and third terms equals 36.

334. Find the four numbers forming a geometric progression in which the sum of the extremes is equal to 27, and the product of the means, to 72.

335. Find the four numbers forming a geometric progression, knowing that the sum of the extremes is equal to 35, and the sum of the means, to 30.

336. Determine a geometric progression in which

$$u_1+u_2+u_3+u_4+u_5=31$$

and

$$u_2+u_3+u_4+u_5+u_6=62$$

337. A geometric progression consists of five terms; their sum less the first term is equal to $19\frac{1}{2}$, and that less the last one equals 13. Compute the extremes of the progression.

338. Find the first term and common ratio of a geometric progression consisting of nine terms, such that the product of its extremes is equal to 2304, and the sum of the fourth and sixth terms equals 120.

339. Three numbers form a geometric progression. The sum of these numbers is equal to 126, and their product, to 13,824. Find these numbers.

340. A geometric progression consists of an even number of terms. The sum of all the terms is three times that of the odd terms. Determine the common ratio of the progression.

INFINITELY DECREASING GEOMETRIC PROGRESSION

341. Prove that the numbers

$$\frac{\sqrt{2}+1}{\sqrt{2}-1};\quad \frac{1}{2-\sqrt{2}};\quad \frac{1}{2};\ \dots$$

constitute an infinitely decreasing geometric progression and find the limit of the sum of its terms.

342. Compute the expression

$$(4\sqrt{3}+8)\left[\sqrt{3}\,(\sqrt{3}-2)+\frac{3-2\sqrt{3}}{\sqrt{3}}+\frac{\sqrt{3}-2}{\sqrt{3}}+\dots\right]$$

after proving that the bracketed addends are the terms of a decreasing geometric progression.

343. Find the sum of the terms of an infinitely decreasing geometric progression in which all the terms are positive, the first term is 4, and the difference between the third and fifth terms is equal to $\frac{32}{81}$.

344. Determine the sum of an infinitely decreasing geometric progression, if it is known that the sum of its first and fourth terms is equal to 54, and the sum of the second and third terms, to 36.

345. In an infinitely decreasing geometric progression the sum of all the terms occupying odd places is equal to 36, and that of all the terms at even places equals 12. Find the progression.

346. The sum of the terms of an infinitely decreasing geometric progression is equal to 56, and the sum of the squared terms of the same progression is 448. Find the first term and the common ratio.

347. The sum of the terms of an infinitely decreasing geometric progression is equal to 3, and the sum of the cubes of all its terms equals $\frac{108}{13}$. Write the progression.

348. Determine an infinitely decreasing geometric progression, the second term of which is 6, the sum of the terms being equal to $\frac{1}{8}$ of that of the squares of the terms.

ARITHMETIC AND GEOMETRIC PROGRESSIONS

349. The second term of an arithmetic progression is 14, and the third one 16. It is required to set up a geometric progression such that its common ratio would be equal to the common difference of the arithmetic progression, and the sum of the first three terms would be the same in both progressions.

350. The first and third terms of an arithmetic and a geometric progressions are equal to each other, respectively, the first terms being equal to 3. Write these progressions, if the second term of the arithmetic progression exceeds by 6 the second term of the geometric progression.

351. In a geometric progression the first, third and fifth terms may be considered as the first, fourth and sixteenth terms of an arithmetic progression. Determine the fourth term of this arithmetic progression, knowing that its first term is 5.

352. Three numbers, whose sum is equal to 93, constitute a geometric progression. They may also be considered as the first, second and seventh terms of an arithmetic progression. Find these numbers.

353. In an arithmetic progression the first term is 1, and the sum of the first seven terms is equal to 2555. Find the medium term of a geometric progression consisting of seven terms, if the first and the last terms coincide with the respective terms of the indicated arithmetic progression.

354. The sum of the three numbers constituting an arithmetic progression is equal to 15. If 1, 4 and 19 are added to them, respectively, we will then obtain three numbers forming a geometric progression. Find these numbers.

355. Find the three numbers constituting a geometric progression, if it is known that the sum of these numbers is equal to 26, and that when 1, 6 and 3 are added to them, respectively, three new numbers are obtained which form an arithmetic progression.

356. Three numbers form a geometric progression. If the third term is decreased by 64, then the three numbers thus obtained will constitute an arithmetic progression. If then the second term of this arithmetic progression is decreased by 8, a geometric progression will be formed again. Determine these numbers.

357. Can three numbers constitute an arithmetic and a geometric progression at the same time?

CHAPTER VI

COMBINATORICS AND NEWTON'S BINOMIAL THEOREM

358. The number of permutations of n letters is to the number of permutations of $n+2$ letters as 0.1 to 3. Find n.

359. The number of combinations of n elements taken three at a time is five times less than the number of combinations of $n+2$ elements taken four at a time. Find n.

360. Find the medium term of the expansion of the binomial $\left(\frac{a}{x}-x^{\frac{1}{2}}\right)^{16}$.

361. Determine the serial number of the term of the expansion of the binomial $\left(\frac{3}{4}\sqrt[3]{a^2}+\frac{2}{3}\sqrt{a}\right)^{12}$, which contains a^7.

362. Find the serial number of the term of the expansion of the binomial $\left(\sqrt[3]{\frac{a}{\sqrt{b}}}+\sqrt{\frac{b}{\sqrt[3]{a}}}\right)^{21}$, which contains a and b to one and the same power.

363. Simplify the expression $\left(\frac{a+1}{a^{\frac{2}{3}}-a^{\frac{1}{3}}+1}-\frac{a-1}{a-a^{\frac{1}{2}}}\right)^{10}$ and determine the term of the expansion that contains no a.

364. The exponent of one binomial exceeds that of the other by 3. Determine these exponents, if the sum of the binomial coefficients in the expansions of both binomials taken together is equal to 144.

365. Find the thirteenth term of the expansion of $\left(9x-\frac{1}{\sqrt{3x}}\right)^{m}$, if the binomial coefficient of the third term of the expansion is 105.

366. In the expansion of $\left(x^2+\frac{a}{x}\right)^m$ the coefficients at the fourth and thirteenth terms are equal to each other. Find the term containing no x.

367. Find the medium term of the expansion of $\left(a\sqrt[-2]{a}-\sqrt[5]{\frac{a^{-2}}{\sqrt{a}}}\right)$, if it is known that the coefficient of the fifth term is to the coefficient of the third term as 14 to 3.

368. The sum of the coefficients of the first, second and third terms of the expansion of $\left(x^2+\frac{1}{x}\right)^m$ is equal to 46. Find the term containing no x.

369. Find the term of the expansion of the binomial $(x\sqrt{x}+\sqrt[-3]{x})^m$ which contains x^5, if the sum of all the binomial coefficients is equal to 128.

370. Find the sixth term of a geometric progression, whose first term is $\frac{1}{i}$ and the common ratio is the complex number $(1+i)$.

371. Find the seventh term of a geometric progression, whose common ratio is $\left(1+\frac{1}{i}\right)$, and the first term, i.

372. At what value of n do the coefficients of the second, third and fourth terms of the expansion of the binomial $(1+x)^n$ form an arithmetic progression?

373. The coefficients of the fifth, sixth and seventh terms of the expansion of the binomial $(1+x)^n$ constitute an arithmetic progression. Find n.

374. In the expression $\left(\frac{\sqrt[5]{a^4}}{\sqrt[x]{a^{x-1}}}+a\sqrt[x+1]{a^{x-1}}\right)^5$ determine x such that the fourth term of the expansion of the binomial be equal to $56a^{5.5}$.

375. In the expression $\left(2\sqrt[x]{2^{-1}}+\frac{4}{\sqrt[4-x]{4}}\right)^6$ determine x such that the third term of the expansion of the binomial be equal to 240.

376. Determine x in the expression $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^x$, if in the expansion of the binomial the ratio of the seventh term from the beginning to the seventh term from the end is equal to $\frac{1}{6}$.

377. Find the value of x in the expression $(x+x^{\log x})^5$, the third term of the expansion of which is 1,000,000.

378. Find the value of x in the expression $\left[(\sqrt{x})^{\frac{1}{\log x+1}}+\sqrt[12]{x}\right]^6$, the fourth term of the expansion of which is 200.

379. In the expression $\left(\frac{1}{\sqrt[7]{x^2}}+x^{\log\sqrt{x}}\right)^9$ determine x such that the third term of the expansion of the binomial is equal to 36,000.

380. The sixth term of the expansion of the binomial $\left(\frac{1}{x^2\sqrt[3]{x^2}}+x^{2\log x}\right)^8$ is 5600. Find x.

381. The ninth term of the expansion of the binomial

$$\left[\frac{\sqrt{10}}{(\sqrt{x})^{5\log x}}+x^{2}\sqrt[\log x]{x}\right]^{10}$$

is 450. Find x.

382. Determine x, if the fourth term of the expansion of the binomial $\left(10^{\log\sqrt{x}}+\frac{1}{\sqrt[\log x]{10}}\right)^7$ is 3,500,000.

383. Determine at what value of x in the expansion of the binomial $\left(\sqrt[6]{x}+\frac{1}{\sqrt{x}}\right)^{12}$ the term containing x to a power twice as large as that of the succeedent term will be less than the latter by 30.

384. Determine at what value of x the fourth term in the expansion of the binomial $\left(\sqrt{2^{x-1}}+\frac{1}{\sqrt[3]{2^x}}\right)^m$ is 20 times greater than the exponent of the binomial, if the binomial coefficient of the fourth term is five times greater than that of the second term.

385. Find out at what values of x the difference between the fourth and sixth terms in the expansion of the binomial $\left(\frac{\sqrt{2^x}}{\sqrt[16]{8}}+\frac{\sqrt[16]{32}}{\sqrt{2^x}}\right)^m$ is equal to 56, if it is known that the exponent of the binomial m is less than the binomial coefficient of the third term in the expansion by 20.

386. Find out at what values of x the sum of the third and fifth terms in the expansion of $\left(\sqrt{2^x}+\frac{1}{\sqrt{2^{x-1}}}\right)^m$ is equal to 135, if the sum of the binomial coefficients of the last three terms is equal to 22.

387. Determine at what x the sixth term in the expansion of the binomial $\left[\sqrt{2^{\log(10-3x)}}+\sqrt[5]{2^{(x-2)\cdot\log 3}}\right]^m$ is equal to 21, if it is known that the binomial coefficients of the second, third and fourth terms in the expansion represent, respectively, the first, third and fifth terms of an arithmetic progression.

388. Determine at what value of x the fourth term in the expansion of the binomial

$$\left[\left(\sqrt[3]{5}\right)^{-\frac{1}{2}\log\left(6-\sqrt{8x}\right)}+\sqrt[6]{\frac{5^{\log(x-1)}}{25^{\log 5}}}\right]^{m}$$

is equal to 16.8, if it is known that $\frac{14}{9}$ of the binomial coefficient of the third term and the binomial coefficients of the fourth and fifth terms in the expansion constitute a geometric progression.

389. Determine at what x the difference between the nine-fold third term and the fifth term in the expansion of the binomial

$$\left(\frac{\sqrt{2^{x-1}}}{\sqrt[3]{2}}+\sqrt[3]{4}\cdot 2^{\frac{x}{2}}\right)^{m}$$

is equal to 240, if it is known that the difference between the logarithm of the three-fold binomial coefficient of the fourth term and the logarithm of the binomial coefficient of the second term in the expansion is equal to 1.

CHAPTER VII

ALGEBRAIC AND ARITHMETIC PROBLEMS *

390. Find the weight of an artillery round, knowing that the charge weighs 0.8 kg, the weight of the projectile is equal to $\frac{2}{3}$ of the total weight of the round, and the weight of the shell is $\frac{1}{4}$ of the weight of the round.

391. At a certain factory women make 35% of all the workers, the rest of the workers being men. The number of men exceeds that of women by 252 persons. Determine the total amount of workers.

---

* We do not divide the problems into algebraic and arithmetic ones, since arithmetically solvable problems can always be solved algebraically, and vice versa; the problems which are solved with the aid of equations may often have a simpler arithmetic solution. Under "Answers and Solutions" we sometimes give arithmetic, and sometimes, algebraic solutions, but this should not at all lay any restraint on the student's initiative as to the choice of the method of solution.

392. A batch of goods was sold for 1386 roubles at a 10% profit. Determine the prime cost of the goods.

393. A factory sold 3348 roubles worth of goods at a loss of 4%. What was the prime cost of the goods?

394. If 34.2 kg of copper is extracted from 225 kg of ore, what percentage of copper does the ore contain?

395. Prior to a price reduction, a package of cigarettes cost 29 kopecks. After the reduction, it cost 26 kopecks. What was the price reduction in percent?

396. One kilogram of a commodity cost 6 roubles and 40 kopecks. The price was then cut to 5 roubles and 70 kop. What was the price reduction in percent?

397. The raisins obtained in drying some grapes amount to 32% of the total weight of the grapes. What quantity of grapes must we take to obtain 2 kg of raisins?

398. A group of tourists have to collect money for an excursion. If each pays in 75 kopecks, there will be a deficit of 4.4 roubles; if each pays in 80 kopecks, there will be an excess of 4.4 roubles. How many persons take part in the excursion?

399. A number of persons were to pay equal amounts to a total of 72 roubles. If there were 3 persons less, then each would have to contribute 4 roubles more. How many people were there?

400. Sixty copies of the first volume of a book and 75 copies of the second volume cost a total of 405 roubles. However, a 15% discount on the first volume and a 10% discount on the second volume reduce the overall price to 355 roubles and 50 kopecks. Determine the price of each volume.

401. An antique shop bought two items for 225 roubles and then sold them and made a profit of 40%. What did the shop pay for each item, if the first of them yielded a profit of 25% and the second, a profit of 50%?

402. Sea water contains 5% (by weight) of salt. How many kilograms of fresh water should be added to 40 kg of sea water for the latter to contain 2% of salt?

403. The hypotenuse of a right-angled triangle measures $3\sqrt{5}$ metres. Determine the legs, if it is known that when one of them is increased by $133\frac{1}{3}$ % and the other, by $16\frac{2}{3}$%, the sum of their lengths is equal to 14 metres.

404. Two sacks contain 140 kg of flour. Each will contain one and the same amount, if we take 12.5% of the flour of the first sack and put it into the second. How many kilograms of flour does each sack contain?

405. Two factories, A and B, undertook to fulfil an order in 12 days. After two days factory A was closed down for repairs, while factory B continued fulfilment of the order. Knowing that B has an efficiency of $66\frac{2}{3}$ % of that of A, determine in how many days the order will be completed.

406. In a mathematics test, 12% of the students of a class did not solve the problems at all, 32% solved them with certain mistakes, and the remaining 14 students obtained correct solutions. How many students are there in the class?

407. A piece of a rail making 72% of the rail length is cut off. The remaining part weighs 45.2 kg. Determine the weight of the cut-off piece.

408. A piece of a silver-copper alloy weighs 2 kg. The weight of silver comes to $14\frac{2}{7}$% of that of copper. How much silver is there in this piece?

409. Three workers received a total of 4080 roubles for a job. The sums received by the first and the second workers stand in a ratio of $7\frac{1}{2}$ to $1\frac{3}{4}$. The money received by the third worker is $43\frac{1}{3}$ % of that of the first. What was each worker paid?

410. Three boxes contain 64.2 kg of sugar. The second box contains $\frac{4}{5}$ of the contents of the first, and the third contains $42\frac{1}{2}$ % of what there is in the second box. How much sugar is there in each box?

411. There is scrap of two grades of steel containing 5% and 40% of nickel. How much of each grade is required to obtain 140 tons of steel containing 30% of nickel?

412. A piece of a copper-tin alloy weighing 12 kg contains 45% of copper. How much pure tin must be added to this piece to obtain a new alloy with 40% of copper?

413. How much pure alcohol must be added to 735 grams of a 16% alcohol solution of iodine to obtain a 10% solution?

414. A piece of a copper-zinc alloy weighing 24 kg was immersed in water and lost $2\frac{8}{9}$ kg in weight. Determine the amount of copper and zinc in the alloy, if it is known that in water, copper loses $11\frac{1}{9}$% and zinc, $14\frac{2}{7}$ % of its weight.

415. Rails are to be laid in a 20 km long section of a single-track railroad line. Rails are available in lengths of 25 and 12.5 metres. If all 25-metre lengths are used, then 50% of the 12.5-metre lengths will have to be added. If all 12.5-metre lengths are laid, then $66\frac{2}{3}\%$ of the 25-metre lengths will have to be added. Determine the number of rail lengths of each kind available.

416. After the graduation exercises at a school the students exchanged photographs. How many students were there, if a total of 870 photographs were exchanged?

417. The geometric mean of two numbers is greater by 12 than the smaller number and the arithmetic mean of the same numbers is smaller by 24 than the larger number. Find the two numbers.

418. Find three numbers, the second of which is greater than the first by the amount the third number is greater than the second, if we know that the product of the two smaller numbers is equal to 85 and the product of the two larger numbers equals 115.

419. The number a is the arithmetic mean of three numbers, and b is the arithmetic mean of their squares. Express the arithmetic mean of their pairwise products in terms of a and b.

420. A rectangular sheet of tin with a perimeter of 96 cm is used to make an open-top box so that a 4-cm square is cut out of each corner of the sheet and the edges are soldered together. What is the size of the sheet used, if the box has a volume of 768 cm^3?

421. Find a two-digit number, if the quotient obtained by dividing this number by the product of its digits is equal to $2\frac{2}{3}$ and, besides, the difference between the desired number and the number obtained by reversing the order of the same digits is 18.

422. Find a two-digit number, if we know that the number of units therein exceeds by two the number of tens and that the product of the desired number by the sum of its digits is equal to 144.

423. Determine a certain positive integer on the basis of the following data: if we adjoin the figure 5 on the right of it, the resulting number is exactly divisible by a number exceeding the desired one by 3, the quotient being equal to the divisor minus 16.

424. Find two two-digit numbers having the following property: if we adjoin 0 followed by the smaller number on the right of the larger one, and adjoin the larger number followed by 0 on the right of the smaller one, then of the two five-digit numbers thus obtained the first number divided by the second yields a quotient of 2 and a remainder of 590. It is also known, that the sum of the two-fold larger desired number and the three-fold smaller desired number is equal to 72.

425. A student was asked to multiply 78 by a two-digit number in which the tens digit was three times as large as the units digit; by mistake, he interchanged the digits in the second factor and thus obtained a product smaller than the true product by 2808. What was the true product?

426. Two railway stations are at a distance of 96 km from each other. One train covers this distance 40 minutes faster than does the other. The speed of the first train is 12 km/h higher than that of the second. Determine the speed of both trains.

427. Two persons simultaneously leave cities A and B and travel towards each other. The first person travels 2 km/h faster than does the second and arrives in B one hour before the second arrives in A. A and B are 24 km apart. How many kilometres does each person make in one hour?

428. The distance between A and B by railway is 66 km and by water, 80.5 km. A train leaves A four hours after the departure of a boat and arrives in B 15 minutes before the boat. Determine the mean speeds of the train and the boat, if the former runs 30 km/h faster than does the latter.

429. A tailor shop has an order for 810 suits, another shop has to make 900 suits in the same period of time. The first shop has completed its task 3 days before the target date, and the second, 6 days ahead of time. How many suits does each shop produce per day, if the second shop makes 4 suits per day more than the first?

430. Two ships meet, one going off to the south and the other, to the west. Two hours after their encounter, they are 60 km apart. Find the speed of each ship, if it is known that the speed of one of them is 6 km/h higher than that of the other.

431. A dog at point A goes in pursuit of a fox 30 metres away. The dog makes 2 m and the fox, 1 m long leaps. If the dog makes two leaps to the fox's three, at what distance from A will the dog catch up with the fox?

432. Assuming that the hands of a clock move without jerks, how long will it take for the minute hand to catch up with the hour hand if it was 4 o'clock at the starting time.

433. A train left station A for C via B. The speed of the train in the section from A to B was as required, but it fell off by 25% in the section between B and C. On the return trip, the required speed was maintained between C and B, but decreased 25% between B and A. How long did it take for the train to cover the distance from A to C, if we know that the same time was spent on the *A*-*B* section as on the *B*-*C* section and that on the *A*-to-*C* section the train spent $\frac{5}{12}$ of an hour less than on the return trip (from *C* to *A*)?

434. A cyclist has to make a trip of 30 km. He leaves 3 minutes late, but travels 1 km/h faster and arrives in time. Determine the speed of the cyclist.

435. A fast train was held up by a red-light signal for 16 minutes and made up for the lost time on a 80-km stretch travelling 10 km/h faster than called for by schedule. What is the scheduled speed of the train?

436. A train has to cover 840 km in a specified time. At the half-distance point it was held up for half an hour and so, in the remaining section of the route, it increased its speed by 2 km/h. How much time did the train spend en route?

437. Two trains start out towards each other from points 650 km apart. If they start out at the same time, they will meet in 10 hours, but if one of them starts out 4 hours and 20 minutes before the other, they will pass each other 8 hours following the departure of the latter. Determine the mean speed of each train.

438. Two trains start out at the same time from stations *A* and *B* 600 km apart and run towards each other. The first train arrives at *B* three hours before the second arrives at *A*. The first train travels 250 km in the time required for the second to cover 200 km. Find the speed of each train.

439. A commuter walking to his train had covered 3.5 km in one hour and then figured out that at such a rate he would be one hour late. Therefore, over the remainder of the distance he made 5 km/h and arrived 30 minutes before the train's leaving time. Determine the distance the commuter had to walk.

440. The distance between *A* and *B* is 19 km by highway. A cyclist starts out from *A* at a constant speed in the direction of *B*. A motor car leaves *A* 15 minutes later in the same direction. In 10 minutes it catches up with the cyclist and continues on to *B*, then turns around and in 50 minutes after leaving *A* encounters the cyclist a second time. Determine the speeds of the car and cyclist.

441. A mail train leaves station *A* at 5 a.m. for station *B*, 1080 km away. At 8 a.m. a fast train leaves *B* for *A* and runs 15 km/h faster than the mail train. When do the trains pass each other if this occurs midway between *A* and *B*?

442. *A* is 78 km distant from *B*. A cyclist leaves *A* in the direction of *B*. One hour later, another cyclist leaves *B* in the direction of A and cycles 4 km/h faster than the first one. They meet 36 km from B. How long is each one en route prior to the encounter and what are their speeds?

443. Two hikers start out at the same time and, walking towards each other, meet in 3 hours and 20 minutes. How long will it take for each hiker to cover the whole distance, if the first of them arrives at the starting point of the second 5 hours after the second arrives at the starting point of the first?

444. Two hikers start out towards each other, one from A and the other, from B. The first hiker starts from A six hours after the second leaves B and when they meet it turns out that he has covered 12 km less than the second hiker. After the encounter the hikers continue walking at the same rate as before and the first of them arrives at B eight hours later, the second arriving at A in 9 hours. Determine the distance between A and B and the speed of the two hikers.

445. A dirigible and an airplane are flying towards each other, having left their terminals at the same time. When they meet, the dirigible has made 100 km less than the airplane, and it arrives at the departure point of the airplane three hours after they pass each other. The airplane arrives at the airport of the dirigible 1 hour and 20 minutes after they pass each other. Find the speeds of the airplane and the dirigible and the distance between the airports.

446. Two hikers leave A and B at the same time in the direction towards each other. When they meet, it turns out that the first hiker has covered a km more than the second. If they continue on their ways at the same rate as before, the first hiker will arrive at B in m hours and the second will arrive at A in n hours after they meet. Find the speed of each hiker.

447. Two bodies are moving along the circumference of a circle. The first body makes the whole circle 5 seconds faster than the second. If they both move in one direction, they will come together every 100 seconds. What portion of the circumference (in degrees) does each body make in one second?

448. Two bodies moving along the circumference of a circle in the same direction come together every 56 minutes. If they were moving with the same speeds as before, but in opposite directions, they would meet every 8 minutes. Also, when moving in opposite directions, the distance (along the circumference) between the approaching bodies decreases from 40 metres to 26 metres in 24 seconds. What is the speed of each body in metres per minute and how long is the circumference?

449. Two points are uniformly moving in the same direction along the circumference of a circle of length c and come together every t seconds. Find the speed of each point, knowing that one of them makes the whole circle n seconds faster than the other.

450. The distance between two towns along a river is 80 km. A ship makes a round trip between the towns in 8 hours and 20 minutes. Find the speed of the ship in still water, if the rate of the current of water is taken to be 4 km/h.

451. A motor boat goes 28 km downstream and then returns immediately. The round trip takes 7 hours. Find the speed of the boat in still water, if the rate of the current of water is 3 km/h.

452. A person boats from town A to town B and back in 10 hours. The towns are 20 km apart. Find the rate of the current of water, if we know that he boats 2 km upstream during the same time as he does 3 km downstream.

453. A ship covers the distance between A and B in two days. The return trip takes 3 days. Determine the time a raft will take to float down the river from A to B.

454. Two bodies, M_1 and M_2, are uniformly moving towards each other from A and B 60 metres apart. M_1 starts out from A 15 seconds before M_2 starts out from B. At their respective terminals the two bodies turn around and immediately go back at the same speeds as before. Their first encounter takes place in 21 seconds and the second, in 45 seconds after the start of M_1. Find the speed of each body.

455. A road leading from city A to city B first runs uphill for 3 km, then it is level for 5 km and then runs downhill for 6 km. A messenger sets out from A in the direction of B and having covered half the distance, finds out that he must return to pick up some packages he has forgotten. In 3 hours and 36 minutes after leaving he returns to A. Leaving A a second time, he reaches B in 3 hours and 27 minutes and makes the return trip to A in 3 hours and 51 minutes. What is the speed of the messenger when going uphill, over the level ground and downhill, assuming that within the bounds of each road section the speed remains constant?

456. A typist figures out that if she types 2 pages above her work quota daily, she will complete her work 3 days ahead of schedule, and if she makes 4 pages extra per day, she will finish 5 days ahead of time. How many pages does she have to type and in what time?

457. A worker made a certain number of identical parts in a specified time. If he had produced 10 parts more every day, he would have completed the job $4\frac{1}{2}$ days ahead of schedule, and if he had produced 5 parts less every day, he would have been 3 days behind time. How many parts did he make and in what time?

458. A typist had to do a job in a specified time by typing a certain number of pages every day. She calculated that if she had typed 2 pages more than required per day, she would have completed the task 2 days ahead of time, but if she had turned out 60% of her work quota, then she would have finished the job 4 days ahead of time and made 8 pages more than required. What was the daily work quota and in what time had the job to be completed?

459. Two workers together complete a certain task in 8 hours. Working individually, the first worker can do the job 12 hours faster than can do the second. How many hours would it take each worker to do the job individually?

460. A swimming pool is filled by two pipes in 6 hours. One pipe alone fills it 5 hours faster than does the other pipe alone. How long will it take for each pipe operating individually to fill the pool?

461. Two workers are given a task to make a batch of identical parts. After the first had worked for 7 and the second, for 4 hours, they found out that $^5/_9$ of the task had been completed. Having worked together for another 4 hours, they figured out that $^1/_{18}$ of the job had yet to be done. How long would it take each worker to do the whole job individually?

462. Four identical hoisting cranes were being used to load a ship. After they had worked for 2 hours, another two cranes of a lower capacity were put into operation, with the result that the loading operation was completed in three hours. If all the cranes had begun working at the same time, the loading would have been completed in 4.5 hours. Determine the time (in hours) required for one high-power and one low-power crane to do the job.

463. A task was set to deliver a building material from a railway station to a construction site in 8 hours. The material had to be delivered with 30 three-ton trucks. These trucks worked for two hours and then 9 five-ton trucks were added to help out. The task was completed in time. If the five-ton trucks had begun the operation, and the three-ton trucks had been brought two hours later, then only $^{13}/_{15}$ of the material would have been delivered in the allotted time. Determine how many hours it would take one three-ton truck alone, one five-ton truck alone, and 30 five-ton trucks to deliver all the material.

464. Two typists undertake to do a job. The second typist begins working one hour after the first. Three hours after the first typist has begun working there is still $^9/_{20}$ of the work to be done. When the assignment is completed, it turns out that each typist has done half the work. How many hours would it take each one to do the whole job individually?

465. Two trains start out from stations A and B towards each other, the second train leaving half an hour later than does the first. Two hours after the first train had started, the distance between the trains came to $^{19}/_{30}$ of the entire distance between A and B. The trains met midway between A and B. How much time would it take each train to cover the distance between A and B?

466. A rectangular bath 20 cm × 90 cm × 25 cm (a rectangular parallelepiped) is used to wash photographic negatives. Water flows in through one pipe and, at the same time, out through another pipe to ensure its constant agitation in the bath. It requires 5 minutes less time to empty the bath through the second pipe than it does to fill it through the first pipe, the second being closed. If both pipes are open, a full bath will be emptied in one hour. Find the amount of water each pipe lets pass through in one minute.

467. A construction job required the digging out of 8000 m^3 of earth in a specified time. The operation was completed 8 days ahead of time because the team of navvies overfulfilled their plan by 50 cubic metres daily. Determine the original time limit for the assignment and daily overfulfillment of the plan in percent.

468. A railway was being repaired by two teams of workers. Each repaired 10 km of the track despite the fact that the second team worked one day less than did the first. How many kilometres of the track did each team repair per day if both teams together repaired 4.5 km daily?

469. Two workers together did a job in 12 hours. If at the beginning the first worker had done half the assignment, and then the second had completed the other half, the whole job would have been done in 25 hours. How long would it take each worker to do the whole job individually?

470. Two tractors of different performance characteristics, working together, ploughed a field in t days. If at first one tractor had ploughed half the field, and then the other one had completed the other half, the ploughing operation would have been completed in k days. How many days would it take each tractor to plough the field individually?

471. Three different dredgers were at work, deepening the entrance channel to a port. The first dredger, working alone, would have taken 10 days longer to do the job; the second, working alone, would have required an extra 20 days, and the third dredger, working alone, would have required six times more time tian needed for all the three machines operating simultaneously. How long would it have taken each dredger to do the job individually?

472. Two workers, the second one beginning working $1\frac{1}{2}$ days after the first. can complete an assignment in 7 days. If each of them had done the job individually, the first worker would have required 3 days more than would have the second. How many days would it take each worker to do the job individually?

473. Two different tractors, working together, ploughed a field in 8 days. If at first one tractor had ploughed half the field and then both tractors together had ploughed the other half, the whole job would have been done in 10 days. How many days would it take each tractor to plough the field individually?

474. A number of men undertook to dig a ditch and could have finished the job in 6 hours, if they had begun working simultaneously, but they began one after another, the intervals between their starting times being equal. After the last worker had begun working, a time interval of the same length elapsed and the job was finished, each one of the participants working till the completion of the job. How long did they work, if the first worker to begin worked 5 times as long as the last one to begin?

475. Three workers together can complete a task in t hours. The first of them, working alone, can do the job twice as fast as the third and one hour faster than the second. How long would it take each worker to do the job individually?

476. A tank is filled with water from two taps. At the beginning the first tap was open for one third of the time which would have been needed to fill the tank, if the second tap alone had been open. Then the second tap was open for one third of the time required to fill the tank, if the first tap alone were open. This done, the tank was $^{13}/_{18}$ full. Compute the time required to fill the tank by each tap separately, if both taps together fill it in 3 hours and 36 minutes.

477. In the construction of an electric power station, a team of bricklayers was assigned the task of laying 120,000 bricks in a specified time. The team completed the task 4 days ahead of time. Determine the daily quota of bricklaying and the actual number of bricks laid, if it is known that in three days the team laid 5000 bricks more than required by the work quota for 4 days.

478. Three vessels contain water. If $^1/_3$ of the water of the first vessel is poured into the second, and then $^1/_4$ of the water now in the second vessel is poured into the third, and, finally, $^1/_{10}$ of the water now in the third vessel is poured into the first, then each vessel will contain 9 litres. How much water was there originally in each vessel?

479. A tank is filled with pure alcohol. Some of the alcohol is poured out and replaced by an equal amount of water; the same amount of the alcohol-water mixture thus obtained is then poured out, leaving 49 litres of pure alcohol in the tank. The tank has a capacity of 64 litres. How much alcohol was poured out for the first time and how much for the second time? (It is assumed that the volume of the mixture is equal to the sum of the volumes of the alcohol and water; actually it is somewhat lesser.)

480. A 20-litre vessel is filled with alcohol. Some of the alcohol is poured out into another vessel of an equal capacity, which is then made full by adding water. The mixture thus obtained is then poured into the first vessel to capacity. Then $6^2/_3$ litres is poured from the first vessel into the second. Both vessels now contain equal amounts of alcohol. How much alcohol was originally poured from the first vessel into the second?

481. An 8-litre vessel is filled with air containing 16% of oxygen. Some of the air is let out and replaced by an equal amount of nitrogen; then the same amount of the gas mixture as before is let out and again replaced by an equal amount of nitrogen. There is now 9% of oxygen in the mixture. Determine the amount of the gas mixture released from the vessel each time.

482. Two collective farmers together brought 100 eggs to market. Having sold their eggs at different prices, both farmers made equal sums of money. If the first farmer had sold as many eggs as the second, she would have received 72 roubles; if the second farmer had sold as many eggs as the first, she would have received 32 roubles. How many eggs did each one of them have originally?

483. Two collective farmers with a total of a litres of milk, though selling the milk at different prices, made equal sums of money. If the first farmer had sold as much milk as the second, she would have received m roubles, and if the second farmer had sold as much milk as the first, she would have received n roubles ($m > n$). How many litres of milk did each one of them have originally?

484. Two internal combustion engines of the same power output were subjected to an efficiency test and it was found that one of them consumed 600 grams of petrol, while the other, which was in operation 2 hours less, consumed 384 grams. If the first engine had consumed as much petrol per hour as the second, and the second, as much as the first, then both engines would have consumed equal amounts of petrol during the same period of operation as before. How much petrol does each engine consume per hour?

485. There are two grades of gold-silver alloy. In one of them the metals are in a ratio of 2 : 3 and in the other, in a ratio of 3 : 7. How much of each alloy need we take to get 8 kg of a new alloy in which the gold-to-silver ratio will be 5 to 11?

486. One barrel contains a mixture of alcohol and water in a ratio of 2 to 3, another barrel, in a ratio of 3 to 7. How many pails need we take from each barrel to obtain 12 pails of a mixture in which the alcohol-to-water ratio is 3 to 5?

487. A certain alloy consists of two metals in a ratio of 1 to 2, another alloy contains the same metals in a ratio of 2 to 3. How many parts of both alloys are needed to produce a third alloy containing the metals in a ratio of 17 to 27?

488. Two wheels are set in rotation by an endless belt; the smaller wheel makes 400 revolutions per minute more than does the larger wheel. The larger wheel makes 5 revolutions in a time interval that is 1 second longer than that required for the smaller wheel to make 5 revolutions. How many revolutions per minute does each wheel make?

489. Over a distance of 18 metres the front wheel of a vehicle makes 10 revolutions more than does the rear wheel. If the circumference of the front wheel were increased by 6 decimetres, and the circumference of the rear wheel, reduced by 6 decimetres, then over the same distance the front wheel would complete 4 revolutions more than would the rear one. Find the circumferences of both wheels.

490. A barge with 600 tons of goods was unloaded in three days, $^2/_3$ of the goods being unloaded during the first and third days. The amount of goods unloaded during the second day was less than that unloaded on the first day, and the amount unloaded on the third day was less than that unloaded on the second day. The difference between the percent reduction of the amount of goods unloaded on the third day with respect to that unloaded on the second day and the percent reduction of the amount unloaded on the second day with respect to that unloaded on the first day is equal to 5. Determine how much was unloaded each day.

491. Two solutions, the first containing 800 grams and the second, 600 grams of anhydrous sulphuric acid, are mixed to produce 10 kg of a new solution of sulphuric acid. Determine the weights of the first and second solutions in the mixture, if it is known that the content of anhydrous sulphuric acid in the first solution is 10 percent greater than that in the second solution.

8091

492. There were two different copper alloys, the first containing 40 per cent less copper than the second. When these were melted together, the resulting alloy contained 36 per cent of copper. Determine the percentage of copper in the first and second alloys, if it is known that there were 6 kg of copper in the first alloy and 12 kg in the second.

493. Two trains—a freight train 490 metres long and a passenger train 210 metres long—were travelling along parallel tracks towards each other. The driver of the passenger train noticed the freight train when it was 700 metres away; 28 seconds later they passed each other. Determine the speed of each train, if we know that the freight train takes 35 seconds longer to pass the signal lights than does the passenger train.

494. A freight train consists of four- and eight-wheel tank-cars with oil. The train weighs 940 tons. It is required to determine the number of the eight- and four-wheel tank-cars and also their weight, if it is given that the number of the four-wheel cars is 5 more than that of the eight-wheel cars; the eight-wheel car weighs three times as much as the four-wheel car and the net weight of oil (that is, minus the weight of the cars) in all the eight-wheel cars is 100 tons more than the weight of all the loaded four-wheel cars. The eight-wheel tank-car carries 40 tons of oil and the weight of the oil in the four-wheel tank-car is 0.3 of that in the eight-wheel car.

495. The tunnel boring machines, working at the two ends of a tunnel have to complete the driving in 60 days. If the first machine does 30% of the work assigned to it, and the second, $26\frac{2}{3}$ %, then both will drive 60 metres of the tunnel. If the first machine had done $^{2}/_{3}$ of the work assigned to the second one, and the second, 0.3 of the work assigned to the first one, then the first machine would have needed 6 days more than would have the second. Determine how many metres of the tunnel are driven by each machine per day.

496. Two railway crews working together completed a repair job on a track section in 6 days. To do 40% of the work the first crew alone would require two days more than the second crew alone would require to complete $13\frac{1}{3}$ % of the whole job. Determine how many days it would take each crew to repair the whole track section individually.

497. Six hundred and ninety tons of goods were to be delivered from a wharf to a railway station by five 3-ton trucks and ten $1\frac{1}{2}$-ton trucks. In a few hours, the trucks transported $^{25}/_{46}$ of the goods.

To complete the delivery in time, the remaining goods had to be transported in a time interval 2 hours less than that already spent. The transportation was completed in time because the truck drivers had begun making one trip per hour more than before. Determine how many hours it took to transport all the goods, and also the number of trips per hour that were made originally, if it is known that the $1\frac{1}{2}$-ton trucks made one trip per hour more than did the three-ton trucks.

Note. It is assumed that all the trucks were fully loaded on each trip.

498. A sports ground has the shape of a rectangle with sides of a and b metres. It is bordered by a running-track whose outer rim is also a rectangle whose sides are parallel to and equally spaced from the sides of the ground. The area of the track is equal to that of the ground. Find the width of the track.

499. An auditorium has a chairs arranged in rows, the number of chairs in each row being the same. If b chairs are added to each row and the number of rows is reduced by c, then the total number of places in the hall will increase by one-tenth of their original number. How many chairs are there in each row?

500. Two bodies spaced at d metres are moving towards each other and meet in a seconds. If they move at the same speeds as before, but in one direction, they will meet in b seconds. Determine the speed of each body.

501. A motorcyclist and a cyclist simultaneously start out towards each other from points A and B d kilometres apart. In two hours they pass each other and continue on their ways. The motorcyclist arrives at B t hours before the cyclist arrives at A. Find the speed of the two vehicles.

502. A hiker starts out from point A in the direction of B; a hours later a cyclist starts out from B to meet the hiker and meets him b hours after the start. How long will it take the cyclist and the hiker to cover the whole distance between A and B, if the cyclist requires c hours less than does the hiker?

503. Train A, whose speed is v km/h, departs after train B, whose speed is v_1 km/h. The difference between the departure times (the lag of train A) is calculated so that both trains simultaneously arrive at the destination. Train B covers $^2/_3$ of the distance and then has to reduce its speed to half. As a result the trains meet a km from the destination. Determine the distance to the terminal station.

504. A man puts money in a savings bank and one year later earns an interest of 15 roubles. Having added another 85 roubles, he deposits the money for another year. After the expiry of this period the sum-total of the principal and its interest is 420 roubles. What sum of money was originally deposited and what interest does the savings bank pay?

505. The output of machine-tool A is $m\%$ of the sum of the outputs of machines B and C, and the output of B is $n\%$ of the sum of the outputs of A and C. What is the percentage of the output of C with respect to the overall output of A and B?

506. An increase in the output of a factory as compared to that in the preceding year is $p\%$ for the first year and $q\%$ for the second year. What should the percent increase of the output be for the third year for the average annual increase of the output for three years to be equal to $r\%$?

507. $a\%$ of some quantity of goods is sold at a profit of $p\%$ and $b\%$ of the rest of the goods is sold at a profit of $q\%$. What profit is made on selling the remaining goods, if the total profit is $r\%$?

508. Equal (by weight) pieces are cut off two chunks of alloys of different copper content, the chunks weighing m kg and n kg. Each of the cut-off pieces is melted together with the remainder of the other chunk and the copper contents of both alloys then become equal. Find the weight of each of the cut-off pieces.

509. A certain sum of money was arranged in n piles. An nth part of the money in the first pile was taken from it and put into the second pile. Then an nth part of the money in the enlarged second pile was taken from it and put into the third pile. The same operation was continued from the third to the fourth pile, and so on. Finally, an nth part of the money in the nth pile was taken from it and put into the first pile. After this, final operation each pile had A roubles. How much money was there in each pile prior to the shifting operation (you may confine yourself to $n=5$)?

PART TWO

GEOMETRY AND TRIGONOMETRY

CHAPTER VIII

PLANE GEOMETRY

510. The perimeter of a right triangle is equal to 132, and the sum of the squares of its sides, to 6050. Find the sides.

511. Given in a parallelogram are: the acute angle α and the distances m and p between the point of intersection of the diagonals and the unequal sides. Determine the diagonals and the area of the parallelogram.

512. The base of an isosceles triangle is equal to 30 cm, and the altitude, to 20 cm. Determine the altitude dropped to one of the sides.

513. The base of a triangle is equal to 60 cm, altitude, to 12 cm and the median drawn to the base, to 13 cm. Determine the sides.

514. On the sides of an isosceles right triangle with the leg b three squares are constructed outwards. The centres of these squares are joined through straight lines. Find the area of the triangle thus obtained.

515. The sides of a square are divided in the ratio m to n, a large and a small segments being adjacent to each vertex. The successive points of division are joined by straight lines. Find the area of the quadrilateral obtained, if the side of the given square is equal to a.

516. Inscribed in a square is another square, whose vertices lie on the sides of the former square and the sides form 30-degree angles with those of the former square. What portion of the area of the given square is the area of the inscribed square equal to?

517. Inscribed in a square with side a is another square, whose vertices lie on the sides of the former. Determine the segments into which the sides of the first square are divided by the vertices of the second square, if the area of the latter is equal to $\frac{25}{49}$ of that of the former.

518. Inscribed in a rectangle with sides 3 m and 4 m long is another rectangle, whose sides are in the ratio 1 : 3. Find the sides of this rectangle.

519. Inscribed in an equilateral triangle ABC with side a is another equilateral triangle LMN, whose vertices lie on the sides of the first triangle and divide each of them in the ratio 1 : 2. Find the area of the triangle LMN.

520. Find the sides of a right-angled triangle, given its perimeter $2p$ and altitude h.

521. Two equal segments CM and CN are marked on the sides CA and CB of an isosceles triangle ABC. Determine the length of the segments, knowing the perimeter $2P$ of the triangle ABC, its base $AB = 2a$ and the perimeter $2p$ of the rectangle $AMNB$ cut off by the straight line MN.

522. Given a right-angled trapezoid with bases a, b and shorter side c. Determine the distance between the point of intersection of the diagonals of the trapezoid and the base a, and between the point of intersection and the shorter side.

523. Find the area of an isosceles triangle, if its base is 12 cm, and the altitude is equal to the line-segment joining the mid-points of the base and of one of the sides.

524. The perimeter of a rhombus is equal to $2p$ cm, and the sum of its diagonals, to m cm. Find the area of the rhombus.

525. The longer base of a trapezoid is equal to a, and the shorter, to b; the angles at the longer base are 30° and 45°. Find the area of the trapezoid.

526. Compute the area of a trapezoid, whose parallel sides are equal to 16 cm and 44 cm, and nonparallel ones, to 17 cm and 25 cm.

527. Find the area of a square inscribed in a regular triangle with side a.

528. The base of a triangle is divided by the altitude into two parts equal to 36 cm and 14 cm. A straight line drawn perpendicular to the base divides the area of the given triangle into two equal parts. Into what parts is the base of the triangle divided by this line?

529. The altitude of a triangle is equal to 4; it divides the base into two parts in the ratio 1 : 8. Find the length of the line-segment which is parallel to the altitude and divides the triangle into equal parts.

530. A triangle ABC is divided into three equal figures by straight lines which are parallel to the side AC. Compute the parts into which the side AB, equal to a, is divided by the parallel lines.

531. A straight line parallel to the base of a triangle, whose area is equal to S, cuts off it a triangle with an area equal to q. Determine the area of a quadrilateral, whose three vertices coincide with those of the smaller triangle and the fourth one lies on the base of the larger triangle.

532. Parallel sides of a trapezoid are equal to a and b. Find the length of the line-segment which is parallel to them and divides the area of the trapezoid into two equal parts.

533. Perpendiculars are drawn from the vertex of the obtuse angle of a rhombus to its sides. The length of each perpendicular is equal to a, the distance between their feet being equal to b. Determine the area of the rhombus.

534. Find the area of a triangle, if two of its sides are equal to 27 cm and 29 cm, respectively, and the median drawn to the third side is equal to 26 cm.

535. Given two sides b and c of a triangle and its area $S = \frac{2}{5} bc$. Find the third side a of the triangle.

536. Given the bases a and b and sides c and d of a trapezoid. Determine its diagonals m and n.

537. Given a parallelogram, whose acute angle is equal to 60°. Determine the ratio of the lengths of its sides, if the ratio of the squared lengths of its diagonals is equal to $\frac{19}{7}$.

538. From an arbitrary point taken inside an isosceles triangle perpendiculars are drawn to all the sides. Prove that the sum of the three perpendiculars is equal to the altitude of the triangle.

539. Two secant lines are drawn from a point outside a circle. The internal segment (the chord) of the first secant is equal to 47 m, and the external one, to 9 m; the internal segment of the second secant exceeds its external segment by 72 m. Determine the length of the second secant line.

540. From a point m cm distant from the centre of a circle two lines are drawn tangent to the circle. The distance between the points of tangency is equal to a cm. Determine the radius of the circle.

541. Given inside a circle, whose radius is equal to 13 cm, is a point M 5 cm distant from the centre of the circle. A chord $AB = 25$ cm is drawn through the point M. Find the length of the segments into which the chord AB is divided by the point M.

542. In an isosceles triangle the vertex angle is equal to α. Determine the ratio of the radii of the inscribed and circumscribed circles.

543. The sides of a triangle are: $a = 13$, $b = 14$, $c = 15$. Two of them (a and b) are tangent to a circle, whose centre lies on the third side. Determine the radius of the circle.

544. An isosceles triangle with a vertex angle of 120° is circumscribed about a circle of radius R. Find its sides.

545. On the larger leg of a right triangle, as on the diameter, a semicircle is described. Find the semicircumference if the smaller leg is equal to 30 cm, and the chord joining the vertex of the right angle with the point of intersection of the hypotenuse and the semicircle is equal to 24 cm.

546. In a right-angled triangle a semicircle is inscribed so that its diameter lies on the hypotenuse and its centre divides the latter into two segments equal to 15 cm and 20 cm. Determine the length of the arc of the semicircle between the points at which the legs touch the semicircle.

547. In an isosceles triangle with the base equal to 4 cm and altitude equal to 6 cm a semicircle is constructed on one of the sides as on the diameter. The points at which the semicircle intersects the base and the other side are joined by a straight line. Determine the area of the quadrilateral thus obtained, which is inscribed in the semicircle.

548. Given an isosceles triangle with the base $2a$ and altitude h. Inscribed in it is a circle, and a line tangent to the circle and parallel to the base of the triangle. Find the radius of the circle and the length of the segment of the tangent line contained between the sides of the triangle.

549. From a point lying without a circle two secant lines are drawn, whose external portions are 2 m long. Determine the area of the quadrilateral, whose vertices are the points of intersection of the secants and the circle, if the lengths of its two opposite sides are equal to 6 m and 2.4 m.

550. The sides of a triangle are equal to 6 cm, 7 cm, and 9 cm. From its vertices, as from centres, three mutually tangent circles are described: the circle, whose centre lies at the vertex of the least angle of the triangle, is internally tangent to the remaining two circles, the latter being externally tangent to each other. Find the radii of the three circles.

551. An exterior tangent to two circles of radii 5 cm and 2 cm is 1.5 times longer than their interior tangent. Determine the distance between the centres of the circles.

552. The distance between the centres of two circles, whose radii are equal to 17 cm and 10 cm, is 21 cm. Determine the distances between the centres and the point at which the centre line intersects a common tangent to the circles.

553. To two externally tangent circles of radii R and r common tangent lines are drawn: one interior and two exterior ones. Determine the length of the segment of the interior tangent line contained between the exterior tangents.

554. To two externally tangent circles of radii R and r common exterior tangent lines are drawn. Find the area of the trapezoid bounded by the tangent lines and chords joining the points of tangency.

555. Two circles of radii R and r are externally tangent. A common exterior tangent is drawn to these circles, thus forming a curvilinear triangle. Find the radius of the circle inscribed in this triangle.

556. Through one and the same point of a circle two chords (equal to a and b) are drawn. The area of the triangle formed by joining their ends is equal to S. Determine the radius of the circle.

557. In a circle of radius R three parallel chords are drawn on one side of its centre, whose lengths are respectively equal to those of the sides of a regular hexagon, quadrilateral and triangle inscribed in the circle. Determine the ratio of the area of the portion of the circle contained between the second and third chords to that contained between the first and second ones.

558. Determine the area of a circle inscribed in a right-angled triangle, if the altitude drawn to the hypotenuse divides the latter into two segments equal to 25.6 cm and 14.4 cm.

559. A circle is inscribed in a rhombus with side a and acute angle equal to 60°. Determine the area of the rectangle, whose vertices lie at the points of tangency of the circle and the sides of the rhombus.

560. Drawn to a circle of radius R are four tangent lines which form a rhombus, whose larger diagonal is equal to $4R$. Determine the area of each of the figures bounded by two tangents drawn from a common point and the smaller arc of the circle contained between the points of tangency.

561. The area of an isosceles trapezoid circumscribed about a circle is equal to S. Determine the side of the trapezoid, if the acute angle at its base is equal to $\pi/6$.

562. An isosceles trapezoid with an area of 20 cm^2 is circumscribed about a circle of a radius of 2 cm. Find the sides of the trapezoid.

563. About a circle a trapezoid is circumscribed, whose nonparallel sides form acute angles α and β with the larger of the parallel sides. Determine the radius of the circle, if the area of the trapezoid is equal to Q.

564. About a circle of radius r a right-angled trapezoid is circumscribed, whose least side is equal to $3r/2$. Find the area of the trapezoid.

565. The centre of a circle inscribed in a right-angled trapezoid is 2 cm and 4 cm distant from the end points of the larger of the nonparallel sides. Find the area of the trapezoid.

566. A circle is inscribed in an equilateral triangle with side a. Then three more circles are inscribed in the same triangle so that they are tangent to the first one and to the sides of the triangle, and then another three circles tangent to the above three circles and to the sides of the triangle, and so forth. Find the total area of all the inscribed circles (that is the limit of the sum of the areas of the inscribed circles).

567. A triangle ABC is inscribed in a circle; through the vertex A a tangent line is drawn to intersect the extension of the side BC at the point D. From the vertices B and C perpendiculars are dropped to the tangent line, the shorter of these perpendiculars being equal to 6 cm. Determine the area of the trapezoid formed by the perpendiculars, side BC and the segment of the tangent line, if $BC = 5$ cm, $AD = 5\sqrt{6}$ cm.

568. Three equal circles tangent to one another are inscribed in a regular triangle, whose side is equal to a. Each of them is in contact with two sides of the given triangle. Determine the radii of the circles.

569. Inside an equilateral triangle with side a there are three equal circles tangent to the sides of the triangle and mutually tangent to one another. Find the area of the curvilinear triangle formed by the arcs of the mutually tangent circles (its vertices being the points of tangency).

570. Inside a square with side a four equal circles are situated, each of them touching two adjacent sides of the square and two circles (out of the remaining three). Find the area of the curvilinear quadrangle formed by the arcs of the tangent circles (its vertices being the points of tangency of the circles).

571. Find the area of a segment, if its perimeter is equal to p, and the arc, to 120°.

572. A circle of a radius of 4 cm is inscribed in a triangle. One of its sides is divided by the point of tangency into two portions equal to 6 cm and 8 cm. Find the lengths of the other two sides.

573. In an isosceles triangle a perpendicular dropped from the vertex of an angle at the base to the opposite side divides the latter in the ratio $m : n$. Find the angles of the triangle.

574. A chord perpendicular to the diameter divides it in the ratio $m : n$. Determine each of the arcs (arc measure) into which the circle is divided by the chord and diameter.

575. Determine the angle of a parallelogram, given its altitudes h_1 and h_2 and perimeter $2p$.

576. In a right triangle find the ratio of the legs, if the altitude and median emanating from the vertex of the right angle are in the ratio $40 : 41$.

577. In a right triangle the hypotenuse is equal to c, and one of the acute angles, to α. Determine the radius of the inscribed circle.

578. The sides of a triangle are equal to 25 cm, 24 cm and 7 cm. Determine the radii of the inscribed and circumscribed circles.

579. Determine the radii of two externally tangent circles, if the distance between their centres is equal to d, and the angle between the common exterior tangents, to φ.

580. Determine the angle of a rhombus, given its area Q and the area of the inscribed circle S.

581. A regular $2n$-gon is inscribed in a circle, and a regular n-gon is circumscribed about the same circle. The difference between the areas of the polygons is P. Determine the radius of the circle.

582. The midpoints of the sides of a regular n-gon are joined by straight lines to form a new regular n-gon inscribed in the given one. Find the ratio of their areas.

583. A circle is circumscribed about a regular n-gon with side a, another circle is inscribed in it. Determine the area of the annulus bounded by the circles and its width.

584. A circle is inscribed in a sector of radius R with a central angle α. Determine the radius of the circle.

585. From one point two lines are drawn tangent to a circle of radius R. The angle between the tangents is 2α. Determine the area bounded by the tangents and the arc of the circle.

586. A rhombus with the acute angle α and side a is divided into three equal parts by straight lines emanating from the vertex of this angle. Determine the lengths of the line-segments.

587. A point is situated inside an angle of 60° at distances a and b from its sides. Find the distance of this point from the vertex of the given angle.

588. Determine the area of a triangle, given the lengths of its sides a and b, and the length t of the bisector of the angle between these sides.

589. In an isosceles triangle the length of the side is equal to a, and the length of the line-segment, drawn from the vertex of the triangle to its base and dividing the vertex angle in the ratio 1 : 2, is t. Find the area of the triangle.

590. Given the angles of a triangle, determine the angle between the median and altitude drawn from the vertex of any angle.

591. The side of a regular triangle is equal to a. A circle of radius $\frac{a}{3}$ is drawn from its centre. Determine the area of the portion of the triangle outside this circle.

592. In a right-angled trapezoid, whose altitude is h, on the side, which is not perpendicular to the base, as on the diameter, a circle is drawn touching the opposite side of the trapezoid. Find the area of the right-angled triangle, whose legs are the bases of the trapezoid.

593. Prove that in a right-angled triangle the bisector of the right angle bisects the angle between the median and altitude dropped to the hypotenuse.

594. Prove that in a right-angled triangle the sum of the legs is equal to the sum of the diameters of the inscribed and circumscribed circles.

595. Determine the angles of a right-angled triangle if the ratio of the radii of the circumscribed and inscribed circles is 5 : 2.

596. Prove that the straight lines successively joining the centres of the squares constructed on the sides of a parallelogram and adjoining it from outside also form a square.

CHAPTER IX

POLYHEDRONS

597. The sides of the base of a rectangular parallelepiped are a and b. The diagonal of the parallelepiped is inclined to the plane of the base at an angle α. Determine the lateral area of the parallelepiped.

598. In a regular hexagonal prism the longest diagonal having length d forms an angle α with the lateral edge of the prism. Determine the volume of the prism.

599. In a regular quadrangular pyramid the lateral edge of length m is inclined to the plane of the base at an angle α. Find the volume of the pyramid.

600. The volume of a regular quadrangular pyramid is equal to V. The angle of inclination of its lateral edge to the plane of the base is equal to α. Find the lateral edge of the pyramid.

601. The lateral area of a regular quadrangular pyramid is equal to S cm^2, its altitude, to H cm. Find the side of its base.

602. Find the volume and lateral area of a regular hexagonal pyramid, given the lateral edge l and diameter d of the circle inscribed in the base of the pyramid.

603. Find the altitude of a regular tetrahedron, whose volume is equal to V.

604. In a right parallelepiped the sides of the base are equal to a and b, and the acute angle, to α. The larger diagonal of the base is equal to the smaller diagonal of the parallelepiped. Find the volume of the parallelepiped.

605. The diagonals of a right parallelepiped are equal to 9 cm and $\sqrt{33}$ cm. The perimeter of its base is equal to 18 cm. The lateral edge is equal to 4 cm. Determine the total surface area and volume of the parallelepiped.

606. The lateral edge of a regular triangular pyramid is equal to l, its altitude, to h. Determine the dihedral angle at the base.

607. Determine the volume of a regular quadrangular pyramid, given the angle α between its lateral edge and the plane of the base, and the area S of its diagonal section. Find also the angle formed by the lateral face and the plane containing the base.

608. The base of a regular pyramid is a polygon, the sum of interior angles of which is equal to 540°. Determine the volume of the pyramid if its lateral edge, equal to l, is inclined to the plane of the base at an angle α.

609. Determine the angles between the base and lateral edge, and between the base and lateral face in a regular pentagonal pyramid, whose lateral faces are equilateral triangles.

610. Given the volume V of a regular n-gonal pyramid in which the side of the base is equal to a, determine the angle of inclination of the lateral edge of the pyramid to the plane containing the base.

611. The base of a quadrangular pyramid is a rectangle with the diagonal equal to b and the angle α between the diagonals. Each of the lateral edges forms an angle β with the base. Find the volume of the pyramid.

612. The base of a pyramid is an isosceles triangle with the equal sides of a and the angle between them equal to α. All lateral edges are inclined to the base at an angle β. Determine the volume of the pyramid.

613. The base of a rectangular parallelepiped is a rectangle inscribed in a circle of radius R, the smaller side of this rectangle subtending a circular arc equal to $(2\alpha)^\circ$. Find the volume of the parallelepiped, given its lateral area S.

614. The base of a right prism is an isosceles triangle, whose base is equal to a and the angle at the base, to α. Determine the volume of the prism if its lateral area is equal to the sum of the areas of its bases.

615. The slant height of a regular hexagonal pyramid is equal to m. The dihedral angle at the base is equal to α. Find the total surface area of the pyramid.

616. Through the hypotenuse of a right-angled isosceles triangle a plane P is drawn at an angle α to the plane of the triangle. Determine the perimeter and area of the figure obtained by projecting the triangle on the plane P. The hypotenuse of the triangle is equal to c.

617. In a regular n-gonal pyramid the area of the base is equal to Q, and the altitude forms an angle φ with each of the lateral faces. Determine the lateral and total surface areas of the pyramid.

618. The side of the base of a regular triangular pyramid is equal to a, the lateral face is inclined to the plane of the base at an angle of φ. Find the volume and total surface area of the pyramid.

619. The total surface area of a regular triangular pyramid is equal to S. Find the side of its base, if the angle between the lateral face and the base of the pyramid is equal to α.

620. The base of a pyramid is a rhombus with the acute angle α. The lateral faces are inclined to the plane of the base at an angle β. Determine the volume and total surface area of the pyramid, if the radius of the circle inscribed in the rhombus is equal to r.

621. Determine the angle of inclination of the lateral face of a regular pentagonal pyramid to the plane of the base, if the area of the base of the pyramid is equal to S, and its lateral area, to σ.

622. The base of a right parallelepiped is a rhombus. A plane drawn through one of the sides of the lower base and the opposite side of the upper base forms an angle β with the plane containing the base. The area of the section thus obtained is equal to Q. Determine the lateral area of the parallelepiped.

623. The base of a pyramid is an isosceles triangle with the base angle α. Each of the dihedral angles at the base is equal to φ. The distance between the centre of the circle inscribed in the base of the pyramid and the midpoint of the height of the lateral face is equal to d. Determine the total surface area of the pyramid.

624. The base of a pyramid is a polygon circumscribed about a circle of radius r; the perimeter of the polygon is equal to $2p$, the lateral faces of the pyramid are inclined to the base at an angle φ. Find the volume of the pyramid.

625. The lateral edges of a frustum of a regular triangular pyramid are inclined to the base at an angle α. The side of the lower base is equal to a, and that of the upper one, to b $(a > b)$. Find the volume of the frustum.

626. The bases of a frustum of a regular pyramid are squares with sides a and b $(a > b)$. The lateral edges are inclined to the base at an angle α. Determine the volume of the frustum and the dihedral angles at the sides of the bases.

627. The base of a pyramid is a right-angled triangle, whose hypotenuse is equal to c and acute angle, to α. All the lateral edges are inclined to the base at an angle β. Find the volume of the pyramid and the face angles at its vertex.

628. The base of an oblique prism is a right-angled triangle ABC the sum of the legs of which is equal to m, and the angle at the vertex A, to α. The lateral face of the prism passing through the leg AC is inclined to the base at an angle β. A plane is drawn through the hypotenuse AB and the vertex C_1 of the opposite trihedral angle. Determine the volume of the cut-off triangular pyramid, if it is known that it has equal edges.

629. The base of a pyramid is an isosceles triangle with the base angle α. All the lateral edges are inclined to the plane containing the base at equal angles $\varphi = 90° - \alpha$. The area of the section passing through the altitude of the pyramid and the vertex of the base (isosceles triangle) is equal to Q. Determine the volume of the pyramid.

630. The base of a pyramid is a rectangle. Two of the lateral faces are perpendicular to the base, the other two forming angles α and β with it. The altitude of the pyramid is equal to H. Determine the volume of the pyramid.

631. The base of a pyramid is a square. Out of two opposite edges one is perpendicular to the base, the other is inclined to it at an angle β and has a length l. Determine the lengths of the remaining lateral edges and the angles of their inclination to the base of the pyramid.

632. The base of a pyramid is a regular triangle with side a. One of the lateral edges is perpendicular to the base, the other two being inclined to the base at equal angles β. Find the surface area of the largest lateral face of the pyramid and the angle of its inclination to the base.

633. The base of a pyramid is an isosceles triangle; the equal sides of the base are of length a and form an angle of 120°. The lateral edge of the pyramid, passing through the vertex of the obtuse angle, is perpendicular to the plane of the base, the other two being inclined to it at an angle α. Determine the area of the section of the pyramid by a cutting plane which passes through the largest side of the base of the pyramid and bisects the edge perpendicular to the base.

634. A regular triangular pyramid is cut by a plane perpendicular to the base and bisecting two sides of the base. Determine the volume of the cut-off pyramid, given the side a of the base of the original pyramid and dihedral angle α at the base.

635. Through the vertex of a regular quadrangular pyramid a cutting plane is drawn parallel to a side of the base and at an angle φ to the base of the pyramid. The side of the base of the pyramid is equal to a, and the face angle at the vertex of the pyramid, to α. Find the area of the section.

636. A plane is drawn through the vertex of a regular triangular pyramid and the midpoints of two sides of the base. Determine the area of the section figure and volumes of the portions of the given pyramid into which it is divided by the cutting plane, given the side a of the base and angle α formed by the cutting plane with the base.

637. A regular tetrahedron, whose edge is equal to a, is cut by a plane containing one of its edges and dividing the opposite edge in the ratio $2:1$. Determine the area of the section figure and its angles.

638. Determine the volume of a frustum of a regular quadrangular pyramid, if the side of the larger base is equal to a, the side of the smaller base, to b, and the acute angle of the lateral face, to α.

639. Determine the volume of a regular quadrangular prism, if its diagonal forms an angle α with the lateral face, and the side of the base is equal to b.

640. The base of a right prism is a right-angled triangle with hypotenuse c and acute angle α. Through the hypotenuse of the lower base and the vertex of the right angle of the upper base a plane is drawn to form an angle β with the base. Determine the volume of the triangular pyramid cut off the prism by the plane.

641. The base of a right prism is a right-angled triangle in which the sum of a leg and the hypotenuse is equal to m, and the angle between them, to α. Through the other leg and the vertex of the opposite trihedral angle of the prism a plane is drawn at an angle β to the base. Determine the volume of the portions into which the prism is divided by the cutting plane.

642. The base of a pyramid is an isosceles triangle with the base angle α. Each dihedral angle at the base is equal to $\varphi = 90° - \alpha$.

The lateral area of the pyramid is S. Determine the volume of the pyramid and its total surface area.

643. The base of a pyramid is an isosceles triangle with the side a and the base angle α ($\alpha > 45°$). The lateral edges are inclined to the base at an angle β. A cutting plane is drawn through the altitude of the pyramid and the vertex of one of the angles α. Find the area of the section figure.

644. The base of a right prism is a quadrilateral in which two opposite angles are right ones. Its diagonal joining the vertices of oblique angles has a length l and divides one of them into portions α and β. The area of the section figure contained in a cutting plane passing through the other diagonal of the base and perpendicular to it is equal to S. Find the volume of the prism.

645. The base of a pyramid is a square. Two opposite faces are isosceles triangles; one of them forms an interior angle β with the base, the other, an exterior acute angle α. The altitude of the pyramid is equal to H. Find the volume of the pyramid and the angles formed by the other two lateral faces with the plane containing the base.

646. The base of a pyramid is a rectangle. One of the lateral faces is inclined to the base at an angle $\beta = 90° - \alpha$ and the face opposite it is perpendicular to the base and represents a right-angled triangle with the right angle at the vertex of the pyramid and an acute angle equal to α. The sum of the heights of these two faces is equal to m. Determine the volume of the pyramid and the sum of the areas of the other two lateral faces.

647. The base of a pyramid is a rectangle. One of the lateral faces is an isosceles triangle perpendicular to the base; in the other face, which is opposite the first one, the lateral edges, equal to b, form an angle 2α and are inclined to the first face at an angle α. Determine the volume of the pyramid and the angle between the above two faces.

648. In a regular triangular pyramid, with the side of the base equal to a, the angles between the edges at its vertex are equal to one another, each being equal to α ($\alpha \leqslant 90°$). Determine the angles between the lateral faces of the pyramid and the area of a section drawn through one of the sides of the base and perpendicular to the opposite lateral edge.

649. Determine the volume of a regular octahedron with edge a and also the dihedral angles at its edges.

650. The dihedral angle at a lateral edge of a regular hexagonal pyramid is equal to φ. Determine the face angle at the vertex of the pyramid.

651. The base of a pyramid is a regular hexagon $ABCDEF$. The lateral edge MA is perpendicular to the base, and the opposite edge MD is inclined to the base at an angle α. Determine the angle of inclination of the lateral faces to the base.

652. The base of a pyramid is an isosceles triangle ABC in which $AB = AC$. The altitude of the pyramid SO passes through the midpoint of the altitude AD of the base. Through the side BC a plane is drawn perpendicular to the lateral edge AS and at an angle α to the base. Determine the volume of the pyramid cut off the given one and having vertex S in common with it, if the volume of the other cut-off portion is equal to V.

653. The side of the base of a regular triangular pyramid is equal to a. A section bisecting an angle between the lateral faces represents a right-angled triangle. Determine the volume of the pyramid and the angle between its lateral face and the plane containing the base.

654. Through a side of the base of a regular triangular pyramid a plane is drawn perpendicular to the opposite lateral edge. Determine the total surface area of the pyramid, if the plane divides the lateral edge in the ratio $m:n$, and the side of the base is equal to q.

655. The diagonal of a rectangular parallelepiped is equal to d and forms equal angles α with two adjacent lateral faces. Determine the volume of the parallelepiped and the angle between the base and a plane passing through the end points of three edges emanating from one vertex.

656. In a rectangular parallelepiped the point of intersection of the diagonals of the lower base is joined with the midpoint of one of the lateral edges by a straight line, whose length is equal to m. This line forms an angle α with the base and angle $\beta = 2\alpha$ with one of the lateral faces. Taking the other adjacent lateral face for the base of the parallelepiped, find its lateral area and volume. (Prove that $\alpha < 30°$.)

657. The base of a right prism is a trapezoid inscribed in a semicircle of radius R so that its larger base coincides with the diameter, and the smaller one subtends an arc equal to 2α. Determine the volume of the prism, if the diagonal of a face passing through one of the nonparallel sides of the base is inclined to the latter at an angle α.

658. The diagonal of a rectangular parallelepiped, equal to d, forms an angle $\beta = 90° - \alpha$ with the lateral face. The plane drawn through this diagonal and the lateral edge intersecting with it forms an angle α with the same lateral face (prove that $\alpha > 45°$). Determine the volume of the parallelepiped.

659. In a regular triangular prism two vertices of the upper base are joined with the midpoints of the opposite sides of the lower base by straight lines. The angle between these lines which faces the base is equal to α. The side of the base is equal to b. Determine the volume of the prism.

660. In a regular triangular prism the angle between a diagonal of a lateral face and another lateral face is equal to α. Determine the lateral area of the prism, if the edge of the base is equal to a.

661. The base of a right prism is a right-angled triangle ABC in which $\angle C = 90°$, $\angle A = \alpha$ and the leg $AC = b$. The diagonal of the lateral face of the prism which passes through the hypotenuse AB, forms an angle β with the lateral face passing through the leg AC. Find the volume of the prism.

662. The total surface area of a regular quadrangular pyramid is equal to S, and the face angle at the vertex, to α. Find the altitude of the pyramid.

663. In a regular n-gonal pyramid the face angle at the vertex is equal to α, and the side of the base, to a. Determine the volume of the pyramid.

664. In a regular quadrangular prism a plane is drawn through a diagonal of the lower base and one of the vertices of the upper base, which cuts off a pyramid with a total surface area S. Find the total surface area of the prism, if the angle at the vertex of the triangle obtained in the section is equal to α.

665. The lateral edges of a triangular pyramid are of equal length l. Out of the three face angles formed by these edges at the vertex of the pyramid two are equal to α, and the third, to β. Find the volume of the pyramid.

666. The base of a pyramid is a right-angled triangle, which is a projection of the lateral face passing through a leg. The angle opposite this leg in the base of the pyramid is equal to α, and the one lying in the lateral face is equal to β. The area of this lateral face exceeds that of the base by S. Determine the difference between the areas of the other two faces and the angles formed by the lateral faces with the base.

667. In a triangular pyramid two lateral faces are isosceles right-angled triangles, whose hypotenuses are equal to b and form an angle α. Determine the volume of the pyramid.

668. In a pyramid with a rectangular base each of the lateral edges is equal to l; one of the face angles at the vertex is equal to α, the other, to β. Determine the area of the section passing through the bisectors of the angles equal to β.

669. In a parallelepiped the lengths of three edges emanating from a common vertex are respectively equal to a, b and c. The edges a and b are mutually perpendicular, and the edge c forms an angle α with each of them. Determine the volume of the parallelepiped, its lateral area and the angle between the edge c and the plane containing the base. (For what values of the angle α is the problem solvable?)

670. All the faces of a parallelepiped are equal rhombuses with sides a and acute angles α. Determine the volume of the parallelepiped.

671. The base of an oblique parallelepiped is a rhombus $ABCD$ with the side a and acute angle α. The edge AA_1 is equal to b and forms an angle φ with the edges AB and AD. Determine the volume of the parallelepiped.

672. In a rectangular parallelepiped a plane is drawn through a diagonal of the base and a diagonal of the larger lateral face, both emanating from one vertex. The angle between these diagonals is equal to β. Determine the lateral area of the parallelepiped, the area of the section figure and the angle of inclination of the cutting plane to the base, if it is known that the radius of the circle circumscribed about the base of the parallelepiped is equal to R and the smaller angle between the diagonals of the base, to 2α.

673. The base of a right prism is a right-angled triangle ABC. The radius of the circle circumscribed about it is equal to R, the leg AC subtends an arc equal to 2β. Through a diagonal of the lateral face passing through the other leg BC a plane is drawn perpendicular to this face and inclined to the base at an angle β. Determine the lateral area of the prism and the volume of the cut-off quadrangular pyramid.

674. The base of a pyramid is a trapezoid, whose nonparallel sides and smaller base are of equal length. The larger base of the trapezoid is equal to a, and the obtuse angle, to α. All the lateral edges of the pyramid are inclined to the base at an angle β. Determine the volume of the pyramid.

675. The base of a pyramid is a trapezoid, whose diagonal is perpendicular to one of the nonparallel sides and forms an angle α with the base. All the lateral edges are of equal length. The lateral face passing through the larger base of the trapezoid has an angle $\varphi = 2\alpha$ at the vertex of the pyramid and its area is equal to S. Determine the volume of the pyramid and the angles at which the lateral faces are inclined to the base.

676. The base of a pyramid is a regular triangle, whose side is equal to a. The altitude, dropped from the vertex of the pyramid, passes through one of the vertices of the base. The lateral face passing through the side of the base opposite this vertex is at an angle φ to the base. Determine the lateral area of the pyramid, if one of the equal lateral faces is taken as the base.

677. The base of a right prism is an isosceles triangle with the equal sides of length a and the base angle α. Through the base of the triangle, which is the upper base of the prism, and the opposite vertex of the lower base a cutting plane is drawn at an angle β to the base. Determine the lateral area of the prism and the volume of the cut-off quadrangular pyramid.

678. The base of a pyramid is a square. Its two lateral faces are perpendicular to the base, and the remaining two are inclined to it at an angle α. The radius of the circle circumscribed about the lateral face perpendicular to the base is equal to R. Determine the total surface area of the pyramid.

679. The base of a right prism is a right-angled triangle with a leg a and angle α opposite it. Through the vertex of the right angle of the lower base a plane is drawn which is parallel to the hypotenuse and intersects the opposite lateral face at an angle $\beta = 90° - \alpha$. Determine the volume of the portion of the prism contained between its base and the cutting plane and the lateral area of the prism, if the area of the lateral face passing through the leg a is equal to the area of the section figure. Determine the value of the angle α at which the cutting plane intersects the lateral face passing through the hypotenuse of the base.

680. The base of a pyramid is a rectangle. One lateral edge is perpendicular to the base, and two lateral faces are inclined to it at angles α and β, respectively. Determine the lateral area of the pyramid, if its altitude is equal to H.

681. The base of a pyramid is a right-angled triangle with an acute angle α; the radius of the inscribed circle is equal to r. Each lateral face is inclined to the base at an angle α. Determine the volume and the lateral and total surface areas of the pyramid.

682. The base of a prism $ABCA_1B_1C_1$ is an isosceles triangle ABC ($AB = AC$ and $\angle ABC = \alpha$). The vertex B_1 of the upper base of the prism is projected into the centre of the circle of radius r inscribed in the lower base. Through the side AC of the base and the vertex B_1 a cutting plane is drawn at an angle α to the base. Find the total surface area of the cut-off triangular pyramid $ABCB_1$ and the volume of the prism.

683. The base of a pyramid is a right-angled triangle. The altitude of the pyramid passes through the point of intersection of the hypotenuse and the bisector of the right angle of the base. The lateral edge passing through the vertex of the right angle is inclined to the base at an angle α. Determine the volume of the pyramid and the angles of inclination of the lateral faces to the base, if the bisector of the right angle of the base is equal to m and forms an angle of $45° + \alpha$ with the hypotenuse.

684. The base of a pyramid is a rhombus with the side a. Two adjacent faces are inclined to the plane of the base at an angle α, the third one, at an angle β (prove that the fourth lateral face is inclined to the base at the same angle). The altitude of the pyramid is H. Find its volume and total surface area.

685. The base of a quadrangular pyramid is a rhombus, whose side is equal to a and acute angle, to α. The planes passing through the vertex of the pyramid and diagonals of the base are inclined to the base at angles φ and ψ. Determine the volume of the pyramid, if its altitude intersects a side of the base.

686. The base of an oblique prism is a right-angled triangle ABC with the leg $BC = a$. The vertex B_1 of the upper base is projected into the midpoint of the leg BC. The dihedral angle formed by the lateral faces passing through the leg BC and hypotenuse AB is equal to α. The lateral edges are inclined to the base at an angle β. Determine the lateral area of the prism.

687. The base of a prism $ABCA_1B_1C_1$ is an isosceles triangle ABC ($AB = AC$ and $\angle BAC = 2\alpha$). The vertex A_1 of the upper base is projected into the centre of the circle of radius R circumscribed about the lower base. The lateral edge AA_1 forms with the side AB of the base an angle equal to 2α. Determine the volume and the lateral area of the prism.

688. Determine the volume of a regular quadrangular pyramid, whose lateral edge is equal to l and the dihedral angle between two adjacent lateral faces is equal to β.

689. Given in a frustum of a regular quadrangular pyramid: diagonal d, dihedral angle α at the lower base and altitude H. Find the volume of the frustum.

690. The lateral edge of a frustum of a regular quadrangular pyramid is equal to l and inclined to the base at an angle β. The diagonal of the pyramid is perpendicular to its lateral edge. Determine the volume of the pyramid.

691. The altitude of a frustum of a regular quadrangular pyramid is equal to H, the lateral edge and diagonal of the pyramid are inclined to the base at angles α and β, respectively. Find the lateral area of the frustum.

692. The sides of the bases of a frustum of a regular quadrangular pyramid are equal to a and $a\sqrt{3}$, respectively; the lateral face is inclined to the base at an angle γ. Determine the volume and total surface area of the frustum.

693. A cube is inscribed in a regular quadrangular pyramid so that its four vertices are found on the lateral edges, and the remaining four, in the plane of its base. Determine the edge of the cube, if the altitude of the pyramid is equal to H, and the lateral edge, to l.

694. A cube is inscribed in a regular quadrangular pyramid so that its vertices lie on the slant heights of the pyramid. Find the ratio of the volume of the pyramid to the volume of the cube, if the angle between the altitude of the pyramid and its lateral face is equal to α.

695. The base of a pyramid is a right-angled triangle with legs equal to 6 and 8, respectively. The vertex of the pyramid is at a distance of 24 from the base and is projected onto its plane at a point lying inside the base. Find the edge of the cube, whose four vertices lie in the plane of the base of the given pyramid, and the edges joining these vertices are parallel to the corresponding legs of the base triangle of the pyramid. The other four vertices of the cube lie on the lateral faces of the given pyramid.

696. In a regular quadrangular pyramid the dihedral angle at the base is equal to α. Through its edge a cutting plane is drawn at an angle β to the base. The side of the base is equal to a. Determine the area of the section figure.

697. In a regular quadrangular pyramid the side of the base is equal to a, and the dihedral angle at the base, to α. Through two opposite sides of the base of the pyramid two planes are drawn at right angles to each other. Determine the length of the line of intersection of the planes contained inside the pyramid, if it is known that it intersects the axis of the pyramid.

698. In a regular quadrangular pyramid a plane is drawn through a vertex of the base perpendicular to the opposite lateral edge. Determine the area of the section figure thus obtained, if the side of the base of the pyramid is equal to a, and the lateral edge is inclined to the plane containing the base at an angle φ ($\varphi > 45°$; prove this).

699. It is required to cut a regular quadrangular prism with a plane to obtain a section yielding a rhombus with the acute angle α. Find the angle of inclination of the cutting plane to the base.

700. The base of a right parallelepiped is a rhombus with the acute angle α. At what angle to the base must a cutting plane be drawn to obtain a section yielding a square with its vertices lying on the lateral edges of the parallelepiped?

701. A right parallelepiped, whose base is a rhombus with the side a and acute angle α is cut with a plane passing through the vertex of the angle α, the section yielding a rhombus with the acute angle $\frac{\alpha}{2}$. Determine the area of this section.

702. The edge of a tetrahedron is equal to b. Through the midpoint of one of the edges a plane is drawn parallel to two non-intersecting edges. Determine the area of the section thus obtained.

703. The base of a pyramid is a right-angled triangle with a leg a. One of the lateral edges of the pyramid is perpendicular to the base, the other two being inclined to it at one and the same angle α. A plane perpendicular to the base cuts the pyramid, yielding a square. Determine the area of this square.

704. In a frustum of a regular quadrangular pyramid the sides of the upper and lower bases are respectively equal to a and $3a$ and the lateral faces are inclined to the plane containing the lower base at an angle α. Through a side of the upper base a plane is drawn parallel to the opposite lateral face. Determine the volume of the quadrangular prism cut off the given frustum and the total surface area of the remaining portion of the frustum.

705. Two planes are drawn through a point taken on a lateral edge of a regular triangular prism with the side of the base a. One of them passes through a side of the lower base of the prism at an angle α to the base, the other, through the parallel side of the upper base and at an angle β to it. Determine the volume of the prism and the sum of the areas of the sections thus obtained.

706. In a regular quadrangular prism a plane is drawn through the midpoints of two adjacent sides of the base at an angle α to the latter to intersect three lateral edges. Determine the area of the section figure obtained and its acute angle, if the side of the prism's base is equal to b.

707. The base of a right prism is an isosceles trapezoid (with the acute angle α) circumscribed about a circle of radius r. Through one of the nonparallel sides of the base and the opposite vertex of the acute angle of the upper base a plane is drawn at an angle α to the base. Determine the lateral area of the prism and the area of the section figure thus obtained.

708. The base of a right prism $ABCA_1B_1C_1$ is an isosceles triangle ABC with angle α at the base BC. The lateral area of the prism is equal to S. Find the area of the section by a plane passing through a diagonal of the face BCC_1B_1 parallel to the altitude AD of the base of the prism and at an angle β to the base.

709. The base of a right prism $ABCA_1B_1C_1$ is a right-angled triangle ABC with an angle β at the vertex B ($\beta < 45°$). The difference between the areas of its lateral faces passing through the legs BC and AC is equal to S. Find the area of the section by a plane forming an angle φ with the base and passing through three points: the vertex B_1 of the angle β of the upper base, midpoint of the lateral edge AA_1 and point D situated on the base and symmetrical to the vertex B with respect to the leg AC.

710. Non-intersecting diagonals of two adjacent lateral faces of a rectangular parallelepiped are inclined to its base at angles α and β. Find the angle between these diagonals.

711. Given three plane angles of the trihedral angle $SABC$: $\angle BSC = \alpha$; $\angle CSA = \beta$; $\angle ASB = \gamma$. Find the dihedral angles of this trihedral angle.

712. One of the dihedral angles of a trihedral angle is equal to A; the plane angles adjacent to the given dihedral angle are equal to α and β. Find the third plane angle.

713. Given in a trihedral angle are three plane angles: 45°, 60° and 45°. Determine the dihedral angle contained between the two faces with plane angles of 45°.

714. A line-segment AB is given on the edge of a dihedral angle. In one of the faces a point M is given, at which a straight line drawn from A at an angle α to AB intersects a line drawn from B perpendicular to AB. Determine the dihedral angle, if the straight line AM is inclined to the second face of the dihedral angle at an angle β.

715. Given two skew lines inclined at an angle φ to each other and having a common perpendicular $PQ = h$ which intersects both of them. Given on these lines are two points A and B, from which the line-segment PQ is seen at angles α and β, respectively. Determine the length of the line-segment AB.

716. Given on two mutually perpendicular skew lines, the perpendicular distance between which $PQ = h$, are two points A and B, from which the line-segment PQ is seen at angles α and β, respectively. Determine the angle of inclination of the segment AB to PQ.

717. A cutting plane divides the lateral edges of a triangular pyramid in the ratios (as measured from the vertex): $\frac{m_1}{n_1}$, $\frac{m_2}{n_2}$, $\frac{m_3}{n_3}$. In what ratio is the volume of the pyramid divided by this plane?

718. From the midpoint of the altitude of a regular quadrangular pyramid a perpendicular, equal to h, is dropped to a lateral edge, and another perpendicular, equal to b, to a lateral face. Find the volume of the pyramid.

CHAPTER X

SOLIDS OF REVOLUTION

719. The generator of a cone is equal to l and forms an angle of 60° with the plane of the base. Determine the volume of the cone.

720. The length of the generator of a cone is equal to l, and the circumference of the base, to c. Determine the volume.

721. The lateral surface of a cylinder is developed into a square with the side a. Find the volume of the cylinder.

722. When developed, the curved surface of a cylinder represents a rectangle, whose diagonal is equal to d and forms an angle α with the base. Determine the volume of the cylinder.

723. The angle at the vertex of an axial section of a cone is equal to 2α, and the sum of the lengths of its altitude and the generator, to m. Find the volume and surface of the cone.

724. The volume of a cone is V. Its altitude is trisected and through the points of division two planes are drawn parallel to the base. Find the volume of the medium portion.

725. Determine the volume of a cone, if a chord, equal to a, drawn in its base circle subtends an arc α, and the altitude of the cone forms an angle β with the generator.

726. Two cones (one inside the other) are constructed on one and the same base; the angle between the altitude and the generator of the smaller cone is equal to α, and that of the larger cone, to β. The difference between the altitudes is equal to h. Find the volume of the solid bounded by the curved surfaces of the cones.

727. The curved surface of a cone is equal to S, and the total one, to P. Determine the angle between the altitude and the generator.

728. When developed on a plane, the curved surface of a cone represents a circular sector with the angle α and chord a. Determine the volume of the cone.

729. A plane, drawn through the vertex of a cone and at an angle φ to the base, cuts off the circle of the base an arc α; the distance between the plane and the centre of the base is equal to a. Find the volume of the cone.

730. A square, whose side is equal to a, is inscribed in the base of a cone. A plane drawn through the vertex of the cone and a side of the square intersects the surface of the cone along a triangle, the angle at the vertex of which is α. Determine the volume and surface of the cone.

731. The element l of a frustum of a cone forms an angle α with the plane of the lower base and is perpendicular to the straight line joining its upper end point with the lower end point of the opposite element. Find the curved surface of the frustum.

732. Given a cone of volume V, whose generator is inclined to the base at an angle α. At what height should a cutting plane be drawn perpendicular to the axis of the cone to divide the curved surface of the cone into two equal parts? The same question for the total surface.

733. Determine the volume and surface of a spherical sector cut off a sphere of radius R and having an angle α in the axial section.

734. The surface of a spherical segment of radius R is S. Find its altitude.

735. The area of a triangle ABC is equal to S, the side $AC = b$ and $\angle CAB = \alpha$. Find the volume of the solid formed by rotating the triangle ABC about the side AB.

736. Given in a triangle ABC: the side a, angle B and angle C. Determine the volume of a solid obtained by rotating the triangle about the given side.

737. A rhombus with the larger diagonal d and acute angle γ rotates about an axis passing outside it through its vertex and perpendicular to its larger diagonal. Determine the volume of the solid thus obtained.

738. Given in a triangle: sides b and c and the angle α between them. The triangle rotates about an axis which passes outside it through the vertex of the angle α and is inclined to the sides b and c at equal angles. Determine the volume of the solid thus generated.

739. In an isosceles trapezoid a diagonal is perpendicular to one of the nonparallel sides. The side is equal to b and forms an angle α with the larger base. Determine the surface of the solid generated by rotating the trapezoid about the larger base.

740. Two planes are drawn through the vertex of a cone. One of them is inclined to the base of the cone at an angle α and intersects it along a chord of length a, the other is inclined to the base at an angle β and intersects it along a chord of length b. Determine the volume of the cone.

741. A sphere is inscribed in a cone. Find the volume of the sphere, if the generator of the cone is equal to l and is inclined to the base at an angle α.

742. A straight line, tangent to the curved surface of a cone, forms an angle θ with the element passing through the point of tangency. What angle (φ) does this line form with the plane of the base P of the cone, if its generator is inclined to the plane P at an angle α?

743. An obtuse triangle with acute angles α and β and the smaller altitude h rotates about the side opposite the angle β. Find the surface of the solid thus generated.

744. In a cone (whose axial section represents an equilateral triangle) installed with its base up and filled with water a ball of radius r is placed flush with the water level. Determine the height of the water level in the cone after the ball is removed.

745. In a cone, the radius of the base circle of which is equal to R and whose generator is inclined to the base at an angle $\frac{\alpha}{2}$, a right triangular prism is inscribed so that its lower base lies on the base of the cone, and the vertices of the upper base are on the curved surface of the cone. Determine the lateral area of the prism, if the base of the prism is a right-angled triangle with an acute angle α, and its altitude is equal to the radius of the circle along which the plane passing through the upper base of the prism intersects the cone.

746. In a triangular pyramid, whose base is a regular triangle with the side a, a cylinder is inscribed so that its lower base is found on the base of the pyramid, its upper base touching all the lateral faces. Find the volumes of the cylinder and the pyramid cut-off by the plane passing through the upper base of the cylinder, if the altitude of the cylinder is equal to $\frac{a}{2}$, one of the lateral edges of the pyramid is perpendicular to the base, and one of its lateral faces is inclined to the base at an angle α (define the values of α for which the problem is solvable).

747. A right triangular prism is inscribed in a sphere of radius R. The base of the prism is a right-angled triangle with an acute angle α and its largest lateral face is a square. Find the volume of the prism.

748. The base of a pyramid is a rectangle with an acute angle α between the diagonals, and its lateral edges form an angle φ with the base. Determine the volume of the pyramid, if the radius of the circumscribed sphere is equal to R.

749. The radius of the base circle of a cone is equal to R and the angle at the vertex of its axial section is α. Find the volume of a reqular triangular pyramid circumscribed about the cone.

750. A sphere of radius r is inscribed in a frustum of a cone. The generator of the cone is inclined to the base at an angle α. Find the curved surface of the frustum.

751. Circumscribed about a sphere is a frustum of a cone, whose elements are inclined to the base at an angle α. Determine the surface of the frustum, if the radius of the sphere is equal to r.

752. A sphere of radius r is inscribed in a frustum of a cone, whose generator is inclined to the plane of the base at an angle α. Find the volume of the frustum.

753. From a point on the surface of a sphere of radius R three equal chords are drawn at an angle α to one another. Determine their lengths.

754. A frustum of a cone is inscribed in a sphere of radius R. The bases of the frustum cut off the sphere two segments with arcs in the axial section equal to α and β, respectively. Find the curved surface of the frustum.

755. The lateral faces of a regular quadrangular pyramid are inclined to the base at an angle α. The slant height of the pyramid is equal to m. Find the surface of a cone inscribed in the pyramid and the angle of inclination of the lateral edge to the base.

756. A cone is circumscribed about a regular hexagonal pyramid. Find its volume, if the lateral edge of the pyramid is equal to l and the face angle between two adjacent lateral edges is equal to α.

757. A cone is inscribed in a regular triangular pyramid. Find the volume of the cone if the lateral edge of the pyramid is equal to l and the face angle between two adjacent lateral edges is equal to α.

758. A cone is inscribed in a sphere and its volume is equal to one fourth of that of the sphere. Find the volume of the sphere, if the altitude of the cone is equal to H.

759. A sphere is inscribed in a regular triangular prism. It touches the three lateral faces and both bases of the prism. Find the ratio of the surface of the sphere to the total surface area of the prism.

760. A sphere of radius R is inscribed in a pyramid, whose base is a rhombus with the acute angle α. The lateral faces of the pyramid are inclined to the base at an angle φ. Find the volume of the pyramid.

761. A hemisphere is inscribed in a regular quadrangular pyramid so that its base is parallel to the base of the pyramid and the spherical surface is in contact with it. Determine the total surface area of the pyramid, if its lateral faces are inclined to the base at an angle α and the radius of the sphere is equal to r.

762. A hemisphere is inscribed in a regular quadrangular pyramid so that its base lies on the base of the pyramid and the spherical surface touches the lateral faces of the pyramid. Find the ratio of the surface of the hemisphere to the total surface area of the pyramid and the volume of the hemisphere, if the lateral faces are inclined to the base at an angle of α and the difference between the lengths of the side of the base and the diameter of the sphere is equal to m.

763. In a cone, with the radius of the base circle R and an angle α between the altitude and generator, a sphere is inscribed which touches the base and the curved surface of the cone. Determine the volume of the portion of the cone situated above the sphere.

764. The surface of a right circular cone is n times as large as the surface of the sphere inscribed in it. At what angle is the generator of the cone inclined to the base?

765. A sphere is inscribed in a cone. The ratio of their volumes is equal to n. Find the angle of inclination of the generator to the base (calculate for $n = 4$).

766. Determine the angle between the axis and generator of a cone, whose surface is n times as large as the area of its axial section.

767. Inscribed in a cone is a hemisphere, whose great circle lies on the base of the cone. Determine the angle at the vertex of the cone, if the ratio of the surface area of the cone to the curved surface area of the hemisphere is $18:5$.

768. Determine the angle between the altitude and generator of a cone, if the volume of the cone is $1\frac{1}{3}$ times as large as that of the hemisphere inscribed in the cone so that the base of the hemisphere lies on the base of the cone and the spherical surface touches the curved surface of the cone.

769. Determine the angle between the altitude and generator of a cone, whose curved surface is divided into two equal parts by the line of its intersection with a spherical surface, whose centre is located at the vertex of the cone and the radius is equal to the altitude of the cone.

770. A cone with the altitude H and the angle between the generator and altitude equal to α is cut by a spherical surface with the centre at the vertex of the cone to divide the volume of the cone into two equal portions. Find the radius of the sphere.

771. On the altitude of a cone, equal to H, as on the diameter, a sphere of radius $\frac{H}{2}$ is constructed. Determine the volume of the portion of the sphere situated outside the cone, if the angle between the generator and altitude is equal to α.

772. Given two externally tangent spheres O and O_1, and a cone circumscribed about them. Compute the area of the curved surface of the frustum, whose bases are the circles along which the spheres contact the surface of the cone, if the radii of the spheres are equal to R and R_1.

773. Four balls of one and the same radius r lie on a table so that they touch one another. A fifth ball of the same radius is placed on them at the centre. Find the distance between the top point of the fifth ball and the plane of the table.

774. Determine the angle at the vertex of the axial section of a cone circumscribed about four equal balls arranged so that each of them is in contact with the three remaining ones.

775. The faces of a frustum of a regular triangular pyramid touch a sphere. Determine the ratio of the surface of the sphere to the total surface area of the pyramid, if the lateral faces of the pyramid are inclined to the base at an angle α.

776. Inscribed in a cone is a cylinder, whose altitude is equal to the radius of the base circle of the cone. Find the angle between the axis of the cone and its generator, if the ratio of the surface of the cylinder to the area of the base of the cone is $3:2$.

777. The radius of a sphere inscribed in a regular quadrangular pyramid is equal to r. The dihedral angle formed by two adjacent lateral faces of the pyramid is equal to α. Determine the volume of the pyramid, whose vertex is at the centre of the sphere and the vertices of the base lie at the four points of tangency of the sphere and the lateral faces of the pyramid.

778. A sphere of radius r is inscribed in a cone. Find the volume of the cone, if it is known that a plane tangent to the sphere and perpendicular to the generator of the cone is drawn at a distance d from the vertex of the cone.

779. The edge of a cube is a, AB being its diagonal. Find the radius of a sphere tangent to the three faces converging at the vertex A and to the three edges emanating from the vertex B. Also find the area of the portion of the spherical surface outside the cube.

780. In a regular tetrahedron, whose edge is equal to a, a sphere is inscribed so that it is in contact with all the edges. Determine the radius of the sphere and the volume of its portion outside the tetrahedron.

CHAPTER XI

TRIGONOMETRIC TRANSFORMATIONS

Prove the following identities:

781. $\sec\left(\frac{\pi}{4}+\alpha\right)\sec\left(\frac{\pi}{4}-\alpha\right)=2\sec 2\alpha$

782. $\frac{\sin(2\alpha+\beta)}{\sin\alpha}-2\cos(\alpha+\beta)=\frac{\sin\beta}{\sin\alpha}$

783. $2(\csc 2\alpha+\cot 2\alpha)=\cot\frac{\alpha}{2}-\tan\frac{\alpha}{2}$

784. $\dfrac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}=\tan(45^\circ+\alpha)$; **785.** $\dfrac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}=\tan 2\alpha+\sec 2\alpha$

786. $\sin^2\left(\dfrac{\pi}{8}+\alpha\right)-\sin^2\left(\dfrac{\pi}{8}-\alpha\right)=\dfrac{\sin 2\alpha}{\sqrt{2}}$

787. $\dfrac{2\cos^2\alpha-1}{2\tan\left(\dfrac{\pi}{4}-\alpha\right)\cdot\sin^2\left(\dfrac{\pi}{4}+\alpha\right)}=1$

788. $\tan^2\left(\dfrac{\pi}{4}-\alpha\right)=\dfrac{1-\sin 2\alpha}{1+\sin 2\alpha}$

789. $\dfrac{\cos 2\alpha}{\cot^2\alpha-\tan^2\alpha}=\dfrac{1}{4}\sin^2 2\alpha$

790. $\dfrac{\sin\alpha+\cos(2\beta-\alpha)}{\cos\alpha-\sin(2\beta-\alpha)}=\cot\left(\dfrac{\pi}{4}-\beta\right)$

791. $\dfrac{1+\sin 2\alpha}{\cos 2\alpha}=\dfrac{1+\tan\alpha}{1-\tan\alpha}=\tan\left(\dfrac{\pi}{4}+\alpha\right)$

792. $\dfrac{\sin x+\cos(2y-x)}{\cos x-\sin(2y-x)}=\dfrac{1+\sin 2y}{\cos 2y}$

793. $\tan^2\alpha-\tan^2\beta=\sin(\alpha+\beta)\sin(\alpha-\beta)\sec^2\alpha\sec^2\beta$

794. $\dfrac{\tan\left(\dfrac{\pi}{4}-\dfrac{\alpha}{2}\right)\cdot(1+\sin\alpha)}{\sin\alpha}=\cot\alpha$

795. $\tan\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right)\cdot\dfrac{1-\sin\alpha}{\cos\alpha}=1$

796. $\dfrac{2(\sin 2\alpha+2\cos^2\alpha-1)}{\cos\alpha-\sin\alpha-\cos 3\alpha+\sin 3\alpha}=\csc\alpha$

797. $\dfrac{\sin\alpha-\sin 3\alpha+\sin 5\alpha}{\cos\alpha-\cos 3\alpha+\cos 5\alpha}=\tan 3\alpha$

798. $\sin(a-b)+\sin(a-c)+\sin(b-c)=4\cos\dfrac{a-b}{2}\sin\dfrac{a-c}{2}\cos\dfrac{b-c}{2}$

799. $2(\sin^6 x+\cos^6 x)-3(\sin^4 x+\cos^4 x)+1=0$

800. $\sin\alpha+\sin\left(\alpha+\dfrac{2\pi}{3}\right)+\sin\left(\alpha+\dfrac{4\pi}{3}\right)=0$

801. $\sin^2(45^\circ+\alpha)-\sin^2(30^\circ-\alpha)-\sin 15^\circ\cos(15^\circ+2\alpha)=\sin 2\alpha$

802. Show that

$$\frac{1-2\cos^2\varphi}{\sin\varphi\cos\varphi}=\tan\varphi-\cot\varphi$$

803. Show that

$$\tan^2\frac{\alpha}{2}=\frac{2\sin\alpha-\sin 2\alpha}{2\sin\alpha+\sin 2\alpha}$$

804. Prove the identity

$$\cos^2\varphi+\cos^2(\alpha+\varphi)-2\cos\alpha\cos\varphi\cos(\alpha+\varphi)=\sin^2\alpha$$

805. Simplify the expression

$$\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta\cos(\alpha+\beta)$$

806. Prove that

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma-2\cos\alpha\cos\beta\cos\gamma=2,$$

if $\alpha+\beta+\gamma=\pi$.

807. Prove that

$$\cot A\cot B+\cot A\cot C+\cot B\cot C=1,$$

if $A+B+C=\pi$.

808. Prove that

$$\cos\frac{\pi}{5}\cos\frac{2\pi}{5}=\frac{1}{4}$$

809. Prove that

$$\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$$

Reduce to a form convenient for taking logarithms:

810. $1+\cos\alpha+\cos\frac{\alpha}{2}$; **811.** $1-\sqrt{2}\cos\alpha+\cos 2\alpha$

812. $1-\sin^2(\alpha+\beta)-\sin^2(\alpha-\beta)$

813. $1+\sin\alpha+\cos\alpha+\tan\alpha$; **814.** $\dfrac{1+\sin\alpha-\cos\alpha}{\sin\frac{\alpha}{2}}$

815. $1-\tan\alpha+\sec\alpha$; **816.** $\cos\alpha+\sin 2\alpha-\cos 3\alpha$

817. $\tan\left(\alpha+\frac{\pi}{4}\right)+\tan\left(\alpha-\frac{\pi}{4}\right)$

818. $\dfrac{2\sin\beta-\sin 2\beta}{2\sin\beta+\sin 2\beta}$; **819.** $\dfrac{\sqrt{2}-\cos\alpha-\sin\alpha}{\sin\alpha-\cos\alpha}$

820. $\cot\alpha+\cot 2\alpha+\csc 2\alpha$; **821.** $\cos 2\alpha+\sin 2\alpha\tan\alpha$

822. $2\sin^2\alpha+\sqrt{3}\sin 2\alpha-1$; **823.** $\dfrac{1+\tan 2\alpha\tan\alpha}{\cot\alpha+\tan\alpha}$

824. $2+\tan 2\alpha+\cot 2\alpha$

825. $\tan x-1+\sin x\,(1-\tan x)+\frac{1}{1+\tan^2 x}$

826. $\frac{1+\cos\alpha+\cos 2\alpha+\cos 3\alpha}{\cos\alpha+2\cos^2\alpha-1}$; 827. $1-\frac{1}{4}\sin^2 2\alpha-\sin^2\beta-\cos^4\alpha$

828. $\tan\ x+\tan y+\tan z-\frac{\sin(x+y+z)}{\cos x\cos y\cos z}$

829. $\sin\alpha+\sin\beta+\sin\gamma$ if $\alpha+\beta+\gamma=180^\circ$

CHAPTER XII

TRIGONOMETRIC EQUATIONS

Solve the following equations:

830. $1-\sin 5x=\left(\cos\frac{3x}{2}-\sin\frac{3x}{2}\right)^2$

831. $\sin x+\sin 2x+\sin 3x+\sin 4x=0$

832. $\sin(x+30^\circ)+\cos(x+60^\circ)=1+\cos 2x$

833. $\sin x+\sin 2x+\sin 3x=\cos x+\cos 2x+\cos 3x$

834. $\cos 2x-\cos 8x+\cos 6x=1$; 835. $\cos x-\cos 2x=\sin 3x$

836. $\sin(x-60^\circ)=\cos(x+30^\circ)$; 837. $\sin 5x+\sin x+2\sin^2 x=1$

838. $\sin^2 x\,(\tan x+1)=3\sin x\,(\cos x-\sin x)+3$

839. $\cos 4x=-2\cos^2 x$; 840. $\sin x+\cos x=\frac{1}{\sin x}$

841. $\sin 3x=\cos 2x$; 842. $\sin^4\frac{x}{3}+\cos^4\frac{x}{3}=\frac{5}{8}$

843. $3\tan^2 x-\sec^2 x=1$; 844. $(1+\cos 4x)\sin 4x=\cos^2 2x$

845. $\sin^4 x+\cos^4 x=\cos 4x$; 846. $3\cos^2 x-\sin^2 x-\sin 2x=0$

847. $\cos^2 x+3\sin^2 x+2\sqrt{3}\sin x\cos x=1$

848. $6\sin^2 x+3\sin x\cos x-5\cos^2 x=2$

849. $\sin^2 x+\frac{3}{2}\cos^2 x=\frac{5}{2}\sin x\cos x$

850. $\sin x+\sqrt{3}\cos x=1$; 851. $\sin x+\cos x=1$

852. $\sin x+\cos x=1+\sin 2x$; 853. $\sin 3x+\cos 3x=\sqrt{2}$

854. $\sin x \sin 7x = \sin 3x \sin 5x$; **855.** $\cos x \sin 7x = \cos 3x \sin 5x$

856. $\sin x \sin 2x \sin 3x = \frac{1}{4} \sin 4x$; **857.** $2\cos^2 x + 4\cos x = 3\sin^2 x$

858. $5\cos 2x = 4\sin x$; **859.** $\tan\left(\frac{\pi}{4} + x\right) + \tan x - 2 = 0$

860. $8\tan^2\frac{x}{2} = 1 + \sec x$; **861.** $\frac{\cos\left(\frac{\pi}{2} - x\right)}{1 + \cos x} = \sec^2\frac{x}{2} - 1$

862. $1 - \cos(\pi - x) + \sin\frac{\pi + x}{2} = 0$

863. $2\left[1 - \sin\left(\frac{3\pi}{2} - x\right)\right] = \sqrt{3}\tan\frac{\pi - x}{2}$

864. $\sin x - \cos x - 4\cos^2 x \sin x = 4\sin^3 x$

865. $\cot x + \frac{\sin x}{1 + \cos x} = 2$

866. $2\cot(x - \pi) - (\cos x + \sin x)(\csc x - \sec x) = 4$

867. $\sin(\pi - x) + \cot\left(\frac{\pi}{2} - x\right) = \frac{\sec x - \cos x}{2\sin x}$

868. $\frac{1 - \tan\frac{x}{2}}{1 - \cot\frac{x}{2}} = 2\sin\frac{x}{2}$

869. $\sin(\pi - x) + \cot\left(\frac{3\pi}{2} + x\right) = \sec(-x) - \cos(2\pi - x)$

870. $\sec^2 x - \tan^2 x + \cot\left(\frac{\pi}{2} + x\right) = \cos 2x \sec^2 x$

871. $\sin^3 x(1 + \cot x) + \cos^3 x(1 + \tan x) = \cos 2x$

872. $\sin^3 x \cos 3x + \sin 3x \cos^3 x = 0.375$

873. $\tan x + \tan 2x = \tan 3x$

874. $1 + \sin x + \cos x = 2\cos\left(\frac{x}{2} - 45^\circ\right)$

875. $1 - \cos^2 2x = \sin 3x - \cos\left(\frac{\pi}{2} + x\right)$

876. $1 - 3\cos x + \cos 2x = \frac{\csc(\pi - x)}{\cot 2x - \cot x}$

877. $[\cos x - \sin(x - \pi)]^2 + 1 = \frac{2\sin^2 x}{\sec^2 x - 1}$

878. $(\sin x+\cos x)^2=2\sin\left(\frac{\pi}{4}+x\right)\sin\left(\frac{\pi}{4}-x\right)$

879. $2-\sin x\cos 2x-\sin 2x\cos x=$
$$=\left[\cos\left(\frac{\pi}{4}-\frac{3x}{2}\right)-\sin\left(\frac{\pi}{4}-\frac{3x}{2}\right)\right]^2$$

880. $(1-\tan x)(1+\sin 2x)=1+\tan x$

881. $\cos x+\sin x=\frac{\cos 2x}{1-\sin 2x}$

882. $(1+\sin 2x)(\cos x-\sin x)=1-2\sin^2 x$

883. $\frac{\cos^2 x-\sin^2 2x}{4\cos^2 x}=\sin(x+30^\circ)\sin(x-30^\circ)$

884. $\frac{\sin(60^\circ+x)+\sin(60^\circ-x)}{2}=\frac{\tan x}{(1+\tan^2 x)^2}+\frac{\cot x}{(1+\cot^2 x)^2}$

885. $\sec^2 x-\left(\cos x+\sin x\tan\frac{x}{2}\right)=\frac{\sin(x-30^\circ)+\cos(60^\circ-x)}{\cos x}$

886. $\sin\left(\frac{\pi}{4}+x\right)-\sin\left(\frac{\pi}{4}-x\right)=\frac{\tan\frac{x}{2}+\cot\frac{x}{2}}{2\sqrt{2}}$

887. $2\sqrt{2}\sin(45^\circ+x)=\frac{1+\cos 2x}{1+\sin x}$

888. $1-\frac{2(\sin 2x-\cos 2x\tan x)}{\sqrt{3}\sec^2 x}=\cos^4 x-\sin^4 x$

889. $\sin 3x=4\sin x\cos 2x$

890. $\sec x+1=\sin(\pi-x)-\cos x\tan\frac{\pi+x}{2}$

891. $\frac{\tan 2x\tan x}{\tan 2x-\tan x}-2\sin(45^\circ+x)\sin(45^\circ-x)=0$

892. $\tan(x-45^\circ)\tan x\tan(x+45^\circ)=\frac{4\cos^2 x}{\tan\frac{x}{2}-\cot\frac{x}{2}}$

893. $\frac{\tan(x+45^\circ)+\tan(x-45^\circ)}{2}=\tan(x-45^\circ)\tan(x+45^\circ)\tan x$

894. $\tan(x+\alpha)+\tan(x-\alpha)=2\cot x$

895. $\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2=\frac{2}{\tan\frac{x}{2}-\tan\frac{x+\pi}{2}}$

896. $\frac{\sin x}{\sin(30^\circ+x)+\sin(30^\circ-x)}=1+\tan\left(\frac{x}{2}+45^\circ\right)-\tan\left(45^\circ-\frac{x}{2}\right)$

897. $\sin^4 x+\sin^4\left(x+\frac{\pi}{4}\right)=\frac{1}{4}$

897a. $\sin^4 x+\sin^4\left(x+\frac{\pi}{4}\right)+\sin^4\left(x-\frac{\pi}{4}\right)=\frac{9}{8}$

Solve the following systems of equations:

898. $\cos\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{1}{2}$; $\cos x\cos y=\frac{1}{4}$

899. $x+y=\alpha$; $\sin x\sin y=m$; **900.** $x+y=\alpha$; $\tan x+\tan y=m$

901. $x+y=\frac{\pi}{4}$; $\tan x+\tan y=1$

902. $2^{\sin x+\cos y}=1$; $16^{\sin^2 x+\cos^2 y}=4$

903. $\sin x\sin y=\frac{1}{4\sqrt{2}}$; $\tan x\tan y=\frac{1}{3}$

904. $\sin x=2\sin y$; $\cos x=\frac{1}{2}\cos y$

CHAPTER XIII

INVERSE TRIGONOMETRIC FUNCTIONS

905. Compute

$$2\arcsin\left(-\frac{\sqrt{3}}{2}\right)+\operatorname{arccot}(-1)+\arccos\frac{1}{\sqrt{2}}+\frac{1}{2}\arccos(-1)$$

906. Prove that

$$\tan(\arccos x)=\frac{\sqrt{1-x^2}}{x}$$

907. Prove that

$$\tan(\arcsin x)=\frac{x}{\sqrt{1-x^2}}$$

Compute:

908. $\sin\left[\frac{1}{2}\operatorname{arccot}\left(-\frac{3}{4}\right)\right]$; **909.** $\sin\left[\frac{1}{2}\arcsin\left(-\frac{2\sqrt{2}}{3}\right)\right]$

910. $\cot\left[\frac{1}{2}\arccos\left(-\frac{4}{7}\right)\right]$; **911.** $\tan\left(5\arctan\frac{\sqrt{3}}{3}-\frac{1}{4}\arcsin\frac{\sqrt{3}}{2}\right)$

912. $\sin\left(3\arctan\sqrt{3}+2\arccos\frac{1}{2}\right)$

913. $\cos\left[3\arcsin\frac{\sqrt{3}}{2}+\arccos\left(-\frac{1}{2}\right)\right]$

Prove the following identities:

914. $\arctan(3+2\sqrt{2})-\arctan\frac{\sqrt{2}}{2}=\frac{\pi}{4}$

915. $\arccos\sqrt{\frac{2}{3}}-\arccos\frac{\sqrt{6}+1}{2\sqrt{3}}=\frac{\pi}{6}$

916. $\arcsin\frac{4}{5}+\arcsin\frac{5}{13}+\arcsin\frac{16}{65}=\frac{\pi}{2}$

917. $\arccos\frac{1}{2}+\arccos\left(-\frac{1}{7}\right)=\arccos\left(-\frac{13}{14}\right)$

918. $2\arctan\frac{1}{5}+\arctan\frac{1}{4}=\arctan\frac{32}{43}$

919. $\arctan\frac{1}{3}+\arctan\frac{1}{5}+\arctan\frac{1}{7}+\arctan\frac{1}{8}=\frac{\pi}{4}$

Solve the following equations:

920. $4\arctan(x^2-3x-3)-\pi=0$

921. $6\arcsin(x^2-6x+8.5)=\pi$

922. $\arctan(x+2)-\arctan(x+1)=\frac{\pi}{4}$

923. $2\arctan\frac{1}{2}-\arctan x=\frac{\pi}{4}$

924. $\arcsin\frac{2}{3\sqrt{x}}-\arcsin\sqrt{1-x}=\arcsin\frac{1}{3}$

925. $\arctan\frac{a}{b}-\arctan\frac{a-b}{a+b}=\arctan x$

926. $\arcsin 3x=\arccos 4x$; **927.** $2\arcsin x=\arcsin\frac{10x}{13}$

928. Solve the system of equations

$$x+y=\arctan\frac{2a}{1-a^2},\quad \tan x\tan y=a^2\quad (|a|<1)$$

ANSWERS AND SOLUTIONS

PART ONE

ARITHMETIC AND ALGEBRA

CHAPTER I

ARITHMETIC CALCULATIONS

1. 6.5625
2. $29\frac{7}{12}$
3. $365\frac{5}{8}$
4. $3\frac{4}{15}$
5. $18\frac{1}{3}$
6. 50
7. 23.865
8. $36\frac{25}{72}$
9. 599.3
10. 84.075
11. 2.5
12. $2\frac{17}{21}$
13. 0.0115
14. $\frac{157}{280}$
15. $38\frac{15}{64}$
16. 6
17. 700
18. 100
19. 10
20. $7\frac{1}{2}$
21. 5
22. 3
23. $2\frac{3}{80}$
24. 5
25. $1\frac{17}{84}$
26. 10
27. 1
28. 1320
29. 11
30. 250
31. 4
32. 4000
33. 66
34. 2
35. 9.5
36. 0.09
37. $\frac{35}{48}$
38. 2
39. $-\frac{1}{16}$
40. $2\frac{1}{3}$
41. $\frac{1}{8}$
42. 1301
43. -20.384
44. 2.25
45. $1\frac{1}{8}$

CHAPTER II

ALGEBRAIC TRANSFORMATIONS

Preliminaries

In solving problems of the present chapter (beginning with Problem 62) the following should be taken into consideration.

1. The radical $\sqrt[n]{a}$ is called principal, or arithmetical, if the radicand a is positive (or equal to zero) and if, furthermore, the root itself is taken positive.

Examples. The expression $\sqrt[3]{-27}$ cannot represent an arithmetical root, since the radicand is negative. The expression $\sqrt[4]{16}$ is an arithmetical root, if we consider only its positive value (i.e. 2). The expression $\sqrt[3]{27}$ represents an arithmetical root (i.e. 3), if we consider only its real value $\left(\text{it also has two imaginary values } -\frac{3}{2}+\frac{3\sqrt{3}}{2}i \text{ and } -\frac{3}{2}-\frac{3\sqrt{3}}{2}i\right)$. The expression $\sqrt{-16}$ cannot represent an arithmetical root, since the radicand is a negative number.

2. The rules for transformation of radicals set forth in algebra are true only for arithmetical roots.

For instance, the equality $\sqrt[3]{x}=\sqrt[6]{x^2}$ is not true for negative values of x. Thus, at $x=-8$ the left member of the equality has only one real value $\sqrt[3]{-8}=-2$, while the right member has two real values $\sqrt[6]{64}=\pm 2$ (if imaginary values are also considered, then $\sqrt[3]{-8}$ has three values, and $\sqrt[6]{64}$—six values).

In view of this, in the present section dedicated to the identity transformations of irrational expressions we assume that all radicands may have only positive (and zero) values*, which means that literal quantities entering the expressions to be simplified should meet some additional conditions. In a number of cases (see, for instance, notes to Problems 65 to 71) we indicate such conditions. Sometimes the conditions to be satisfied by literal quantities are given in the problem itself. Then in solving such a problem one is to prove that under these conditions all radicands are positive.

3. It should be particularly noted that the equality $\sqrt{x^2}=x$ (where $\sqrt{x^2}$ is an arithmetical root) holds true only for $x \geqslant 0$. For a negative x it is invalid; instead we have the equality $\sqrt{x^2}=-x$. Both cases may be covered by the equality $\sqrt{x^2}=|x|$. Thus, if $x=-3$, then $\sqrt{(-3)^2}=\sqrt{9}=-(-3)$ (and $\sqrt{(-3)^2}$ is an arithmetical root, since the radicand $(-3)^2$ is positive and the root is taken in its positive value). We may also write $\sqrt{(-3)^2}=|3|$. The importance of this note is seen from the following examples.

Example 1. Simplify the expression $\sqrt{m^2-2mn+n^2}$. The solution

$$\sqrt{m^2-2mn+n^2}=\sqrt{(m-n)^2}=m-n$$

is true only for $m \geqslant n$. For $m<n$ it should be written instead

$$\sqrt{m^2-2mn+n^2}=-(m-n), \text{ i. e. } \sqrt{m^2-2mn+n^2}=n-m$$

---

* With the exception of Problem 64, where the radicand of the cube radical can never be positive (see the solution of this problem on pages 96-97).

Thus, if $m=2$ and $n=3$, then $m-n=-1$, whereas

$$\sqrt{m^2-2mn+n^2}=\sqrt{4-12+9}=\sqrt{1}=1$$

A general formula has the form

$$\sqrt{m^2-2mn+n^2}=|m-n| \text{ or } \sqrt{m^2-2mn+n^2}=|n-m|.$$

Example 2. Simplify the expression

$$\frac{\sqrt{4+4p+p^2}-\sqrt{4-4p+p^2}}{\sqrt{4+4p+p^2}+\sqrt{4-4p+p^2}}$$

Denoting (for the sake of brevity) the given expression by A, we have (for $p\neq-2$):

$$A=\frac{|2+p|-|2-p|}{|2+p|+|2-p|}=\frac{1-\left|\frac{2-p}{2+p}\right|}{1+\left|\frac{2-p}{2+p}\right|}$$

If the fraction $\frac{2-p}{2+p}$ is positive, then

$$A=\frac{1-\frac{2-p}{2+p}}{1+\frac{2-p}{2+p}}=\frac{p}{2}$$

and if it is negative, then

$$A=\frac{1+\frac{2-p}{2+p}}{1-\frac{2-p}{2+p}}=\frac{2}{p}$$

Let us analyse what values of p yield the first or/and the second cases. The fraction $\frac{2-p}{2+p}$ is positive when $2-p$ and $2+p$ are of the same sign. Let us first require that both quantities $2-p$ and $2+p$ be positive. The quantity $2-p$ is positive for $p<2$, the quantity $2+p$ is positive for $p>-2$. Consequently, both quantities are positive for $-2<p<2$. Now we require that both quantities $2-p$ and $2+p$ be negative, but soon find out that this requirement cannot be fulfilled, since $2-p$ is negative for $p>2$, and $2+p$ is negative for $p<-2$, but these conditions are incompatible. Hence, the fraction $\frac{2-p}{2+p}$ is positive only for $-2<p<2$. For $p>2$ and $p<-2$ this fraction is negative.

Thus, $A=\frac{p}{2}$ for $|p|<2$ and $A=\frac{2}{p}$ for $|p|>2$. At $|p|=2$ both expressions are valid.

Example 3. The equality $\sqrt{a^6}=a^3$ is true only for $a\geqslant 0$. For negative values of a we have instead of it the equality $\sqrt{a^6}=-a^3$. Thus, at $a=-1$

we have $\sqrt{(-1)^6} = -(-1) = +1$. Here $\sqrt{(-1)^6}$ is an arithmetical root, since the radicand $(-1)^6 = 1$ is positive and we take the positive value of the root.

Example 4. Take the factors outside the radical sign in the expression $\sqrt{(a-5)^6 (a-3)^3}$.

The given root may be arithmetical only for $a \geqslant 3$, since for $a < 3$ the factor $(a-3)^3$, and, hence, the whole radicand are negative. The equality

$$\sqrt{(a-5)^6 (a-3)^3} = (a-5)^3 (a-3) \sqrt{a-3}$$

holds true for $a \geqslant 5$. For $a < 5$ the following should be written instead

$$\sqrt{(a-5)^6 (a-3)^3} = -(a-5)^3 (a-3) \sqrt{a-3}$$

A general formula is

$$\sqrt{(a-5)^6 (a-3)^3} = |a-5|^3 (a-3) \sqrt{a-3} \quad (\text{for } a \geqslant 3).$$

4. In general, the equality $\sqrt[n]{x^n} = x$ (where the left member denotes an arithmetical root) holds true only for positive values of x (and at $x = 0$). If n is an even number, then at a negative x instead of $\sqrt[n]{x^n} = x$ we have the equality $\sqrt[n]{x^n} = -x$. If n is an odd number, then at negative values of x there exists no arithmetical root at all.

46. Grouping the last three terms of the expression in parentheses, factor it

$$a^2 - b^2 - c^2 + 2bc = a^2 - (b-c)^2 = (a+b-c)(a-b+c)$$

The given expression takes the form

$$(a+c+b)(a+c-b) = (a+c)^2 - b^2$$

Answer: $(a+c)^2 - b^2$; $139\frac{91}{225}$.

47. The expression in parentheses is equal to $\frac{1}{n-1}$. It is convenient to reverse all the signs both in the numerator and denominator of the last fraction, and then to factor the numerator; the fraction takes the form

$$\frac{(a+n)(n-1)(n^2+n+1)}{a^2-1}$$

Answer: $\frac{n^2+n+1}{n}$.

48. The denominator of the second fraction is equal to $(1+x)(x-2a)$. The parenthesized expression is equal to $1+x$. The given expression is equal to

$$\frac{x}{a(x-2a)} - \frac{2}{x-2a} = \frac{1}{a}$$

Answer: $\frac{1}{a}$.

49. *Answer*: $\frac{1}{a+2x}$.

50. Represent the second addend in the form $\frac{a-130}{3a-1}$. Reduce the parenthesized fractions to a common denominator; this yields $\frac{-3(2a^2+9a+10)}{a(3a-1)}$; equating the trinomial $2a^2+9a+10$ to zero and finding the roots $a_1=-2$; $a_2=-\frac{5}{2}$, factor it

$$2a^2+9a+10=2(a+2)\left(a+\frac{5}{2}\right)$$

Now the expression in parentheses takes the form

$$\frac{-3(a+2)(2a+5)}{a(3a-1)}$$

Multiplying it by

$$\frac{3a^3+8a^2-3a}{1-\frac{1}{4}a^2}=\frac{4a(a+3)(3a-1)}{(2+a)(2-a)}$$

Answer: $\frac{12(2a+5)(a+3)}{a-2}$.

51. Reduce each fraction, factoring both the numerator and denominator.

Answer: $\frac{ab}{a+b}$.

52. Factor the denominator of the second fraction and reduce the latter

$$\frac{x}{x^2+y^2}-\frac{y(x-y)}{(x^2+y^2)(x+y)}=\frac{1}{x+y}$$

Answer: $\frac{1}{x+y}$.

53. After simplification the denominators of the fractions take the form

$$\frac{4(x^2+x+1)}{3} \quad \text{and} \quad \frac{4(x^2-x+1)}{3}$$

The given expression is transformed in the following way

$$\frac{2}{3}\cdot\frac{3}{4}\left(\frac{1}{x^2+x+1}+\frac{1}{x^2-x+1}\right)=\frac{x^2+1}{(x^2+1)^2-x^2}=\frac{x^2+1}{x^4+x^2+1}$$

Answer: $\frac{x^2+1}{x^4+x^2+1}$.

54. Factor the denominators of the first four fractions, reduce the first fraction by $a-1$. The expression in parentheses takes the form

$$\frac{1}{a-1}+\frac{2(a-1)}{(a+2)(a-2)}-\frac{4(a+1)}{(a-1)(a+2)}+\frac{a}{(a-1)(a-2)}=$$

$$=\frac{2(a+3)}{(a-1)(a+2)(a-2)}$$

This should be multiplied by the fraction $\frac{36a^3-144a-36a^2+144}{a^3+27}$. Factor the numerator of the last fraction by grouping the terms, and the denominator, as the sum of cubes a^3+3^3; then this fraction takes the form

$$\frac{36(a-1)(a+2)(a-2)}{(a+3)(a^2-3a+9)}$$

Answer: $\frac{72}{a^2-3a+9}$.

55. Let us denote the dividend (the sum of fractions) by A and the divisor, by B. Factoring the polynomials which enter A, we find

$$A=\frac{3(x+2)}{2(x+1)(x^2+1)}+\frac{(x+2)(2x-5)}{2(x-1)(x^2+1)}$$

Taking $\frac{x+2}{2(x^2+1)}$ outside parentheses, we have

$$A=\frac{x+2}{2(x^2+1)}\cdot\left(\frac{3}{x+1}+\frac{2x-5}{x-1}\right)=\frac{(x+2)(x^2-4)}{(x^2+1)(x+1)(x-1)}$$

Then we find

$$B=\frac{2(x^2-4)}{(x^2+1)(x+1)(x-1)}$$

Dividing A by B, we get $\frac{x+2}{2}$.

Answer: $\frac{x+2}{2}$.

56. Let A denote the dividend and B, the divisor. Equating the trinomial $x^2-xy-2y^2$ entering the expression A to zero, we solve the obtained equation for one of the unknowns, say, for the unknown x; on finding $x_1=-y$ and $x_2=2y$, we get the following factorization of the trinomial: $x^2-xy-2y^2=(x+y)(x-2y)$. Now we have

$$A=\frac{x-y}{2y-x}-\frac{x^2+y^2+y-2}{(x+y)(x-2y)}$$

Write in the subtrahend $2y-x$ instead of $x-2y$, simultaneously reversing the signs in the numerator of this fraction. Reducing the fractions to a common denominator, we get

$$A=\frac{2x^2+y-2}{(2y-x)(x+y)}$$

In the expression B factor the numerator by representing it in the form $(2x^2+y)^2-2^2$, and the denominator, by grouping x^2+xy and $y+x$. Then

$$B=\frac{(2x^2+y+2)(2x^2+y-2)}{(x+y)(x+1)}$$

Dividing A by B, we get $\frac{x+1}{(2y-x)(2x^2+y+2)}$.

Answer: $\dfrac{x+1}{(2y-x)(2x^2+y+2)}$.

57. Factoring the polynomials contained in the given expression, we obtain

$$\frac{(a+2)(a-1)}{a^n(a-3)}\cdot\left[\frac{4(a+1)}{4(a+1)(a-1)}-\frac{3}{a(a-1)}\right]$$

Answer: $\dfrac{a+2}{a^{n+1}}$.

58. Let A denote the dividend and B, the divisor. The numerator of the fraction A is

$$\frac{1}{2}[4a^2(b+c)^{2n}-1]=\frac{1}{2}[2a(b+c)^n+1][2a(b+c)^n-1]$$

and the denominator

$$a(n^2-a^2-2a-1)=a[n^2-(a+1)^2]=a(n+a+1)(n-a-1)$$

Leave the numerator of the fraction B unchanged and write its denominator in the form $-ac(n-a-1)$.

Answer: $-\dfrac{[2a(b+c)^n+1]c}{2(n+a+1)}$.

59. *First method*. Reduce all the fractions to a common denominator:

$$\frac{bc(b-c)-ac(a-c)+ab(a-b)}{abc(a-b)(a-c)(b-c)} \tag{a}$$

Having multiplied the binomials in the denominator, we get $a^2b-ab^2+b^2c-a^2c+ac^2-bc^2$, i.e. the same expression as in the numerator. After reduction we obtain $\dfrac{1}{abc}$.

Second method. Putting in the numerator of the fraction (a) $a=b$, we find that in this case the numerator vanishes. Consequently, according to Bézout's theorem, it is divisible by $(a-b)$. The quotient is

$$a(b-c)-c(b-c)=(b-c)(a-c)$$

Thus, the numerator is equal to $(a-b)(b-c)(a-c)$.

Third method. Let us reduce to a common denominator only the first two fractions of the given expression. We get

$$\frac{b^2-bc-a^2+ac}{ab(a-b)(a-c)(b-c)}$$

Grouping the terms in the numerator (the first one with the third and the second with the fourth), we arrive at the expression

$$(b+a)(b-a)-c(b-a)=(a-b)(c-a-b).$$

Now reduce the fraction by $(a-b)$ and add the third fraction of the given expression.

Answer: $\dfrac{1}{abc}$.

60. The first factor is equal to $\frac{a+x+1}{a+x-1}$. The expression in brackets is equal to $\frac{(a+x)^2-1}{2ax}=\frac{(a+x+1)(a+x-1)}{2ax}$. Multiplying the given expressions, we find $\frac{(a+x+1)^2}{2ax}$. Substitute $x=\frac{1}{a-1}$, the numerator takes the form $\frac{a^4}{(a-1)^2}$, the denominator becoming equal to $\frac{2a}{a-1}$.

Answer: $\frac{a^3}{2(a-1)}$.

61. Let us denote the expression in brackets by A and the expression in parentheses, by B. We have $A:B^{-1}=AB$. Getting rid of the powers with negative exponents in the expression A, we have

$$A=\frac{2b^2-3ab-2a^2}{a(a+2b)(2b-a)}=\frac{(b-2a)(2b+a)}{a(a+2b)(2b-a)}=\frac{b-2a}{a(2b-a)}$$

Transforming B, we get

$$B=a^n\left(2b+3a-\frac{6a^2}{2a-b}\right)=a^n\cdot\frac{b(a-2b)}{2a-b}$$

Finally, we find $AB=a^{n-1}b$ (reverse the signs of the terms both in the numerator and denominator in one of the fractions to be multiplied).

Answer: $a^{n-1}b$.

62. The numerator is transformed to the form a^2-b^2, the denominator, to $a+b$.

Answer: $a-b$.

Note. For the roots to be arithmetical ones, the numbers a and b must not be negative.

63. The first radical is equal to

$$\sqrt[3]{(a-b)^3(a+b)^2}=(a-b)\sqrt[3]{(a+b)^2}$$

Answer: $b(a^3-b^3)$.

Note. It is assumed that $a>b$ (otherwise the first root will not be arithmetical).

64. This is an exception from the rule considered on page 90 which states that radicands may have only positive values. The thing is that the radicand of the cube radical is always negative. Indeed, we must consider the expressions $\sqrt{6x}$ and $\sqrt{2x}$ (which have real values only for $x\geqslant 0$) to be positive (otherwise the expression $2\sqrt{6x}-4\sqrt{2x}$ loses its uniqueness). But then the difference $2\sqrt{6x}-4\sqrt{2x}=\sqrt{24x}-\sqrt{32x}$ is negative.

And so, we admit that the radicand of the cube radical is a negative number. Then the cube root itself has a negative value. Prior to applying the rule for transforming radicals, we have to accomplish such a transformation:

$$\sqrt[3]{2\sqrt{6x}-4\sqrt{2x}}=-\sqrt[3]{4\sqrt{2x}-2\sqrt{6x}}$$

Now the radical on the right is an arithmetical root. After reduction to the same index as that of the first of the given factors we obtain

$$-\sqrt[3]{4\sqrt{2x}-2\sqrt{6x}}=-\sqrt[6]{(4\sqrt{2x}-2\sqrt{6x})^2}=-\sqrt[6]{8x(7-4\sqrt{3})}$$

Multiplying the roots, we get: $-\sqrt[6]{64x^2\,[49-(4\sqrt{3})^2]} = -2\sqrt[3]{x}$.

Answer: $-2\sqrt[3]{x}$.

Note. If no attention is paid to the fact that the radicand of the cube root is negative, one obtains the wrong answer $2\sqrt[3]{x}$.

65. The first radical is equal to $\sqrt[4]{(a+1)^4\,(a-1)}$. Taking the factor $(a+1)$ outside the radical sign, we get $|a+1|\sqrt[4]{a-1}$. The given expression is equal to

$$\frac{a}{2}\,|a+1|\,\sqrt[4]{a-1}\cdot\frac{\sqrt{a-1}}{(a+1)\,(a+2)}$$

Bring the radicals to a common index:

$$\frac{a}{2}\,\frac{|a+1|\,\sqrt[4]{(a-1)^3}}{(a+1)\,(a+2)}.$$

If the number $a+1$ is positive, then $|a+1| = a+1$, and on reducing, we get $\frac{a}{2}\,\frac{\sqrt[4]{(a-1)^3}}{a+2}$.

Note. The number $a+1$ is just positive. Indeed, since the radicand $(a+1)^4\,(a-1)$ is assumed to be positive (or equal to zero), and the factor $(a+1)^4$ is nonnegative in all cases, then $a-1 \geqslant 0$, i.e. $a \geqslant 1$, and under this condition $a+1 \geqslant 2$.

Answer: $\frac{a}{2}\,\frac{\sqrt[4]{(a-1)^3}}{a+2}$.

66. Assuming all the roots to be arithmetical, bring the factors

$$\sqrt{\frac{(1+a)\sqrt[3]{1+a}}{3a}} \quad\text{and}\quad \sqrt[3]{\frac{\sqrt{3}}{9+18a^{-1}+9a^{-2}}} = \sqrt[3]{\frac{\sqrt{3}\,a^2}{9\,(1+a)^2}}$$

to a common index 6. The first and second factors take the respective form

$$\sqrt[6]{\frac{(1+a)^3\,(1+a)}{27a^3}}\,; \quad \sqrt[6]{\frac{3a^4}{81\,(1+a)^4}}$$

Multiplying them, we get $\frac{1}{3}\sqrt[6]{a}$.

Note. The first factor is an arithmetical root only if $a > 0$ (if $a < 0$, the radicand is negative, at $a = 0$ it loses its sense). The second factor is an arithmetical root at any a (except for $a = -1$). Consequently, the quantity a may have any positive value.

Answer: $\frac{1}{3}\sqrt[6]{a}$.

67. Place ab under the first radical sign. The given expression takes the form

$$\sqrt[n]{a-b}\,\frac{1}{\sqrt[n]{a-b}} = 1$$

Note. For the given radicals to be arithmetical roots the following condition must be observed: $a \geqslant b$. The case $a = b$ is excluded, since the second factor loses its sense.

Answer: 1.

68. Rationalizing the denominators, we get

$$(\sqrt{6}-11)(\sqrt{6}+11)=-115$$

Answer: -115.

69. The dividend is equal to $\frac{\sqrt{a-b}+\sqrt{a+b}}{b}$; the divisor, to $\frac{\sqrt{a-b}+\sqrt{a+b}}{\sqrt{a-b}}$; the quotient, to $\frac{\sqrt{a-b}}{b}$.

Note. For all the given roots to be arithmetical the following three conditions must be simultaneously satisfied: $a \geqslant 0$, $a-b \geqslant 0$, $a+b \geqslant 0$ (they may be replaced by the following two conditions: $a \geqslant 0$, $|b| \leqslant |a|$).

Answer: $\frac{\sqrt{a-b}}{b}$.

70. The dividend is equal to $\frac{2b}{b^2-a}$; the divisor, to $\frac{3b}{b^2-a}$; the quotient, to $\frac{2}{3}$. The quantity a may have any positive value; b may have any value, except for $\pm\sqrt{a}$.

Answer: $\frac{2}{3}$.

71. The numerator of the first fraction is reduced to the form

$$\frac{(\sqrt{1+a})^2+(\sqrt{1-a})^2}{\sqrt{1-a^2}}=\frac{|1+a|+|1-a|}{\sqrt{1-a^2}}$$

If the expressions $1+a$ and $1-a$ are positive, then (see Preliminaries on page 90, Item 3) the numerator is equal to $\frac{2}{\sqrt{1-a^2}}$. Under the same condition the denominator is equal to $\frac{|1+a|-|1-a|}{\sqrt{1-a^2}}=\frac{2a}{\sqrt{1-a^2}}$; the fraction is equal to $\frac{1}{a}$, and the given expression, to 0.

Note. For the radicals contained in the given expression to be arithmetical roots it is necessary that the quantities $1+a$ and $1-a$ be of the same sign. But it is impossible that both of them are negative, since $1+a<0$ if $a<-1$ and $1-a<0$ if $a>1$, but these conditions are incompatible. For the quantities $1+a$ and $1-a$ to be simultaneously positive the following condition should be fulfilled: $-1<a<1$, i.e. $|a|<1$ (the values $a=\pm1$ are excluded, since at each of them one of the given expressions $\frac{1+a}{1-a}$, $\frac{1-a}{1+a}$ loses its sense; the value $a=0$ is also excluded, since the fraction $\frac{1}{a}$ loses its sense).

Answer: 0.

72. Substituting $x=\frac{1}{2}\left(a+\frac{1}{a}\right)$ into the expression $\sqrt{x^2-1}$, we get

$$\sqrt{x^2-1}=\sqrt{\frac{1}{4}\left(a+\frac{1}{a}\right)^2-1}=\sqrt{\frac{1}{4}\left(a-\frac{1}{a}\right)^2}=\frac{1}{2}\left|a-\frac{1}{a}\right|$$

Since, by hypothesis, $a\geqslant 1$, then $a-\frac{1}{a}\geqslant 0$. Therefore

$$\sqrt{x^2-1}=\frac{1}{2}\left(a-\frac{1}{a}\right)$$

Similarly we find

$$\sqrt{y^2-1}=\frac{1}{2}\left(b-\frac{1}{b}\right)$$

Substitute the values of the radicals found into the given expression.

Answer: $\frac{a^2+b^2}{a^2b^2+1}$.

73. Substituting $x=\frac{2am}{b(1+m^2)}$ into the expressions $\sqrt{a+bx}$ and $\sqrt{a-bx}$, we find

$$\sqrt{a+bx}=\sqrt{a+\frac{2am}{1+m^2}}=|1+m|\sqrt{\frac{a}{1+m^2}}$$

and

$$\sqrt{a-bx}=|1-m|\sqrt{\frac{a}{1+m^2}}$$

Since $1+m^2$ is always positive, then a must also be positive (at $a<0$ both roots are imaginary; at $a=0$ they are equal to zero and the given expression is indeterminate). Since, according to the additional condition, $|m|<1$, then both $1+m$ and $1-m$ are positive.

The given expression takes the form

$$\frac{(1+m)\sqrt{\frac{a}{1+m^2}}+(1-m)\sqrt{\frac{a}{1+m^2}}}{(1+m)\sqrt{\frac{a}{1+m^2}}-(1-m)\sqrt{\frac{a}{1+m^2}}}=\frac{1}{m}$$

Answer: $\frac{1}{m}$ (for $a>0$).

74. The problem is similar to the previous one. We have

$$(m-x)^{\frac{1}{2}}=\left(m-\frac{2mn}{n^2+1)}\right)^{\frac{1}{2}}=m^{\frac{1}{2}}\frac{\sqrt{(n-1)^2}}{\sqrt{n^2+1}}=m^{\frac{1}{2}}\frac{|n-1|}{\sqrt{n^2+1}}$$

Since, by hypothesis, $n<1$, then

$$(m-x)^{\frac{1}{2}}=\frac{m^{\frac{1}{2}}(1-n)}{\sqrt{n^2+1}}$$

By analogy,

$$(m+x)^{\frac{1}{2}}=\frac{m^{\frac{1}{2}}(1+n)}{\sqrt{n^2+1}}$$

Answer: $\frac{1}{n}$.

75. Substituting $x=\frac{2\sqrt{k}}{1+k}$ into the expression $1-x^2$, we get

$$1-x^2=\frac{(1+k)^2-4k}{(1+k)^2}=\frac{(1-k)^2}{(1+k)^2}$$

Now we find

$$(1-x^2)^{-\frac{1}{2}}=1:\sqrt{1-x^2}=\frac{|1+k|}{|1-k|}$$

Since, by the additional condition, $k>1$, then the quantity $1+k$ is positive and $1-k$, negative. Therefore $(1-x^2)^{-\frac{1}{2}}=\frac{1+k}{k-1}$. Inside the first brackets we get $\frac{k}{k-1}$, inside the second, $\frac{1}{k-1}$. The given expression is equal to

$$\left(\frac{k}{k-1}\right)^{-\frac{1}{2}}+\left(\frac{1}{k-1}\right)^{-\frac{1}{2}}=\frac{\sqrt{k-1}}{\sqrt{k}}+\sqrt{k-1}$$

Answer: $\sqrt{k-1}\left(1+\frac{1}{\sqrt{k}}\right)$.

76. The expression in the first parentheses is equal to $\frac{1}{2}-\frac{a}{4}-\frac{1}{4a}$ (the exponent -2 refers only to the numerator of the third addend!). Simplifications yield $\frac{-(a-1)^2}{4a}$ or $\frac{-(1-a)^2}{4a}$.

The expressions

$$\sqrt[3]{(a+1)^{-3}}=\frac{1}{a+1} \quad \text{and} \quad (a+1)^{\frac{3}{2}}=\sqrt{(a+1)^3}$$

will be arithmetical roots only if $a>-1$. At this condition the radical

$$\sqrt{(a^2-1)(a-1)}=\sqrt{(a-1)^2(a+1)}$$

will also be an arithmetical root (since the factor $(a-1)^2$ cannot be negative). The equality

$$\sqrt{(a-1)^2(a+1)}=(a-1)\sqrt{a+1}$$

is true only for $a\geqslant 1$. If $a<1$, then

$$\sqrt{(a-1)^2(a+1)}=-(a-1)\sqrt{a+1}$$

(see Preliminaries, Item 3, page 90).

The given expression is equal to

$$-\frac{(a-1)^2}{4a}\cdot\left[\frac{a-1}{a+1}-\frac{a+1}{|a-1|}\right]$$

Note. At $a=\pm 1$ the expression loses its sense.

Answer: $\frac{a-1}{a+1}$ for $a>1$; $\frac{(a^2+1)(1-a)}{2a(a+1)}$ for $-1<a<1$, i.e. for $|a|<1$.

77. The given expression may be represented in the form

$$2\left[\sqrt{x^2(x^2-a^2)}-a^2\sqrt{\frac{x^2}{x^2-a^2}}\right]\cdot\frac{\sqrt{x^2-a^2}}{2ax\sqrt{\left(\frac{x}{a}-2\frac{a}{x}\right)^2}}$$

It is assumed that $x^2-a^2>0$, i.e. $|x|>|a|$ (otherwise the root $\sqrt{x^2-a^2}$ will not be arithmetical, the case $|x|=|a|$ is excluded, since the second radicand loses its sense).

The first factor is reduced to the form

$$2|x|\frac{|x^2-a^2|-a^2}{\sqrt{x^2-a^2}}=2|x|\frac{x^2-2a^2}{\sqrt{x^2-a^2}}$$

(since $x^2-a^2>0$, then $|x^2-a^2|=x^2-a^2$).

The expression $\sqrt{\left(\frac{x}{a}-2\frac{a}{x}\right)^2}$ is transformed to the form

$$\sqrt{\left(\frac{x^2-2a^2}{ax}\right)^2}=\frac{|x^2-2a^2|}{|a||x|}$$

Here the numerator can be written in the form x^2-2a^2 only if $x^2-2a^2\geqslant 0$, i.e. if $|x|\geqslant|a|\sqrt{2}$.

Now the given expression is written in the form

$$2|x|\frac{x^2-2a^2}{\sqrt{x^2-a^2}}\cdot\frac{\sqrt{x^2-a^2}\,|a|\cdot|x|}{2ax\,|x^2-2a^2|}$$

Taking into consideration that $|x|\cdot|x|=|x|^2=x^2$ and reducing, we obtain $\frac{x^2-2a^2}{a}\cdot\frac{|a|}{|x^2-2a^2|}x$, or $\frac{x^2-2a^2}{a}\left|\frac{a}{x^2-2a^2}\right|x$, which is the same.

Answer: If $|x|>|a|$, the given expression is equal to $\pm x$; the plus sign is taken when $\frac{x^2-2a^2}{a}>0$, and the minus one, when $\frac{x^2-2a^2}{a}<0$.

If $\frac{x^2-2a^2}{a}=0$, i.e. if $|x|=|a|\sqrt{2}$, the given expression loses its sense.

78. Get rid of negative exponents. The numerator takes the form

$$\frac{2ab\sqrt{a}}{\sqrt{a}+\sqrt{b}}+\frac{2ab\sqrt{b}}{\sqrt{a}+\sqrt{b}}=2ab,$$

the denominator becomes

$$2ab\left(\frac{1}{a+\sqrt{ab}}+\frac{1}{b+\sqrt{ab}}\right)$$

Noting that

$$a+\sqrt{ab}=\sqrt{a}\,(\sqrt{a}+\sqrt{b}) \text{ and } b+\sqrt{ab}=\sqrt{b}\,(\sqrt{b}+\sqrt{a})$$

represent the denominator in the form

$$2ab\left(\frac{1}{\sqrt{a}\,(\sqrt{a}+\sqrt{b})}+\frac{1}{\sqrt{b}\,(\sqrt{a}+\sqrt{b})}\right)=\frac{2ab}{\sqrt{ab}}=2\sqrt{ab}$$

Now the given expression is equal to

$$\frac{2ab}{2\sqrt{ab}}=\sqrt{ab}$$

Answer: $\sqrt{ab}$.

79. The expression in the first parentheses is reduced to the form

$$\frac{2\sqrt{ax}}{\sqrt{a+x}\,(\sqrt{a}+\sqrt{x})}$$

Raising it to the power -2, we get

$$\frac{(a+x)\,(\sqrt{a}+\sqrt{x})^2}{4ax}$$

Similarly, the expression in the second parentheses is reduced to the form

$$\frac{(a+x)\,(\sqrt{a}-\sqrt{x})^2}{4ax}$$

When subtracting, factor out $\frac{a+x}{4ax}$ (simplifications yield $4\sqrt{ax}$ inside the parentheses).

Answer: $\frac{a+x}{\sqrt{ax}}$.

80. After simplification the last addend takes the form $\frac{a}{2\sqrt{x^2+a}}$. Reducing all the fractions to a common denominator and summing, we get

$$\frac{2\,(x^2+a)}{2\sqrt{x^2+a}}=\sqrt{x^2+a}$$

Answer: $\sqrt{x^2+a}$.

81. *Answer*: $2\,(x+\sqrt{x^2-1})$.

82. Inside the brackets we have $a^{-\frac{3}{2}}ba^{-\frac{1}{2}}ba^{\frac{2}{3}}=a^{-\frac{4}{3}}b^2$. The given expression is equal to $a^{-4}b^6$. Substitute into it

$$a=\frac{\sqrt{2}}{2} \text{ and } b=\frac{1}{\sqrt[3]{2}}$$

Answer: 1.

83. We represent the given expression in the form $\frac{1}{a+1}+\frac{1}{b+1}$, and make the following substitutions

$$a=\frac{1}{2+\sqrt{3}}=2-\sqrt{3} \text{ and } b=\frac{1}{2-\sqrt{3}} \quad 2+\sqrt{3}$$

We find $a+1=3-\sqrt{3}$, $\frac{1}{a+1}=\frac{3+\sqrt{3}}{6}$ and so on.

Answer: 1.

84. *Answer*: $\sqrt{x^2-4x}$.

85. *Answer*: n.

86. For all the roots to be arithmetical it is necessary that $x-a^2>0$. The expression in parentheses is reduced to the form $-\frac{4\sqrt{x}\sqrt{x-a^2}}{a^2}$. The given expression is equal to

$$\frac{\sqrt{x}}{\sqrt{x-a^2}}\cdot\left(-\frac{a^2}{4\sqrt{x}\sqrt{x-a^2}}\right)=-\frac{a^2}{4(x-a^2)}$$

Answer: $-\frac{a^2}{4(x-a^2)}$.

87. The denominator of the second fraction is equal to

$$x^{\frac{3}{2}}-1=(x^{\frac{1}{2}})^3-1=(x^{\frac{1}{2}}-1)(x+x^{\frac{1}{2}}+1)$$

Answer: $x-1$.

88. The dividend is equal to

$$2^{\frac{3}{2}}+27y^{\frac{3}{5}}=(2^{\frac{1}{2}})^3+(3y^{\frac{1}{5}})^3=(2^{\frac{1}{2}}+3y^{\frac{1}{5}})(2-3\cdot 2^{\frac{1}{2}}y^{\frac{1}{5}}+9y^{\frac{2}{5}});$$

the divisor is equal to $2^{\frac{1}{2}}+3y^{\frac{1}{5}}$.

Answer: $2-3\sqrt[10]{32y^2}+9\sqrt[5]{y^2}$.

89. Let us get rid of negative exponents in the second term by multiplying both the numerator and denominator by a^2.

We obtain in the numerator a^3-1, and in the denominator

$$a^2(a^{\frac{1}{2}}-a^{-\frac{1}{2}})=a^{\frac{3}{2}}[a^{\frac{1}{2}}(a^{\frac{1}{2}}-a^{-\frac{1}{2}})]=a^{\frac{3}{2}}(a-1).$$

On reducing we get $\frac{a^2+a+1}{a^{3/2}}$. Similarly, the third term is equal to $\frac{a-1}{a^{3/2}}$.

90. The dividend and divisor are respectively transformed to the form

$$\frac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{a^{\frac{2}{3}}(a-b)^{\frac{2}{3}}}; \quad \frac{(a-b)^{\frac{1}{3}}}{a^{\frac{2}{3}}(a^{\frac{3}{2}}-b)^{\frac{3}{2}}}$$

Taking into account that $(a^{\frac{3}{2}}+b^{\frac{3}{2}})(a^{\frac{3}{2}}-b^{\frac{3}{2}})=a^3-b^3$, we obtain the quotient a^2+ab+b^2. At $a=1.2$ and $b=\frac{3}{5}$ we get 2.52.

Answer: a^2+ab+b^2; 2.52.

91. Removing brackets and collecting like terms, we represent the dividend in the form $6a^{\frac{1}{2}}b^{\frac{1}{2}}+9b=3b^{\frac{1}{2}}(2a^{\frac{1}{2}}+3b^{\frac{1}{2}})$; and the divisor, $a^{\frac{1}{2}}(2a^{\frac{1}{2}}+3b^{\frac{1}{2}})$. The quotient is $3\sqrt{\frac{b}{a}}$; at the given values $a=54$ and $b=6$ it is equal to 1.

Answer: $3\sqrt{\frac{b}{a}}$; 1.

92. Multiplying both the numerator and denominator of the given fraction by

$$[(a+b)^{-\frac{1}{2}}+(a-b)^{-\frac{1}{2}}][(a+b)^{-\frac{1}{2}}-(a-b)^{-\frac{1}{2}}]$$

we get for the numerator

$$[(a+b)^{-\frac{1}{2}}-(a-b)^{-\frac{1}{2}}]+[(a+b)^{-\frac{1}{2}}+(a-b)^{-\frac{1}{2}}]=2(a+b)^{-\frac{1}{2}}$$

and for the denominator

$$[(a+b)^{-\frac{1}{2}}-(a-b)^{-\frac{1}{2}}]-[(a+b)^{-\frac{1}{2}}+(a-b)^{-\frac{1}{2}}]=-2(a-b)^{-\frac{1}{2}}$$

Answer: $-\sqrt{\frac{a-b}{a+b}}$.

93. The first factor of the subtrahend is reduced to the form $1-a^2$, then we have

$$a^2(1-a^2)^{-\frac{1}{2}}-\frac{(1-a^2)[(1-a^2)^{\frac{1}{2}}+a^2(1-a^2)^{-\frac{1}{2}}]}{1-a^2}$$

Reducing the fraction by $(1-a^2)$, we obtain

$$a^2(1-a^2)^{-\frac{1}{2}}-(1-a^2)^{\frac{1}{2}}-a^2(1-a^2)^{-\frac{1}{2}}=-(1-a^2)^{\frac{1}{2}}$$

Answer: $-\sqrt{1-a^2}$.

94. The given expression is equal to

$$\frac{x^3-1}{\sqrt{x}\,(x+1)\,(x^2+1)}-\frac{\sqrt{1+x^2}}{\sqrt{x}}\cdot\frac{-1}{(1+x^2)\,\sqrt{1+x^2}}=$$

$$=\frac{x^3-1}{\sqrt{x}\,(x+1)\,(x^2+1)}+\frac{1}{\sqrt{x}\,(1+x^2)}=\frac{\sqrt{x}}{x+1}$$

Answer: $\frac{\sqrt{x}}{x+1}$.

95. The numerator of the third term is reduced to the form $R^2\,(R^2-x^2)^{-\frac{1}{2}}$. The denominator is equal to R^2. The given expression takes the form

$$(R^2-x^2)^{\frac{1}{2}}-x^2\,(R^2-x^2)^{-\frac{1}{2}}+R^2\,(R^2-x^2)^{-\frac{1}{2}}=$$

$$=(R^2-x^2)^{\frac{1}{2}}+(R^2-x^2)^{-\frac{1}{2}}\,(R^2-x^2)=2\,(R^2-x^2)^{\frac{1}{2}}$$

Answer: $2\sqrt{R^2-x^2}$.

96. The first and second addends are reduced, respectively, to the form

$$\frac{p+q}{pq\,(p^{\frac{1}{2}}+q^{\frac{1}{2}})^2}\,;\quad\frac{2\,(p^{\frac{1}{2}}+q^{\frac{1}{2}})}{(p^{\frac{1}{2}}+q^{\frac{1}{2}})^3\,p^{\frac{1}{2}}q^{\frac{1}{2}}}=\frac{2}{(p^{\frac{1}{2}}+q^{\frac{1}{2}})^2\,p^{\frac{1}{2}}q^{\frac{1}{2}}}$$

Reducing these addends to a common denominator, we get

$$\frac{p+q+2p^{\frac{1}{2}}q^{\frac{1}{2}}}{pq\,(p^{\frac{1}{2}}+q^{\frac{1}{2}})^2}$$

The numerator of this expression is equal to $(p^{\frac{1}{2}}+q^{\frac{1}{2}})^2$.

Answer: $\frac{1}{pq}$.

97. Introduce here fractional exponents. Factor out $a^{\frac{2}{3}}$ in the expression $a+a^{\frac{2}{3}}x^{\frac{1}{3}}$, and $x^{\frac{2}{3}}$ in the expression $x+a^{\frac{1}{3}}x^{\frac{2}{3}}$. Then the numerator of the first fraction will be

$$\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}-1=\frac{a^{\frac{2}{3}}-x^{\frac{2}{3}}}{x^{\frac{2}{3}}}=\frac{(a^{\frac{1}{3}}+x^{\frac{1}{3}})\,(a^{\frac{1}{3}}-x^{\frac{1}{3}})}{x^{\frac{2}{3}}}$$

and thus the first fraction is reduced to the form $\frac{a^{\frac{1}{3}}+x^{\frac{1}{3}}}{x^{\frac{2}{3}}}$. The bracketed expression takes the form

$$\frac{a^{\frac{1}{3}}+x^{\frac{1}{3}}}{x^{\frac{2}{3}}}-\frac{1}{x^{\frac{1}{3}}}=\frac{a^{\frac{1}{3}}}{x^{\frac{2}{3}}}$$

Answer: $\frac{a^2}{x^4}$.

98. Represent the binomial $a-\sqrt{ax}$ in the form $\sqrt{a}(\sqrt{a}-\sqrt{x})$. The numerator of the fraction will be

$$(\sqrt{a}+1)^2-\sqrt{a}=a+\sqrt{a}+1$$

The denominator is equal to $3(a+\sqrt{a}+1)$.

Answer: 27.

99. Get rid of negative exponents; reduced by $2a-3$, the first addend takes the form $\frac{2a+3}{a^{1/2}}$; reduced by $a-1$, the second addend yields $\frac{a-3}{a^{1/2}}$.

Answer: $9a$.

100. Take $a-b$ outside the brackets in the first factor. The quantity $\frac{a+b}{a-b}$ cannot be negative (otherwise the roots are not arithmetical). The given expression then takes the form

$$(a-b)^2\left[\left(\sqrt{\frac{a+b}{a-b}}\right)^2-1\right]=(a-b)^2\left(\frac{a+b}{a-b}-1\right)$$

Answer: $2b(a-b)$.

101. Represent the dividend and divisor, respectively, in the form

$$\frac{a\sqrt{ab}}{a+\sqrt{ab}}=\frac{a\sqrt{b}}{\sqrt{a}+\sqrt{b}};\quad \frac{\sqrt[4]{b}(\sqrt[4]{a}-\sqrt[4]{b})}{a-b}$$

The quotient may be reduced by $(\sqrt{a}+\sqrt{b})(\sqrt[4]{a}-\sqrt[4]{b})$, taking into account that

$$a-b=(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=(\sqrt{a}+\sqrt{b})(\sqrt[4]{a}+\sqrt[4]{b})(\sqrt[4]{a}-\sqrt[4]{b})$$

Answer: $a\sqrt[4]{b}(\sqrt[4]{a}+\sqrt[4]{b})$.

102. Represent the given expression in the form

$$\left[\frac{(\sqrt{a})^3+(\sqrt{b})^3}{\sqrt{a}}:\frac{a-\sqrt{a}\sqrt{b}+b}{\sqrt{a}(\sqrt{a}-\sqrt{b})}\right]^{\frac{2}{3}}$$

Factor the numerator of the dividend as the sum of cubes. On reducing we have in the brackets

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$$

Answer: $\sqrt[3]{(a-b)^2}$.

103. Reduce the fraction $\frac{2\sqrt[3]{x}}{x\sqrt[3]{x}-4\sqrt[3]{x}}$ by $\sqrt[3]{x}$. The bracketed expression is reduced to the form $\frac{\sqrt{x}+2}{x-4}$. Reduce this fraction by $\sqrt{x}+2$. The given expression is equal to

$$\left(\frac{1}{\sqrt{x}-2}\right)^{-2}-\sqrt{(x+4)^2}=(\sqrt{x}-2)^2-|x+4|$$

It is assumed that $x>0$ (at negative x the root $\sqrt[3]{x}$ will not be arithmetical; at $x=0$ the given expression loses its sense). Therefore $x+4>0$.

Answer: $-4\sqrt{x}$.

104. The bracketed fraction is equal to

$$\frac{2(\sqrt{x}+\sqrt{y})}{\sqrt{x}(\sqrt{x}+\sqrt{y})}=\frac{2}{\sqrt{x}}$$

The given expression is equal to

$$x^3\left(\frac{2}{\sqrt{x}}\right)^5\sqrt[3]{x\sqrt{x}}=x^3\cdot 32x^{-\frac{5}{2}}x^{\frac{1}{2}}=32x$$

Answer: $32x$.

105. Factor out $\sqrt[4]{ax}$ in the numerator of the first fraction. Taking into account that $\sqrt[4]{x^2}-\sqrt[4]{a^2}=\sqrt{x}-\sqrt{a}$, reduce the fraction. The first factor takes the form

$$\left[-\sqrt[4]{ax}+\frac{1+\sqrt{ax}}{\sqrt[4]{ax}}\right]^{-2}=\left(\frac{1}{\sqrt[4]{ax}}\right)^{-2}=\sqrt{ax}$$

The second factor is $\sqrt{\left(1+\sqrt{\frac{a}{x}}\right)^2}$. Since $\sqrt{\frac{a}{x}}$ is an arithmetical root, the expression $1+\sqrt{\frac{a}{x}}$ is always positive.

Answer: $\sqrt{a}(\sqrt{x}+\sqrt{a})$.

106. The quantities a and c must be positive. Therefore, the denominator of the first fraction, which is reduced to the form

$$\sqrt{2(a-b^2)^2+8ab^2}=\sqrt{2(a+b^2)^2},$$

is equal to $\sqrt{2}(a+b^2)$. The numerator of this fraction is equal to $\sqrt{3}(a+b^2)$. The second fraction is $-\frac{\sqrt{2}}{\sqrt{3}}\cdot\sqrt{ac}$.

Answer: $-\sqrt{ac}$.

107. The minuend is equal to

$$\left\{\sqrt{1+[a^{\frac{2}{3}}x^{-\frac{2}{3}}-1]}\right\}^{-6}=\frac{x^2}{a^2}$$

The radicand of the subtrahend is equal to $(a^2+x^2)^2$ the quantity a^2+x^2 being positive.

Answer: -1.

108. The bracketed expression is equal to $\dfrac{2\sqrt[4]{x}}{\sqrt{x}-\sqrt{a}}$; raising to the power -2, we get $\dfrac{(\sqrt{x}-\sqrt{a})^2}{4\sqrt{x}}$. On reducing by $\sqrt{x}+\sqrt{a}$ the divisor is equal to $\dfrac{\sqrt{x}-\sqrt{a}}{4}$.

Answer: $\dfrac{\sqrt{x}-\sqrt{a}}{\sqrt{x}}$.

109. Factor out $\sqrt[6]{x}$ in the numerator; reduce the fraction; the given expression takes the form

$$(2\sqrt[6]{x})^3+4x+4+(\sqrt{x}+1)^2=5x+10\sqrt{x}+5$$

Answer: $5(\sqrt{x}+1)^2$.

110. In the first fraction transpose $x^{-\frac{1}{3}}$ from the numerator to the denominator (with a positive exponent); the fraction turns out to be equal to $\dfrac{3}{x-2}$. Reduce the second fraction by $x^{\frac{1}{3}}$. The given expression takes the form

$$\left(\frac{3}{x-2}-\frac{1}{x-1}\right)^{-1}-\left(\frac{1-2x}{3x-2}\right)^{-1}=\frac{(x-2)(x-1)}{2x-1}-\frac{3x-2}{1-2x}$$

Answer: $\dfrac{x^2}{2x-1}$.

111. The first factor is equal to $\dfrac{1}{a}$. Squaring the bracketed expression, we get $2a^2-2ab$.

Answer: $2(a-b)$.

112. Cube the difference $\sqrt[3]{x}-\sqrt[3]{a}=x^{\frac{1}{3}}-a^{\frac{1}{3}}$; the numerator of the fraction is equal to

$$3x-3x^{\frac{2}{3}}a^{\frac{1}{3}}+3x^{\frac{1}{3}}a^{\frac{2}{3}}=3x^{\frac{1}{3}}(x^{\frac{2}{3}}-x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}}),$$

the denominator being equal to

$$-3a-3x^{\frac{2}{3}}a^{\frac{1}{3}}+3x^{\frac{1}{3}}a^{\frac{2}{3}}=-3a^{\frac{1}{3}}(a^{\frac{2}{3}}+x^{\frac{2}{3}}-x^{\frac{1}{3}}a^{\frac{1}{3}})$$

Reducing the fraction, we get $-\dfrac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}}$. The given expression is equal to

$$\left(-\frac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}}\right)^3+\sqrt{[(a+x)^3]^{\frac{2}{3}}}:\ a=1$$

Answer: 1.

113. The numerator of the first fraction is equal to

$$a+2\sqrt{ab}-3b=(\sqrt{a})^2+2\sqrt{a}\sqrt{b}-3(\sqrt{b})^2=(\sqrt{a}+3\sqrt{b})(\sqrt{a}-\sqrt{b})$$

Answer: $\dfrac{1}{2b}$.

114. The denominator of the first fraction in parentheses is

$$(a^{\frac{1}{2}})^2+a^{\frac{1}{2}}b^{\frac{1}{2}}-6(b^{\frac{1}{2}})^2=(a^{\frac{1}{2}}-2b^{\frac{1}{2}})(a^{\frac{1}{2}}+3b^{\frac{1}{2}})$$

The denominator of the second fraction is equal to $(a^{\frac{1}{2}}+3b^{\frac{1}{2}})^2$. The numerators are factored in a similar way.

Answer: $\dfrac{5}{a-9b}$.

115. Reduce the fraction in parentheses by $\sqrt{a}+\sqrt{b}$. The first of the fractions entering the given expression is equal to

$$\frac{3\sqrt{a}\,(a-\sqrt{ab}+b)}{3\sqrt{a}\,[(\sqrt{a})^3+(\sqrt{b})^3]}=\frac{1}{\sqrt{a}+\sqrt{b}}$$

The second fraction is equal to

$$\frac{\sqrt{a}\,(\sqrt{b}-\sqrt{a})}{\sqrt{a}\,(a-b)}=-\frac{1}{\sqrt{a}+\sqrt{b}}$$

Answer: 0.

116. *Answer*: 3.

117. Get rid of negative exponents. Factor the expression

$$a^{\frac{3}{2}}-b^{\frac{3}{2}}=(a^{\frac{1}{2}})^3-(b^{\frac{1}{2}})^3$$

Answer: 1.

118. Transforming the first bracketed addend, we get

$$\frac{1-a^2}{\sqrt[3]{a}\,[(\sqrt[3]{a})^2+\sqrt[3]{a}+1]\,[(\sqrt[3]{a})^2-\sqrt[3]{a}+1]}$$

The numerator of this fraction is equal to

$$(1-a)(1+a)=[1-(\sqrt[3]{a})^3]\,[1+(\sqrt[3]{a})^3]$$

Factor the sum and difference of cubes.

Answer: a.

119. Factor out $\sqrt[3]{a}$ in the numerator of the fraction. The multiplicand is equal to $\sqrt[3]{a}-\sqrt[3]{x}$, and the multiplier, to

$$4(\sqrt[3]{a^2}+\sqrt[3]{ax}+\sqrt[3]{x^2}).$$

Answer: $4(a-x)$.

120. Represent the fraction in the form

$$\frac{\sqrt{a}\,[(\sqrt{a})^3-\sqrt[3]{b})^3]}{\sqrt{a}-\sqrt[3]{b}}=\sqrt{a}\,(a+\sqrt{a}\,\sqrt[3]{b}+\sqrt[3]{b^2})$$

The expression in the first parentheses (the dividend) is equal to

$$\sqrt{a}\,(a+2\sqrt{a}\,\sqrt[3]{b}+\sqrt[3]{b^2})=\sqrt{a}\,(\sqrt{a}+\sqrt[3]{b})^2$$

The divisor is equal to $\sqrt{a}\,(\sqrt{a}+\sqrt[3]{b})$.

Answer: a.

121. The denominator of the minuend is equal to $\sqrt{a}\,\sqrt[4]{x}(\sqrt{a}+\sqrt[4]{x})$, and the numerator, to $\sqrt{a}\,\sqrt[4]{x}\,[(\sqrt{a})^3+(\sqrt[4]{x})^3]$. Reduce the minuend to the form $a-\sqrt{a}\,\sqrt[4]{x}+\sqrt{x}$. The subtrahend is equal to $\sqrt{(a+\sqrt{x})^2}=|a+\sqrt{x}|$. Instead of the latter expression we may write $a+\sqrt{x}$, since $a+\sqrt{x}$ is a positive quantity (the quantity a cannot be negative, since the given expression contains $\sqrt{a}$).

Answer: a^2x.

122. The factors of the denominator are equal to $1+\sqrt[4]{x}$ and $1-\sqrt[4]{x}$. The numerator may be represented in the form $-x(1-\sqrt{x})$.

Answer: $-x^3$.

123. The numerator of the bracketed fraction is equal to

$$\sqrt[4]{a^3}\,(\sqrt[4]{a}+\sqrt{b})+b\,\sqrt[4]{b^2}\,(\sqrt[4]{a}+\sqrt{b})=(\sqrt[4]{a}+\sqrt{b})\,[(\sqrt[4]{a})^3+(\sqrt{b})^3]=$$
$$=(\sqrt[4]{a}+\sqrt{b})\,(\sqrt[4]{a}+\sqrt{b})\,(\sqrt{a}-\sqrt[4]{a}\,\sqrt{b}+b)$$

The given expression is equal to

$$\sqrt{a}\,(\sqrt{a}-\sqrt[4]{a}\,\sqrt{b})^{-1}+\frac{\sqrt[4]{a}}{\sqrt{b}-\sqrt[4]{a}}=\frac{\sqrt{a}}{\sqrt[4]{a}\,(\sqrt[4]{a}-\sqrt{b})}-\frac{\sqrt[4]{a}}{\sqrt[4]{a}-\sqrt{b}}=0$$

Answer: 0.

124. The numerator of the minuend is equal to

$$\frac{(\sqrt[3]{a})^3+(\sqrt[3]{x})^3}{(\sqrt[3]{a})^2-(\sqrt[3]{x})^2}-\frac{\sqrt[3]{ax}\,(\sqrt[3]{a}-\sqrt[3]{x})}{(\sqrt[3]{a}-\sqrt[3]{x})^2}=\sqrt[3]{a}-\sqrt[3]{x}$$

Answer: $\sqrt[6]{a}$.

125. First add the first two fractions; the common denominator is equal to

$$[(a^{\frac{1}{4}}+1)+a^{\frac{1}{8}}]\,[(a^{\frac{1}{4}}+1)-a^{\frac{1}{8}}]-(a^{\frac{1}{4}}+1)^2-a^{\frac{1}{4}}=a^{\frac{1}{2}}+a^{\frac{1}{4}}+1$$

We get $\dfrac{2\left(a^{\frac{1}{4}}+1\right)}{a^{\frac{1}{2}}+a^{\frac{1}{4}}+1}$. Now subtract the third fraction; the common denominator is $a+a^{\frac{1}{2}}+1$.

Answer: $\dfrac{4}{a+a^{\frac{1}{2}}+1}$.

126. $\sqrt{\sqrt{2}-1}\,\sqrt[4]{3+2\sqrt{2}}=\sqrt[4]{(\sqrt{2}-1)^2(3+2\sqrt{2})}=1$. A similar transformation is performed in the denominator. The literal radicand in the numerator is equal to $(\sqrt{x}-2)^3$. The fraction $\dfrac{x-\sqrt{x}}{\sqrt{x}-1}$ is reduced by $\sqrt{x}-1$.

Answer: 1.

127. The numerator of the first fraction is equal to

$$a^2\sqrt[6]{a^5b^3}+ab\sqrt[6]{a^5b^3}=a(a+b)\sqrt[6]{a^5b^3}$$

The denominator is transformed to the form $(b+a)(b-2a)\sqrt[6]{a^3b^3}$. Thus, the first fraction is equal to $\dfrac{a\sqrt[3]{a}}{b-2a}$. The dividend in parentheses is equal to $\dfrac{3a^2}{(b-2a)(3-b)}$. Dividing it by $\dfrac{a+b}{3a-ab}$, we obtain $\dfrac{3a^3}{(b-2a)(a+b)}$. Subtracting then $\dfrac{ab}{a+b}$, we find $\dfrac{a(3a^2+2ab-b^2)}{(a+b)(b-2a)}$.

The given expression is equal to

$$\frac{a\sqrt[3]{a}}{b-2a}-\frac{a^{-\frac{2}{3}}a(3a-b)}{b-2a}=\sqrt[3]{a}$$

Answer: $\sqrt[3]{a}$.

128. The multiplicand is equal to $\dfrac{2x+a}{2x-a}$. The expression in the second brackets is equal to $\sqrt{2x-a}$.

Answer: $2x+a$.

129. *Answer*: $\sqrt{2}$.

130. *Answer*: $\dfrac{1}{x(x-1)}$.

131. The minuend is equal to $\dfrac{1}{a+b}$, and subtrahend, to $\dfrac{b}{(a+b)(a+2b)}$.

Answer: $\dfrac{1}{a+2b}$.

132. The first addend is equal to $\frac{a}{a+b}$; the second, to $\frac{b}{a}\cdot\frac{2a+b}{a+b}$.

Answer: $\frac{a+b}{a}$.

133. The first addend is equal to $\frac{a-b}{ab}$; the second, to $\frac{1}{a}$. *Answer*: $\frac{1}{b}$.

134. *Answer*: $\frac{2b-a}{2b+a}$.

CHAPTER III

ALGEBRAIC EQUATIONS *

135. Write the fraction $\frac{6b+7a}{6b}$ in the form $1+\frac{7}{6}\frac{a}{b}$. Then the given equation takes the form

$$\frac{a(b-3a)}{2b^2(b-a)}\,y=\frac{7}{6}\frac{a}{b}$$

whence $y=\frac{7b(b-a)}{3(b-3a)}$.

Answer: $y=\frac{7b(b-a)}{3(b-3a)}$.

136. Get rid of the denominator (the common denominator is a^2-b^2). *Answer*: $x=0$.

137. Solving the given equation by the general method, we get

$$x=\frac{3abc+ab(a+b)+bc(b+c)+ca(c+a)}{ab+bc+ca}$$

This fraction may be reduced by factoring the numerator (represent the expression $3abc$ in the form of the trinomial $abc+abc+abc$ and group each of the successive terms with abc). We get

$$x=a+b+c$$

The solution is simplified with the aid of the following artificial method. Represent the addend $\frac{x-a-b}{c}$ in the form $\frac{x-(a+b+c)}{c}+1$ and perform similar transformations with the other two addends of the left member. The equation takes the form:

---

* In solving problems of the present chapter we do not consider singular values of the known quantities, at which a given equation loses its sense or has no solution, or acquires more solutions.

For instance, in Problem **135** the given equation loses sense at $b=0$ and at $b-a=0$, since at $b=0$ the denominators of the first and second terms vanish, and at $b-a=0$ the same happens to the last term. Furthermore, at $a=0$ the given equation has an infinite number of solutions, because it takes the form $1=1$, becoming an identity. Finally, at $b=3a$ the given equation has no solutions at all, since it is reduced to the form $0\cdot y=\frac{7}{18}$.

$$[x-(a+b+c)]\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)=0$$

Answer: $x=a+b+c$.

138. The common denominator is $6cd\,(2c+3d)\,(2c-3d)$.

Answer: $z=\frac{c\,(4c^2-9d^2)}{8c^2+27d^2}$.

139. Represent the fraction $\frac{2n^2\,(1-x)}{n^4-1}$ in the form $\frac{2n^2\,(x-1)}{1-n^4}$ (to obtain the same denominator as in the next fraction). It is advisable to transform the fraction $\frac{x-1}{n-1}$ to $\frac{1-x}{1-n}$. Transpose all the terms to the left and group them (the first one with the fourth, and the second with the third one). We get

$$(1-x)\left(\frac{1}{1-n}+\frac{1}{1+n}\right)+\frac{1}{1-n^4}\,[2n^2\,(x-1)-(2x-1)]=0$$

Transforming $\frac{1}{1-n}+\frac{1}{1+n}$ to the form $\frac{2}{1-n^2}$, we get rid of the denominator.

Answer: $x=\frac{3}{4}$.

140. Transpose all the terms containing x to the left side of the equation, and all the known ones, to the right side. Reducing each member to a common denominator, we get

$$\frac{(3ab+1)\,(a+1)^2-(2a+1)}{a\,(a+1)^2}\,x=\frac{3ab\,(a+1)^2+a^2}{(a+1)^3}$$

or

$$\frac{3ab\,(a+1)^2+a^2+2a+1-2a-1}{a\,(a+1)^2}\,x=\frac{a\,[3b\,(a+1)^2+a]}{(a+1)^3}$$

Whence

$$\frac{a\,[3b\,(a+1)^2+a]}{a\,(a+1)^2}\,x=\frac{a\,[3b\,(a+1)^2+a]}{(a+1)^3}$$

Reducing it, we find

$$x=\frac{a}{a+1}$$

Answer: $x=\frac{a}{a+1}$.

141. Group the terms as in Problem 140; on transforming we get

$$\frac{ab\,[3c\,(a+b)^2+ab]}{(a+b)^3}=\frac{a\,[3c(a+b)^2+ab]}{a\,(a+b)^2}\,x$$

Answer: $x=\frac{ab}{a+b}$.

142. The common denominator is $(a+b)^2(a-b)$.

Answer: $x=\frac{m(a+b)}{a}$

143. Rewrite the fraction $\frac{mz}{m^2-z^2}$ in the form $\left(-\frac{mz}{z^2-m^2}\right)$. Then the common denominator is $mz(z^2-m^2)$. Getting rid of it and collecting like terms, we get $m^2z^2-4m^3z=0$. This equation has two roots: $z=0$ and $z=4m$. But when rejecting a denominator containing an unknown quantity, extraneous roots may appear; and namely, these are the roots which, when substituted into the common denominator, nullify it. In the given case $z=0$ is an extraneous root. It does not satisfy the given equation, since the first and third terms lose their sense at $z=0$. The root $z=4m$ does not nullify the common denominator, therefore it is not an extraneous one.

Answer: $z=4m$.

144. The common denominator is b^4-x^2. Getting rid of it, we obtain $2x(a^2+b^2-2ab)=2(a^2-b^2)$, whence $x=\frac{a+b}{a-b}$. There are no extraneous roots, since the denominator b^4-x^2 does not vanish at $x=\frac{a+b}{a-b}$.

Answer: $x=\frac{a+b}{a-b}$.

145. The common denominator is $(x^2-a^2)(x+n)$. Getting rid of it, we find $x=\frac{n^2}{a}$. At this value of x the denominator does not vanish. Hence, $x=\frac{n^2}{a}$ is the root of the given equation.

Answer: $x=\frac{n^2}{a}$.

146. Rewrite $x+a^{-1}$ in the form $x+\frac{1}{a}$. After transformations we get

$$\frac{2a}{ax+1}:\frac{2}{ax+1}=\frac{x}{2}$$

Reducing by $ax+1$, we find $x=2a$.

Note. The reduction by $ax+1$ is possible, provided $ax+1$ is not equal to zero. But at $x=2a$ we have $ax+1=2a^2+1>0$. Therefore, the obtained root is not an extraneous one. But suppose, for example, we have the equation $\frac{2a}{x-2a}:\frac{2}{x-2a}=\frac{x}{2}$, in this case the reduction by $x-2a$ would also give $x=2a$. However, this root is of no use because the fractions $\frac{2a}{x-2a}$ and $\frac{2}{x-2a}$ lose their sense at $x=2a$. Thus, the equation $\frac{2a}{x-2a}:\frac{2}{x-2a}=\frac{x}{2}$ has no solution.

Answer: $x=2a$.

147. Rewrite the equation in the form

$$\frac{a+x}{a^2+x^2+ax}+\frac{a-x}{a^2+x^2-ax}=\frac{3a}{x(a^4+a^2x^2+x^4)}$$

The common denominator of the left member $(a^2+x^2+ax)(a^2+x^2-ax)$ may be transformed to

$$(a^2+x^2)^2-(ax)^2=a^4+a^2x^2+x^4$$

We get

$$\frac{2a^3}{a^4+a^2x^2+x^4}=\frac{3a}{x(a^4+a^2x^2+x^4)}$$

Answer: $x=\frac{3}{2a^2}$.

148. Transpose the terms containing the unknown to the left side of the equation, and the constant terms, to the right:

$$(a-b-1)\sqrt{x}=(a^2-b^2)-(a+b)$$

After factorization of the right member we obtain

$$(a-b-1)\sqrt{x}=(a+b)(a-b-1)$$

Whence we have $\sqrt{x}=a+b$.

Since the expression $\sqrt{x}$ means the positive value of the square root, for $a+b<0$ the problem has no solution.

Answer: $x=(a+b)^2$ (if $a+b\geqslant 0$).

149. Getting rid of the denominator and collecting like terms, we get $2x^2+6ax+3a^2=0$.

Answer: $x_1=\frac{a(\sqrt{3}-3)}{2}$; $x_2=-\frac{a(\sqrt{3}+3)}{2}$.

150. The common denominator is $4(x+b)(x-b)$. Simplification yields

$$12x^2-4bx-b^2=0$$

Answer: $x_1=\frac{b}{2}$; $x_2=-\frac{b}{6}$.

151. The common denominator is $(x-a)^2$. Getting rid of it, we obtain

$$(x-a)^2-2a(x-a)+(a^2-b^2)=0$$

From this quadratic equation we find

$$x-a=a\pm b$$

Answer: $x_1=2a+b$; $x_2=2a-b$.

152. The common denominator is $bc^2(a-2b)$. Rejecting it, we get

$$(cx)^2-(a-2b)\cdot(cx)-b(a-b)=0$$

From this equation we find

$$cx=\frac{(a-2b)\pm a}{2}$$

Answer: $x_1=\frac{a-b}{c}$; $x_2=-\frac{b}{c}$.

153. Rejecting the denominator, we obtain the equation $4x(x-a)+8x(x+a)=5a^2$ or, after simplification,

$$12x^2+4ax-5a^2=0$$

Answer: $x_1=\frac{a}{2}$; $x_2=-\frac{5a}{6}$.

154. The common denominator is $n(nx-2)$. After simplifications the equation takes the form

$$(n-1)x^2-2x-(n+1)=0$$

Answer: $x_1=\frac{n+1}{n-1}$; $x_2=-1$.

155. The common denominator is $a(a-x)^2$. After simplifications we get the equation

$$(a+1)x^2-2ax+(a-1)=0$$

Answer: $x_1=1$; $x_2=\frac{a-1}{a+1}$.

156. The common denominator is $(x-a)^2$. Getting rid of the denominator, we obtain the equation

$$(x-a)^2-2b(x-a)-(a^2-b^2)=0$$

Solving it, we find

$$x-a=b\pm a$$

Answer: $x_1=2a+b$; $x_2=b$.

157. The common denominator is $nx(x-2)(x+2)$. After simplifications we obtain the equation

$$x^2-(2-n)x-(2n^2+4n)=0$$

Answer: $x_1=n+2$, $x_2=-2n$.

158. *First method*. After standard transformations we get the following equation

$$x^2+(a-2n-2a+n)x-(a-2n)(2a-n)=0$$

Its solution can be found at once, if we draw our attention to the fact that the constant term is the product of the quantities $-(a-2n)$ and $(2a-n)$, and the coefficient at x is the sum of the same quantities taken with the reversed sign.

Second method. Transposing unity from the right to the left, we get

$$\frac{a+x-2n}{2a-n}-\frac{a-2n+x}{x}=0$$

or

$$(a-2n+x)\left(\frac{1}{2a-n}-\frac{1}{x}\right)=0$$

whence: (1) $a-2n+x=0$ or $x_1=2n-a$

(2) $\frac{1}{2a-n}-\frac{1}{x}=0$ or $x_2=2a-n$

Answer: $x_1 = 2n - a$; $x_2 = 2a - n$.

159. We get the equation

$$(n-1)^2 x^2 - a(n-1)x + (a-1) = 0$$

to avoid operations with fractions it is advisable to put $(n-1)x = z$ or directly find $(n-1)x$ from the equation

$$[(n-1)x]^2 - a[(n-1)x] + a - 1 = 0$$

We get

$$(n-1)x_1 = a-1; \quad (n-1)x_2 = 1$$

Answer: $x_1 = \dfrac{a-1}{n-1}$; $x_2 = \dfrac{1}{n-1}$.

160. The denominator of the left member is equal to $(a-x)^2$. Multiplying both members of the equation by it, we find

$$\left(\frac{a-x}{x}\right)^2 - \left(\frac{a}{a+b}\right)^2 = \frac{5}{9}\left(\frac{a-x}{x}\right)^2;$$

$$\frac{4}{9}\left(\frac{a-x}{x}\right)^2 = \left(\frac{a}{a+b}\right)^2$$

Taking the square root, we get one of the two equations:

$$\frac{2}{3}\cdot\frac{a-x}{x} = \frac{a}{a+b} \quad \text{and} \quad \frac{2}{3}\cdot\frac{a-x}{x} = -\frac{a}{a+b}$$

Answer: $x_1 = \dfrac{2a(a+b)}{5a+2b}$; $x_2 = \dfrac{2a(a+b)}{2b-a}$.

161. First transform the expression

$$(1+ax)^2 - (a+x)^2 = 1 + a^2x^2 - a^2 - x^2$$

Grouping the first term with the last one, and the second with the third one in the right member, we get $(1-x^2)(1-a^2)$. Now the given equation is reduced to the form

$$x(x+1) = \frac{ab}{(a-b)^2}$$

Answer: $x_1 = \dfrac{a}{b-a}$; $x_2 = \dfrac{b}{a-b}$.

162. The trinomial $ax^2 + bx + c$ is factored into first-degree factors in the following way: $ax^2 + bx + c = a(x-x_1)(x-x_2)$, where x_1 and x_2 are the roots of the equation $ax^2 + bx + c = 0$. In this case $a = -3$; $x_1 = 7$; $x_2 = -\dfrac{10}{3}$; thus we get $-3(x-7)\left(x+\dfrac{10}{3}\right)$.

Answer: $(7-x)(3x+10)$.

163. Since

$$\frac{a}{b} - \frac{b}{a} = \frac{a^2-b^2}{ab} = \frac{(a+b)(a-b)}{ab}$$

we may, by guess, factor $\frac{a}{b}-\frac{b}{a}$ into $\frac{a+b}{a}$ and $\frac{a-b}{b}$ $\left(\text{their sum is equal to } \frac{a}{b}+\frac{b}{a}\right)$. Now it is necessary to find out whether this solution is unique. Let u and v be the required factors. By hypothesis,

$$uv=\frac{a}{b}-\frac{b}{a} \text{ and } u+v=\frac{a}{b}+\frac{b}{a}$$

Consequently, u and v are the roots of the quadratic equation

$$x^2-\left(\frac{a}{b}+\frac{b}{a}\right)x+\left(\frac{a}{b}-\frac{b}{a}\right)=0$$

The expressions for u and v will contain the radical

$$\sqrt{\left(\frac{a}{b}+\frac{b}{a}\right)^2-4\left(\frac{a}{b}-\frac{b}{a}\right)}$$

Knowing for sure that a rational solution is possible here, let us try to get rid of the radical. For this purpose instead of $\left(\frac{a}{b}+\frac{b}{a}\right)^2$ write the expression $\left(\frac{a}{b}-\frac{b}{a}\right)^2$ and, for compensation, add $4\cdot\frac{a}{b}\cdot\frac{b}{a}$, i.e. 4; then under the radical sign we get a perfect square $\left[\left(\frac{a}{b}-\frac{b}{a}\right)-2\right]^2$.

Answer: $\frac{a+b}{a}\cdot\frac{a-b}{b}$.

164. $15x^3+x^2-2x=x(15x^2+x-2)$. The roots of the equation $15x^2+x-2=0$ are $x_1=\frac{1}{3}$ and $x_2=-\frac{2}{5}$. Consequently,

$$15x^2+x-2=15\left(x-\frac{1}{3}\right)\left(x+\frac{2}{5}\right)=(3x-1)(5x+2)$$

Answer: $x(3x-1)(15x+2)$.

165. *First method*. Represent the sum $2x^4+4x^2+2$ in the form $2(x^2+1)^2$.

Second method. Arrange the polynomial terms in the order of decreasing of their exponents and break up the term $4x^2$ into two summands $2x^2+2x^2$; then group the first three terms and the last three ones and carry out factorization.

Answer: $(x^2+1)(2x^2+x+2)$.

165a. Rewrite the left member in the following way

$$(1-x^2)^2+4x^2$$

The equation takes the form

$$(1-x^2)^2-4x(1-x^2)+4x^2=0$$

or

$$[(1-x^2)-2x]^2=0$$

Answer: $x_1=-1+\sqrt{2}$; $x_2=-1-\sqrt{2}$.

166. The required equation is $\left(x-\frac{a}{b}\right)\left(x-\frac{b}{a}\right)=0$.

Answer: $abx^2 - (a^2 + b^2)x + ab = 0$.

167. By Viète's theorem the sum of the roots x_1 and x_2 of the equation $x^2 + px + q = 0$ is $-p$, while their product is equal to q. Hence,

$$p = -\left(\frac{1}{10-\sqrt{72}} + \frac{1}{10+\sqrt{72}}\right) = \frac{-2\cdot 10}{100-72} = -\frac{20}{28}$$

$$q = \frac{1}{10-\sqrt{72}} \cdot \frac{1}{10+\sqrt{72}} = \frac{1}{28}$$

The required equation is $x^2 - \frac{20}{28}x + \frac{1}{28} = 0$.

Answer: $28x^2 - 20x + 1 = 0$.

168. Solved like the preceding problem.

Answer: $bx^2 - 2a\sqrt{ax} + a^2 = 0$.

169. According to Viète's theorem, $x_1x_2 = 12$; by hypothesis, $x_1 - x_2 = 1$. From these equations it is possible to find x_1 and x_2 (4 and 3, or -3 and -4) and then $p = -(x_1 + x_2) = \pm 7$.

But to find $x_1 + x_2$ there is no need to determine separately x_1 and x_2. We may compute

$$(x_1 + x_2)^2 = (x_1 - x_2)^2 + 4x_1x_2 = 1^2 + 4\cdot 12 = 49$$

whence $p = -(x_1 + x_2) = \pm 7$.

Answer: $p = \pm 7$.

170. We have

$$x_1x_2 = \frac{1}{5}; \quad x_1 - x_2 = 1$$

Then, as in the previous problem, find $x_1 + x_2 = \pm\frac{3}{\sqrt{5}}$ and take into account that $x_1 + x_2 = \frac{k}{5}$.

Answer: $k = \pm 3\sqrt{5}$.

171. We have

$$x_1^2 + x_2^2 = 1.75; \quad x_1x_2 = a^2; \quad x_1 + x_2 = 3a$$

There are three unknowns here: x_1, x_2, a. We have to find a. Squaring the third equation and subtracting twice the second one, we find $x_1^2 + x_2^2 = 7a^2$. Comparing this with the first equation, we find $7a^2 = 1.75$.

Answer: $a = \pm\frac{1}{2}$.

172. By Viète's theorem

$$p + q = -p, \text{ and } pq = q$$

This system has two solutions: (1) $p = 0$, $q = 0$; (2) $p = 1$, $q = -2$. In the first case we have the equation $x^2 = 0$, in the second, $x^2 + x - 2 = 0$.

Answer: (1) $p = 0$; $q = 0$
(2) $p = 1$; $q = -2$

173. The roots of the required equation are: $y_1 = \frac{x_1}{x_2}$ and $y_2 = \frac{x_2}{x_1}$. Express $y_1 + y_2$ in terms of coefficients a, b, c. For this purpose transform

$y_1+y_2=\frac{x_1^2+x_2^2}{x_1x_2}$ to $\frac{(x_1+x_2)^2-2x_1x_2}{x_1x_2}$ and substitute $-\frac{b}{a}$ for (x_1+x_2) and $\frac{c}{a}$ for x_1x_2. We get $\frac{b^2-2ac}{ac}$. Besides, we have $y_1y_2=\frac{x_1}{x_2}\cdot\frac{x_2}{x_1}=1$. Consequently, the required equation is

$$y^2-\frac{b^2-2ac}{ac}y+1=0$$

Answer: $acy^2-(b^2-2ac)y+ac=0$.

174. This problem may be solved like the preceding one, but a shorter way is preferable.

In the first case both roots of the required equation must be twice as large as those of the given equation. Hence, we have to find the unknown quantity y whose value is twice the value of the unknown quantity x satisfying the equation $ax^2+bx+c=0$. From the condition $y=2x$ we find $x=\frac{y}{2}$; substituting it into the given equation, we get

$$a\left(\frac{y}{2}\right)^2+b\left(\frac{y}{2}\right)+c=0$$

In the second case make the substitution $x=\frac{1}{y}$. We get

$$a\left(\frac{1}{y}\right)^2+b\left(\frac{1}{y}\right)+c=0.$$

Answer: (1) $ay^2+2by+4c=0$
(2) $cy^2+by+a=0$

175. *First method* (see solution of Problem 173). We have

$$y_1+y_2=x_1^3+x_2^3=(x_1+x_2)^3-3x_1x_2(x_1+x_2)$$

Substituting $x_1+x_2=-\frac{b}{a}$ and $x_1x_2=\frac{c}{a}$, we find $y_1+y_2=-\frac{b^3-3abc}{a^3}$. Then, $y_1y_2=(x_1x_2)^3=\frac{c^3}{a^3}$, and, using Viète's theorem, set up the required equation.

Second method (see solution of Problem 174). By hypothesis, $y=x^3$, i.e. $x=\sqrt[3]{y}$. Substituting it into the given equation, we get

$$a\sqrt[3]{y^2}+b\sqrt[3]{y}=-c$$

To rationalize this equation raise both members to the third power and transform the sum $3(a\sqrt[3]{y^2})^2b\sqrt[3]{y}+3a\sqrt[3]{y^2}(b\sqrt[3]{y})^2$ to $3aby\,[a(\sqrt[3]{y})^2+b\sqrt[3]{y}]$. By virtue of the found equation the bracketed expression is equal to $-c$.

Answer: $a^3y^2+(b^3-3abc)y+c^3=0$.

176. Any equation of the nth degree having the roots $x_1, x_2, \ldots, x_n$, may be represented in the form

$$(x-x_1)(x-x_2)\ldots(x-x_n)=0$$

A biquadratic equation always has two pairs of roots of the same absolute value and of opposite signs. Putting $x_3=-x_1$ and $x_4=-x_2$, we may write the biquadratic equation in the form

$$(x - x_1)(x - x_2)(x + x_1)(x + x_2) = 0, \text{ i.e. } (x^2 - x_1^2)(x^2 - x_2^2) = 0$$

or

$$x^4 - (x_1^2 + x_2^2)\,x^2 + x_1^2 x_2^2 = 0.$$

But, by hypothesis,

$$x_1^2 + x_2^2 + (-x_1)^2 + (-x_2)^2 = 50$$

and

$$x_1 x_2 (-x_1)(-x_2) = 144$$

Hence,

$$x_1^2 + x_2^2 = 25 \quad \text{and} \quad x_1^2 x_2^2 = 144$$

Answer: $x^4 - 25x^2 + 144 = 0$.

177. If an algebraic equation (with real coefficients) has a complex root $a + bi$, then the conjugate complex number $a - bi$ will also be its root. Thus, we know two conjugate roots of the given equation: $3 + i\sqrt{6}$ and $3 - i\sqrt{6}$. Both of them can be verified directly, but it is simpler to accomplish the following transformation beforehand.

According to the remainder theorem, the left member of the equation must be divisible by the expressions $x - (3 + i\sqrt{6})$ and $x - (3 - i\sqrt{6})$, and consequently, by their product as well, i.e. by $[(x - 3) - i\sqrt{6}]\,[(x - 3) + i\sqrt{6}] = x^2 - 6x + 15$. On dividing, we factor the left member into two factors: $4x^4 - 24x^3 + 57x^2 + 18x - 45 = (x^2 - 6x + 15)(4x^2 - 3)$, and the given equation decomposes into the following two:

$$(1)\ x^2 - 6x + 15 = 0 \text{ and } (2)\ 4x^2 - 3 = 0$$

The first one has two roots $x_1 = 3 + i\sqrt{6}$ and $x_2 = 3 - i\sqrt{6}$, $x_3 = \frac{\sqrt{3}}{2}$ and $x_4 = -\frac{\sqrt{3}}{2}$ being the roots of the second.

Answer: $x_1 = 3 + i\sqrt{6}$; $x_2 = 3 - i\sqrt{6}$; $x_3 = \frac{\sqrt{3}}{2}$; $x_4 = -\frac{\sqrt{3}}{2}$.

178. By hypothesis, $x = 2$ must satisfy the given equation. Therefore we have $6 \cdot 2^3 - 7 \cdot 2^2 - 16 \cdot 2 + m = 0$, whence $m = 12$. We get the equation $6x^3 - 7x^2 - 16x + 12 = 0$, one of the roots of which is equal to 2. By the remainder theorem the left member must be divisible by $(x - 2)$. Dividing, we find $6x^2 + 5x - 6$. Consequently, the equation may be represented in the form $(x - 2)(6x^2 + 5x - 6) = 0$. In addition to the root $x_1 = 2$, its roots are also the roots x_2 and x_3 of the equation $6x^2 + 5x - 6 = 0$.

Answer: $m = 12$; $x_2 = \frac{2}{3}$; $x_3 = -\frac{3}{2}$.

179. Substituting $x = 2$ and $x = 3$ into the given equation (see solution of the preceding problem), we get

$$4m + n = 10 \text{ and } 9m + n = -15$$

From this system we find $m = -5$, $n = 30$ and obtain the equation $2x^3 - 5x^2 - 13x + 30 = 0$, whose left member must be divisible by $x - 2$ and $x - 3$, and, consequently, by the product $(x - 2)(x - 3)$. The equation is then rewritten in the form $(x-2)(x-3)(2x+5)=0$. Its roots are: $x_1=2$; $x_2=3$; $x_3=-\frac{5}{2}$.

Answer: $m=-5$; $n=30$; $x_3=-\frac{5}{2}$.

180. The quadratic equation $x^2+px+q=0$ has equal roots when the radicand $\left(\frac{p}{2}\right)^2-q$ is equal to zero. In this case it must be that $\left(a\sqrt{a^2-3}\right)^2-4=0$, i.e. $a^4-3a^2-4=0$. This biquadratic equation has two real roots ($a=2$ and $a=-2$) and two imaginary roots ($a=i$ and $a=-i$). Confining ourselves to the real roots*, we get the following pair of equations: $x^2+4x+4=0$ and $x^2-4x+4=0$. The first equation has the roots $x_1=x_2=-2$, the second one, $x_1=x_2=2$.

Answer: at $a=2$ and $a=-2$.

180a. The roots of the equation are

$$x_{1,2}=m\pm\sqrt{m^2-m^2+1}=m\pm 1$$

By hypothesis, we have

$$\begin{cases}-2<m+1<4\\-2<m-1<4\end{cases}\quad\text{or}\quad\begin{cases}-3<m<3\\-1<m<5\end{cases}$$

Answer: $-1<m<3$.

181. Isolate one of the radicals, for instance, the first one. We get $\sqrt{y+2}=2+\sqrt{y-6}$.

Square both members. Collecting like terms and reducing by 4, we have $\sqrt{y-6}=1$, whence $y=7$. A check shows that this root is valid.

Answer: $y=7$.

Note 1. Here and henceforward we consider square roots and, in general, roots of even indices to be arithmetical roots. See Preliminaries for Chapter II (pages 90 to 92). For roots of odd indices see the footnote to Problem 209.

Note 2. A check is carried out to reveal extraneous roots (they may appear as a result of squaring both members of the equation). There are no extraneous roots in the given problem. But let us take the equation $\sqrt{y+2}+\sqrt{y-6}=2$, which differs from the given one only in sign. Solving it in the same way, we get $\sqrt{y-6}=-1$. Squaring the latter, we find the same root $y=7$. It is invalid; the taken equation has no solution at all. Here it is not necessary to carry out a check, since it is obvious that $\sqrt{y-6}$ cannot be equal to -1 (see *Note* 1). But in other cases (see Problems 184 and 190) such a check is necessary.

182. Solved in the same way as the preceding problem.

Answer: $x=6$.

183. Isolate the first radical and square it. After simplification we obtain $x-1=2\sqrt{x-1}$. Squaring once again, we find $(x-1)^2-4(x-1)=0$. This equation may be divided by $x-1$, taking into account that $x=1$ is one of its roots. We find then the other root $x=5$. We may also remove the parentheses and solve the quadratic equation thus obtained. A check shows that both roots are valid.

Answer: $x_1=1$; $x_2=5$.

---

* We assume that the coefficients of the given equation are real numbers.

184. Proceeding in the same way as in the previous problem, we find $x + 22 = 7\sqrt{3x-2}$, hence, $x^2 - 103x + 582 = 0$. This equation has two roots: $x_1 = 6$ and $x_2 = 97$. The given equation is satisfied only by the first root, the second being an extraneous one (it satisfies the equation $\sqrt{3x-2} - \sqrt{x+3} = 7$, which differs from the given one in the sign at the radical).

Answer: $x = 6$.

185. Solved in the same way as the preceding problem. Out of the two roots $x_1 = -1$; $x_2 = 3$ the second is extraneous.

Note. $x = 3$ is a root of the equation

$$-\sqrt{x+1}+\sqrt{2x+3}=1$$

Answer: $x = -1$.

186. *Answer*: $x_1 = 34$; $x_2 = 2$.

187. *Answer*: $x = 4$.

188. Square it. We get the equation

$$x\sqrt{x^2+24}-x^2-2x=0$$

which decomposes into the following two:

$$x=0 \quad \text{and} \quad \sqrt{x^2+24}-x-2=0$$

The second one gives $x = 5$. Perform a verification.

Answer: $x_1 = 0$; $x_2 = 5$.

189. Reduce the given equation to the form

$$\frac{1}{x}+\frac{1}{3}=\sqrt{\frac{1}{9}+\frac{1}{x}\sqrt{\frac{4}{9}+\frac{2}{x^2}}}$$

Square it and multiply by x^2 (through which an extraneous root $x=0$ may be introduced). We obtain the equation

$$1+\frac{2}{3}x=x\sqrt{\frac{4}{9}+\frac{2}{x^2}}$$

Square it once again.

Answer: $x=\frac{3}{4}$.

190. Multiplying both members of the equation by $\sqrt{(x+2)(x+3)}$, we get

$$\sqrt{(x-5)(x+3)}+\sqrt{(x-4)(x+2)}=7$$

Proceeding in the same way as in Problem **181**, we find

$$\sqrt{(x-4)(x+2)}=4$$

Hence, we get two roots $x_1=6$, $x_2=-4$. A check shows that x_2 is invalid, since it yields a wrong equality $\frac{7}{2}\sqrt{2}=-\frac{7}{2}\sqrt{2}$.

Answer: $x=6$.

191. Getting rid of the denominator, we get

$$\sqrt{x^2-16}+\sqrt{x^2-9}=7$$

This equation has two roots $x = 5$ and $x = -5$. But at $x = -5$ the expression $\sqrt{x-3}$ has no real value (see *Note* 1 to Problem 181).

Answer: $x = 5$.

192. Reduce the left member of the equation to a common denominator

$$\frac{3x-5\sqrt{x^2+x}}{-x}=\frac{3}{x}$$

Hence

$$3(x+1)=5\sqrt{x(x+1)}$$

Squaring it, we obtain

$$9(x+1)^2-25(x+1)x=0$$

or

$$(x+1)[9(x+1)-25x]=0$$

Answer: $x_1=-1;\quad x_2=\frac{9}{16}$.

193. Solved in the same way as the preceding problem.

Answer: $x_1 = 2;\ x_2 = -1.6$.

194. Square both members of the given equation. After the identity transformations we get $\sqrt{28-x}=\sqrt{7}$. In squaring this equation an extraneous root may be introduced which satisfies an equation differing from the given one only in the sign of the right member. The equation $\sqrt{28-x}=\sqrt{7}$ has the only root $x=21$, which is not an extraneous one, since $\sqrt{2\sqrt{7}+\sqrt{21}}>\sqrt{2\sqrt{7}-\sqrt{21}}$.

Answer: $x=21$.

195. Rewrite the equation in the following way:

$$\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\frac{3\sqrt{x}}{2\sqrt{x+\sqrt{x}}}$$

Get rid of the denominator; this may result in that an extraneous root $x = 0$ will be introduced (since the denominator vanishes at $x = 0$). There cannot be other extraneous roots, because $x = 0$ is the unique root of the equation $\sqrt{x+\sqrt{x}} = 0$ (see solution of Problem 143).

After simplification we get the equation

$$2x-2\sqrt{x^2-x}-\sqrt{x}=0$$

one of the roots of which is $x=0$. But this root is an extraneous one, since at $x=0$ the right member of the original equation loses its sense. Factor out $\sqrt{x}$:

$$\sqrt{x}\,(2\sqrt{x}-2\sqrt{x-1}-1)=0$$

Solving the equation $2\sqrt{x}-2\sqrt{x-1}-1=0$ (see solution of Problem 181), we find $x=\frac{25}{16}$. Carry out a check.

Answer: $x=\frac{25}{16}$.

196. First rationalize the denominator. To this end multiply both the numerator and denominator by $\sqrt{27+x}+\sqrt{27-x}$; we get

$$\frac{(\sqrt{27+x}+\sqrt{27-x})^2}{2x}=\frac{27}{x}$$

or, after simplifications,

$$\frac{27+\sqrt{27^2-x^2}}{x}=\frac{27}{x}$$

wherefrom we find $x = \pm 27$. Both roots are valid.

Answer: $x = \pm 27$.

197. Isolating the radical, square both members of the equation. We have

$$x^2-2ax=-x\sqrt{x^2+a^2}$$

One of the roots of this equation is $x = 0$. To find other roots divide both members of the equation by x (it can be done, since now $x \neq 0$). Then square both members once again. We get $x = \frac{3}{4}a$.

When verifying the result, one may arrive at the wrong conclusion that the values $x = 0$ and $x = \frac{3}{4}a$ always satisfy the given equation. For a better understanding of the essence of the error let us consider a numerical example. At $a = -1$ the given equation has the form

$$x=-1-\sqrt{1-x\sqrt{x^2+1}}$$

Neither $x = 0$, nor $x = \frac{3}{4}a = -\frac{3}{4}$ satisfy this equation (it has no solution). The same result is obtained for any other negative value of a.

And here is the mistake: the quantity $\sqrt{a^2}$ is considered to be equal to a, whereas it is true only for $a \geqslant 0$. For $a < 0$ we have $\sqrt{a^2} = -a$: for instance, $\sqrt{(-3)^2} = -(-3)$.

The correct general formula (see Preliminaries, Item 3 on page 90) is:

$$\sqrt{a^2}=|a|$$

Using this formula, we find that at $x = 0$ (when the left member of the equation vanishes) the right member is equal to $a - \sqrt{a^2} = a - |a|$. For $a \geqslant 0$ this expression is also equal to zero, but for $a < 0$ it is equal to $2a$. Consequently, if $a \geqslant 0$, the value $x = 0$ is a root of the equation; but if $a < 0$, then $x = 0$ is not the root. The same refers to the value $x = \frac{3}{4}a$.

Answer: if $a \geqslant 0$, then $x_1 = 0$, $x_2 = \frac{3}{4}a$; if $a < 0$, the equation has no solution.

198. Written without powers having negative exponents the given equation has the form

$$\frac{\sqrt{1+\left(\frac{x}{a}\right)^2}-\frac{x}{a}}{\sqrt{1+\left(\frac{x}{a}\right)^2}+\frac{x}{a}}=\frac{1}{4}$$

First method. Reject the denominator: $3\sqrt{1+\left(\frac{x}{a}\right)^2}=5\cdot\frac{x}{a}$. The left member is positive; hence, the right member is also positive. Squaring yields $\left(\frac{x}{a}\right)^2=\frac{9}{16}$, whence $\frac{x}{a}=\frac{3}{4}$ $\left(\text{the value } -\frac{3}{4} \text{ is rejected, since } \frac{x}{a}>0\right)$.

Second method. Rationalize the denominator

$$\left[\sqrt{1+\left(\frac{x}{a}\right)^2}-\frac{x}{a}\right]^2=\frac{1}{4}$$

The bracketed expression cannot be negative; therefore

$$\sqrt{1+\left(\frac{x}{a}\right)^2}-\frac{x}{a}=\frac{1}{2} \text{ or } \sqrt{1+\left(\frac{x}{a}\right)^2}=\frac{1}{2}+\frac{x}{a}$$

Squaring yields: $1+\left(\frac{x}{a}\right)^2=\frac{1}{4}+\frac{x}{a}+\left(\frac{x}{a}\right)^2$, whence $\frac{x}{a}=\frac{3}{4}$.

Answer: $x=\frac{3}{4}a$.

199. Solved in the same way as the preceding problem. Using the second method, we find

$$(\sqrt{1+a^2x^2}-ax)^2=\frac{1}{c^2}$$

The expression $\sqrt{1+a^2x^2}-ax$ is always positive; therefore

$$\sqrt{1+a^2x^2}-ax=\frac{1}{|c|}, \text{ i.e. } \sqrt{1+a^2x^2}=ax+\frac{1}{|c|}$$

Squaring it, we get $x=\frac{|c|^2-1}{2a|c|}$, or $x=\frac{c^2-1}{2a|c|}$, which is the same.

Check. Substituting $x=\frac{c^2-1}{2a|c|}$, we find

$$1+a^2x^2=\frac{4c^2+(c^2-1)^2}{4c^2}=\frac{(c^2+1)^2}{4c^2}$$

Taking into account that c^2+1 is always positive, we find

$$\sqrt{1+a^2x^2}=\frac{c^2+1}{2|c|}$$

Further computations show that the given equation is always satisfied.

Answer: $x=\frac{c^2-1}{2a|c|}$ i.e. for $c>0$ we have $x=\frac{c^2-1}{2ac}$, for $c<0$ we have $x=\frac{1-c^2}{2ac}$.

200. Factor out the expression $\sqrt{x+c}$ both in the numerator and denominator of the left member, and reduce the fraction by this expression*.

After performing these operations we get

$$\frac{\sqrt{x+c}+\sqrt{x-c}}{\sqrt{x+c}-\sqrt{x-c}}=\frac{9(x+c)}{8c}$$

Then rationalize the denominator. After simplifications we find $8\sqrt{x^2-c^2}=x+9c$. Hence, $x=\frac{5c}{3}$ or $x=-\frac{29}{21}c$.

A check shows that both values satisfy the equation when $c>0$ and do not satisfy it if $c<0$.

Answer: At $c>0$ we have $x_1=\frac{5}{3}c$ and $x_2=-\frac{29}{21}c$; if $c\leqslant 0$ the equation has no solution.

201. Transform the first radicand in the following way:

$$x+3-4\sqrt{x-1}=(x-1)-4\sqrt{x-1}+4=(\sqrt{x-1}-2)^2$$

Similarly, the second radicand is equal to $(\sqrt{x-1}-3)^2$. The given equation takes the form

$$|\sqrt{x-1}-2|+|\sqrt{x-1}-3|=1 \tag{A}$$

(see Preliminaries to Chapter II, Item 3). The following three cases are possible: (1) $\sqrt{x-1}>3$; (2) $\sqrt{x-1}<2$; (3) $2\leqslant\sqrt{x-1}\leqslant 3$.

In the first case the equation (A) takes the form:

$$\sqrt{x-1}-2+\sqrt{x-1}-3=1, \text{ or } \sqrt{x-1}=3$$

This result does not agree with the condition $\sqrt{x-1}>3$.

In the second case the equation (A) takes the form:

$$-(\sqrt{x-1}-2)-(\sqrt{x-1}-3)=1 \quad \text{or} \quad \sqrt{x-1}=2$$

This result does not agree with the condition $\sqrt{x-1}<2$ either. Consider, finally, the third case, when the equation (A) takes the form:

$$(\sqrt{x-1}-2)-(\sqrt{x-1}-3)=1 \tag{B}$$

This equality is an identity, hence, the equation (A) is satisfied by all x for which

$$2\leqslant\sqrt{x-1}\leqslant 3$$

Since $\sqrt{x-1}>0$, all the three members of the inequality may be squared, and we find

$$5\leqslant x\leqslant 10,$$

---

* Reducing by $\sqrt{x+c}$, we assume that $x\neq -c$. If the solution of the obtained equation had yielded $x=-c$, this value would not have been a root of the given equation. But, as we see below, we do not obtain such a root.

i.e. the solutions of the given equation are contained within the range bounded by 5 and 10 (the values 5 and 10 included). All of them are the solutions of the given equation, since they suit the third case, when the given equation (A) becomes identity (B).

Answer: $5 \leqslant x \leqslant 10$.

202. Square both members of the equation, transpose all the terms to the left and factor out $\sqrt{a+x}$:

$$\sqrt{a+x}\,(4\sqrt{a+x}+4\sqrt{a-x}-\sqrt{x})=0$$

This equation decomposes into two. From the first one: $\sqrt{a+x}=0$, we find $x=-a$. A check shows that with $a \geqslant 0$ this value satisfies the given equation. If $a<0$, the equation loses its sense (since $\sqrt{a-x}$ becomes an imaginary value). The second equation is $4(\sqrt{a+x}+\sqrt{a-x})=\sqrt{x}$. If it is solved as in Problems 183 to 187, we get (besides the extraneous root $x=0$) $x=\frac{64a}{1025}$. A check will show that this root is also an extraneous one, which means that the second equation has no solutions at all. We may make sure of this fact more easily, if the following method of solution is applied. Let us transform the second equation to

$$\frac{8x}{\sqrt{a+x}-\sqrt{a-x}}=\sqrt{x}$$

which is done by multiplying and dividing $\sqrt{a+x}+\sqrt{a-x}$ by the conjugate expression $\sqrt{a+x}-\sqrt{a-x}$.

Dividing it by $\sqrt{x}$ (which is possible without losing roots, since $x=0$ is not a root), we get $\sqrt{a+x}-\sqrt{a-x}=8\sqrt{x}$. Subtraoting this equation from the above obtained $\sqrt{a+x}+\sqrt{a-x}=\frac{1}{4}\sqrt{x}$, we find

$$2\sqrt{a-x}=-\frac{31}{4}\sqrt{x}$$

But this equality is impossible, since its left member is a positive number, whereas the right one is negative. Suppose, neglecting this fact, we square both members. This operation would yield an extraneous root $x=\frac{64}{1025}a$.

Answer: If a is positive, then $x=-a$; if a is negative, the equation has no solution.

203. Here we may successfully apply the method of transferring the irrationality to the denominator (see the preceding problem).

Answer: $x=0$.

204. *Answer*: $x_1=a$; $x_2=-b$.

205. *Answer*: $x=\frac{(a-1)^2}{4}$ (for $a \geqslant 1$).

For $a<1$ the equation has no solution.

206. The given equation may be represented in the form

$$\frac{(\sqrt{a+x})^3}{ax}=\sqrt{x}$$

or

$$(a+x)^{\frac{3}{2}}=ax^{\frac{3}{2}}$$

Raising it to the power $\frac{2}{3}$, we get $a+x=a^{\frac{2}{3}}x$, whence $x=\frac{a}{a^{\frac{2}{3}}-1}$.

Check:

$$a+x=\frac{a^{\frac{5}{3}}}{a^{\frac{2}{3}}-1}; \quad (a+x)^{\frac{3}{2}}=\frac{a^{\frac{5}{2}}}{(a^{\frac{2}{3}}-1)^{\frac{3}{2}}};$$

$$\frac{(a+x)^{\frac{3}{2}}}{ax}=\frac{a^{\frac{1}{2}}}{(a^{\frac{2}{3}}-1)^{\frac{1}{2}}}$$

Answer: $x=\frac{a}{a^{\frac{2}{3}}-1}$; if $a\leqslant 1$, the equation has no solution.

207. Put $\sqrt[4]{x}=z$. Then

$$\sqrt{x}=(\sqrt[4]{x})^2=z^2$$

The equation takes the form

$$z^2+z-12=0$$

Hence, $z_1=3$, $z_2=-4$. Since $\sqrt[4]{x}$ must be a positive number, the second root is an extraneous one.

Answer: $x=81$.

208. Put $(x-1)^{\frac{1}{4}}=z$. Then proceed as in the previous problem.

Answer: $x=17$.

209*. Cubing both members, we get

$$\sqrt{10+2x}+\sqrt{15-2x}=7.$$

Here the isolation of one of the radicals is not essential.

Answer: $x_1=3$; $x_2=-\frac{1}{2}$.

210. Cube both members of the equation by using the formula $(a+b)^3=$ $=a^3+3ab(a+b)+b^3$. We obtain

$$x+3\sqrt[3]{x(2x-3)}\,[\sqrt[3]{x}+\sqrt[3]{2x-3}]+2x-3=12(x-1)$$

---

* Here and henceforward we do not consider cube radicals, and, in general, radicals of odd indices to be arithmetical, assuming that the radicand may be negative as well (but obligatorily real). The value of the radical is also considered to be a real number.

By virtue of the given equation the bracketed expression may be replaced by the expression $\sqrt[3]{12(x-1)}$. We get

$$\sqrt[3]{x(2x-3)\cdot 12(x-1)}=3(x-1)$$

Cube it. Transposing all the terms to the left, we find

$$(x-1)[12x(2x-3)-27(x-1)^2]=0$$

This equation is decomposed into the following two:

$$x-1=0 \text{ and } 12x\ (2x-3)-27(x-1)^2=0$$

Check the found roots.

Answer: $x_1=1$; $x_2=3$.

211. Solved in the same way as the preceding problem.

Answer: $x_1=a$; $x_2=b$; $x_3=\frac{a+b}{2}$.

212. Put $\sqrt[3]{x}=z$; then $\sqrt[3]{x^2}=z^2$. Substituting into the original equation, we get $2z^2+z-3=0$, whence $z_1=1$; $z_2=-\frac{3}{2}$.

Answer: $x_1=1$; $x_2=-\frac{27}{8}$.

213. Solved in the same way as the preceding problem.

Answer: $z_1=64$; $z_2=-\frac{125}{8}$.

214. Put $\sqrt[6]{a+x}=z$; then $\sqrt{a+x}=z^3$ and $\sqrt[3]{a+x}=z^2$.

Answer: $x_1=-a$; $x_2=1-a$.

215. Put $\sqrt{\frac{2x+2}{x+2}}=z$; then $\sqrt{\frac{x+2}{2x+2}}=\frac{1}{z}$ and, after a number of transformations, the equation takes the form

$$12z^2-7z-12=0$$

hence,

$$z_1=\frac{4}{3} \quad \text{and} \quad z_2=-\frac{3}{4}$$

The second solution is rejected as a negative one (see *Note 1* to Problem 181 on page 122). To determine x we get an equation $\sqrt{\frac{2x+2}{x+2}}=\frac{4}{3}$.

Answer: $x=7$.

216. *Answer*: $x=\pm 5$.

217. Put $\sqrt[3]{x}=z$; then $\sqrt[3]{x^2}=z^2$ and $x=z^3$. We obtain:

$$\frac{z^4-1}{z^2-1}-\frac{z^2-1}{z+1}=4$$

Reduce the first fraction by z^2-1, and the second, by $z+1$. We get: $z^2-z-2=0$. But the reduction of the first fraction is lawful, provided $z^2-1\neq 0$, and that of the second fraction, if $z+1\neq 0$. Meanwhile, out of the two roots $z_1=2$ and $z_2=-1$ the second one gives $z+1=0$. It does not suit, since at $z=-1$ we have $x=-1$, and the left member of the given equation loses its sense.

Answer: $x = 8$.

218. Putting $\sqrt{x}=z$, transform the equation to

$$\frac{z^2-4}{z+2}=z^2-8$$

Reduce the fraction by $z+2$ (see the explanation to the preceding problem). We get $z^2-z-6=0$, whence $z_1=3$, $z_2=-2$. The second root does not suit, because, firstly, the expression $\frac{z^2-4}{z+2}$ loses its sense and, secondly, z cannot be a negative number.

Answer: $x = 9$.

219. Here the introduction of an auxiliary unknown, used in the previous problems, is of no help. Represent the equation in the form

$$\frac{(\sqrt{a-x})^3+(\sqrt{x-b})^3}{\sqrt{a-x}+\sqrt{x-b}}=a-b$$

and reduce it by $\sqrt{a-x}+\sqrt{x-b}$ (the reduction is lawful, since this number cannot be equal to zero). After simplifications we obtain $\sqrt{(a-x)(x-b)}=0$.

Answer: $x_1=a$; $x_2=b$.

220. Represent the given equation in the form

$$\sqrt{2-x}\cdot\left(\frac{\sqrt{2-x}}{2-\sqrt{x}}-\frac{1}{\sqrt{2}}\right)=0$$

This equation decomposes into the following two: the first equation is $\sqrt{2-x}=0$, its root being $x_1=2$; the second one is $\sqrt{2(2-x)}=2-\sqrt{x}$ (after getting rid of the denominator). Its roots are: $x_2=0$; $x_3=\frac{16}{9}$.

Answer: $x_1=2$; $x_2=0$; $x_3=\frac{16}{9}$.

221. *Answer*: $x = 81$.

222. Isolating the radical and squaring both members of the obtained equation, we get a fourth-degree equation. But in the present case it is possible to apply an artificial method. Rewrite the equation in the form

$$\sqrt{x^2-3x+5}+x^2-3x+5=12$$

Putting $\sqrt{x^2-3x+5}=z$, we get $z^2+z-12=0$. Take only the positive root $z=3$.

Answer: $x_1=4$; $x_2=-1$.

223. We may use the same method as in the preceding problem. But it is obvious that the equation has no solution. Indeed, the quantity $3x^2+5x+1$ exceeds $3x^2+5x-8$ at any x. Therefore

$$\sqrt{3x^2+5x+1}>\sqrt{3x^2+5x-8}$$

which means that the left member of the given equation is negative at any x and, consequently, cannot be equal to unity.

Answer: The equation has no solution.

224. Denote one of the radicands by z; it is most convenient to put $y^2 + 4y + 6 = z$. The equation takes the form

$$\sqrt{z+2}+\sqrt{z-2}=\sqrt{2z}$$

Getting rid of the radicals, we find $z^2 = 4$. Only the root $z = 2$ suits (at $z = -2$ the two radicands are negative). Solve the equation $y^2 + 4y + 6 = 2$. Check on solution.

Answer: $y = -2$.

225.* It may be solved by using the substitution method (from the second equation find $y = 6 - x$ or $x = 6 - y$ and substitute it into the first equation), but the following artificial method is somewhat more effective. The first equation is transformed to $(x - y)^2 = 4$, whence $x - y = 2$ or $x - y = -2$. We obtain two systems of equations:

$$(1) \quad \begin{cases} x-y=2 \\ x+y=6 \end{cases} \qquad (2) \quad \begin{cases} x-y=-2 \\ x+y=6 \end{cases}$$

Answer: (1) $x_1=4,\ y_1=2$
(2) $x_2=2,\ y_2=4$

226. Represent the given system in the form

$$\begin{cases} xy+(x+y)=11 \\ xy\,(x+y)=30 \end{cases}$$

Put for the sake of brevity $xy=z_1$; $x+y=z_2$. Then we have

$$\begin{cases} z_1+z_2=11 \\ z_1z_2=30 \end{cases}$$

By Viète's theorem z_1 and z_2 are the roots of the quadratic equation $z^2-11z+30=0$. We find: $z_1=6,\ z_2=5$ or $z_1=5,\ z_2=6$. We get two systems:

$$\begin{cases} x+y=6 \\ xy=5 \end{cases} \quad \text{and} \quad \begin{cases} x+y=5 \\ xy=6 \end{cases}$$

Each of them may be solved by applying Viète's theorem or the substitution method.

Answer: (1) $x=5,\ y=1$ (2) $x=1,\ y=5$
(3) $x=2,\ y=3$ (4) $x=3,\ y=2$

227. Put $y^2=z$; then we have the following system

$$\begin{cases} x+z=7 \\ xz=12 \end{cases}$$

Answer: (1) $x=4,\ y=\sqrt{3}$
(2) $x=4,\ y=-\sqrt{3}$
(3) $x=3,\ y=2$
(4) $x=3,\ y=-2$

228. Put $x^2=z_1$ and $-y=z_2$. We get the system

$$\begin{cases} z_1+z_2=23 \\ z_1z_2=-50 \end{cases}$$

---

* Most problems of this chapter are successfully solved by using artificial methods. The main difficulty here is to find out an adequate artificial method.

Answer: (1) $x=5,\quad y=2$
(2) $x=-5,\quad y=2$
(3) $x=i\sqrt{2},\quad y=-25$
(4) $x=-i\sqrt{2},\quad y=-25$

229. Put $-xy=z_1$; $x^2-y^2=z_2$. We obtain the system

$$\begin{cases} z_1z_2=-180 \\ z_1+z_2=-11 \end{cases}$$

We find $z_1=9$; $z_2=-20$ or $z_1=-20$; $z_2=9$. Now we have two systems

$$(1)\ \begin{cases} xy=-9 \\ x^2-y^2=-20 \end{cases} \quad \text{and} \quad (2)\ \begin{cases} xy=20 \\ x^2-y^2=9 \end{cases}$$

Solve the first system. From its first equation we find $y=-\frac{9}{x}$. Substitute it into the second one and find the biquadratic equation $x^4+20x^2-81=0$. Its roots are:

$$x_{1,2}=\pm\sqrt{-10+\sqrt{181}}\approx\pm\sqrt{3.45}\approx\pm1.86$$

$$x_{3,4}=\pm\sqrt{-10-\sqrt{181}}\approx\pm\sqrt{-23.45}\approx\pm4.84i$$

Now we find

$$y_{1,2}=\frac{\mp 9}{\sqrt{-10+\sqrt{181}}}\approx\frac{\mp 9}{1.86}\approx\mp4.84$$

$$y_{3,4}\approx\frac{\mp 9}{4{,}84i}\approx\pm1.86i$$

Solve the second system using the same method.

Answer: (1) $x\approx1.86,\quad y\approx-4.84$
(2) $x\approx-1.86,\quad y\approx4.84$
(3) $x\approx4.84i,\quad y\approx1.86i$
(4) $x\approx-4.84i,\quad y\approx-1.86i$
(5) $x=5,\quad y=4$
(6) $x=-5,\quad y=-4$
(7) $x=4i,\quad y=-5i$
(8) $x=-4i,\quad y=5i$

230. Eliminate the constant terms by multiplying the second equation by 7 and subtracting the result from the first one. We get

$$-32x^2-2xy+75y^2=0$$

This is a homogeneous equation of the second degree (i.e. an equation containing only terms of the second degree). Dividing both members of the equation by x^2 (this may be done since $x=0$ is not a root), we transform it to $-32-2\frac{y}{x}+75\left(\frac{y}{x}\right)^2=0$, and, solving this quadratic equation, we find $\frac{y}{x}=\frac{2}{3}$ or or $\frac{y}{x}=-\frac{16}{25}$. By using this method we can find the ratio $\frac{y}{x}$ from any homogeneous equation of the second degree.

Now we solve two systems:

$$(1)\ \begin{cases} 5x^2-10y^2-5=0 \\ \frac{y}{x}=\frac{2}{3} \end{cases} \quad \text{and} \quad (2)\ \begin{cases} 5x^2-10y^2-5=0 \\ \frac{y}{x}=-\frac{16}{25} \end{cases}$$

(by the substitution method).

Answer: (1) $x=3,\quad y=2$ (3) $x=\frac{25}{\sqrt{113}},\quad y=-\frac{16}{\sqrt{113}}$

(2) $x=-3,\quad y=-2$ (4) $x=-\frac{25}{\sqrt{113}},\quad y=\frac{16}{\sqrt{113}}$

231. Rewrite the first equation: $x^2-2xy+y^2=\frac{1}{2}xy$. Then we have $(x-y)^2=\frac{1}{2}xy$. Write the second equation in the form $2(x-y)=\frac{1}{2}xy$. Hence, $(x-y)^2-2(x-y)=0$, wherefrom $x-y=0$ and $x-y=2$. We get two systems:

$$(1)\ \begin{cases} x-y=0 \\ xy=0 \end{cases} \quad \text{and} \quad (2)\ \begin{cases} x-y=2 \\ xy=8 \end{cases}$$

Answer: (1) $x=y=0$; (2) $x=4$, $y=2$; (3) $x=-2$, $y=-4$.

232. Rewrite the first equation in the following way:

$$(x^2+2xy+y^2)=13+xy \text{ or } (x+y)^2-13=xy$$

From the second equation: $x+y=4$; substituting, we get $16-13=xy$. Now we solve the system

$$\begin{cases} xy=3 \\ x+y=4 \end{cases}$$

Answer: (1) $x=3$, $y=1$; (2) $x=1$, $y=3$.

233. Solved in the same way as the preceding problem. We get a new system

$$\begin{cases} xy=6 \\ x-y=1 \end{cases}$$

Answer: (1) $x=3$, $y=2$ (2) $x=-2$, $y=-3$

234. Put $\frac{x}{y}=z$; then $\frac{y}{x}=\frac{1}{z}$, and the first equation takes the form $z+\frac{1}{z}=\frac{25}{12}$ or $12z^2-25z+12=0$. Its roots are $z_1=\frac{4}{3}$ and $z_2=\frac{3}{4}$. Now we

have two systems:

$$(1)\quad \begin{cases} \dfrac{x}{y}=\dfrac{4}{3} \\ x^2-y^2=7 \end{cases} \qquad (2)\quad \begin{cases} \dfrac{x}{y}=\dfrac{3}{4} \\ x^2-y^2=7 \end{cases}$$

They are solved by substituting the value of x obtained from the first equation into the second one.

Answer: (1) $x=4$, $y=3$
(2) $x=-4$, $y=-3$
(3) $x=3i$, $y=4i$
(4) $x=-3i$, $y=-4i$

235. The system can be written in the form

$$\begin{cases} x^m y^n = ca^m b^n \\ x^n y^m = da^m b^n \end{cases}$$

Multiply these equations and divide one of them by the other. We get $(xy)^{m+n}=cda^{2m}b^{2n}$ and $\left(\dfrac{x}{y}\right)^{m-n}=\dfrac{c}{d}$; whence

$$xy=(cd)^{\frac{1}{m+n}}a^{\frac{2m}{m+n}}b^{\frac{2n}{m+n}} \quad \text{and} \quad \frac{x}{y}=\left(\frac{c}{d}\right)^{\frac{1}{m-n}}$$

Multiplying these equations, we find

$$x^2=c^{\frac{2m}{m^2-n^2}}d^{\frac{2n}{n^2-m^2}}a^{\frac{2m}{m+n}}b^{\frac{2n}{m+n}}$$

y^2 may be expressed in a similar way proceeding from the equation $\left(\dfrac{y}{x}\right)^{m-n}=\dfrac{d}{c}$. It differs from the corresponding equation for x only in the order in which follow the letters c and d.

Answer: $x=c^{\frac{m}{m^2-n^2}}d^{\frac{n}{n^2-m^2}}a^{\frac{m}{m+n}}b^{\frac{n}{m+n}}$

$y=c^{\frac{n}{n^2-m^2}}d^{\frac{m}{m^2-n^2}}a^{\frac{m}{m+n}}b^{\frac{n}{m+n}}$

236. In the second equation we factor x^3+y^3 into $(x+y)(x^2-xy+y^2)$ and divide the second equation by the first one. We get $x+y=5$. Adding $3xy$ to both members of the first equation, we obtain $(x+y)^2=7+3xy$. Substituting 5 for $(x+y)$ in this equation, we find $xy=6$. We solve now the system

$$\begin{cases} x+y=5 \\ xy=6 \end{cases}$$

Answer: (1) $x=3$, $y=2$
(2) $x=2$, $y=3$

237. Multiplying the second equation by 3 and adding it to the first one, we get $(x+y)^3=1$. Confining ourselves to real solutions, we find $x+y=1$. Substituting 1 for $x+y$ in the second equation, we have $xy=-2$. We solve now the system

$$\begin{cases} x+y=1 \\ xy=-2 \end{cases}$$

Answer: (1) $x=2, \quad y=-1$
(2) $x=-1, \quad y=2$

238. Solved in the same way as the preceding problem.

Answer: (1) $x=3, \quad y=2$
(2) $x=2, \quad y=3$

239. Put $\frac{x+y}{x-y}=z$. The first equation takes the form $z+\frac{1}{z}=5\frac{1}{5}$. Hence, $z=5$ and $z=\frac{1}{5}$, i.e.

$$\frac{x+y}{x-y}=5 \quad \text{and} \quad \frac{x+y}{x-y}=\frac{1}{5}$$

From the equation $\frac{x+y}{x-y}=5$ we find $y=\frac{2}{3}x$. Solve this equation together with the given equation $xy=6$. Use the equation $\frac{x+y}{x-y}=\frac{1}{5}$ in the same way.

Answer: (1) $x=3, \quad y=2$
(2) $x=-3, \quad y=-2$
(3) $x=3i, \quad y=-2i$
(4) $x=-3i, \quad y=2i$

240. Eliminate the unknown z from the given system. To this end (1) subtract the second equation from the first one multiplied by c, and (2) subtract the third equation from the second one multiplied also by c. As a result, we get the following system

$$\begin{cases} (c-a)\,x+(c-b)\,y=(c-d) \\ a\,(c-a)\,x+b\,(c-b)\,y=d\,(c-d) \end{cases}$$

wherefrom we find x and y; z is found in a similar way.

Answer: $x=\frac{(c-d)(b-d)}{(c-a)(b-a)}$; $y=\frac{(a-d)(c-d)}{(a-b)(c-b)}$; $z=\frac{(b-d)(a-d)}{(b-c)(a-c)}$.

241. First eliminate u; for this purpose: (1) multiply the second equation by 2 and add it to the first one; (2) multiply the third equation by (-2) and add it to the second one; (3) multiply the third equation by (-3) and add the result to the fourth one. Finally we obtain the following system

$$\begin{cases} 5x-4y+13z=36 \\ -4x-11y+9z=1 \\ -5x-13y+12z=5 \end{cases}$$

Eliminate x from this system, subtracting the third equation from the second one beforehand. We get

(a) $5x-4y+13z=36$
(b) $x+2y-3z=-4$
(c) $-5x-13y+12z=5$

Add (a) to (c), multiply (b) by 5 and add it to (c). We obtain the following:

$$\begin{cases} -17y+25z=41 \\ -3y-3z=-15 \end{cases} \quad \text{or} \quad \begin{cases} -17y+25z=41 \\ y+z=5 \end{cases}$$

Hence we find $z = 3$ and $y = 2$; x is found from (b) and u, from the third of the given equations.

Answer: $x = 1$; $y = 2$; $z = 3$; $u = 4$.

242. Subtracting the first equation from the second one, we get $y + 2z = 1$. Hence, $y = 1 - 2z$. Substituting this value of y into the first equation, we find $x = z + 3$. Substituting then the found values of x and y into the third equation, we get $3z^2 + z - 2 = 0$. Its roots are $z_1 = \frac{2}{3}$ and $z_2 = -1$. Substituting now the values of z into the equations $x = z + 3$ and $y = 1 - 2z$, we find two values for each of the unknowns x and y.

Answer: (1) $x=\frac{11}{3}$, $y=-\frac{1}{3}$, $z=\frac{2}{3}$

(2) $x=2$, $y=3$, $z=-1$

243. Square the first equation, cube the second one, and square the third one on having transposed the second term to the right member of the equation. And so we get the following system.

$$\begin{cases} 4x+y-3z=-3 \\ 5x+2y+z=1.5 \\ 6x-y-z=0. \end{cases}$$

Answer: $x=\frac{9}{58}$; $y=-\frac{6}{29}$; $z=\frac{33}{29}$.

244. Squaring the first equation and subtracting the second one from it, we obtain $xy + xz + yz = 54$. By virtue of the third equation the first two addends may be replaced by $2yz$. We get $3yz = 54$, i.e.

$$yz = 18 \tag{a}$$

Now the third equation may be written in the form $xy + xz = 2\cdot 18$, i.e.

$$x(y+z) = 36 \tag{b}$$

Since the first equation has the form

$$x + (y+z) = 13 \tag{c}$$

x and $y + z$ may be found from (b) and (c). We get

$$\begin{cases} x=9 \\ y+z=4 \end{cases} \quad \text{or} \quad \begin{cases} x=4 \\ y+z=9 \end{cases}$$

To find y and z separately, make use of (a). Thus we obtain two systems:

$$(1)\ \begin{cases} y+z=4 \\ yz=18 \end{cases} \quad \text{and} \quad (2)\ \begin{cases} y+z=9 \\ yz=18 \end{cases}$$

Note. When squaring the first equation there appears a danger of introducing extraneous roots. But if they had appeared, they would have satisfied the equation $x + y + z = -13$, which contradicts the equation (c).

Answer: (1) $x=9,\quad y=2+i\sqrt{14},\quad z=2-i\sqrt{14}$

(2) $x=9,\quad y=2-i\sqrt{14},\quad z=2+i\sqrt{14}$

(3) $x=4,\quad y=6,\quad z=3$

(4) $x=4,\quad y=3,\quad z=6$

245. Represent the third equation in the form

$$z^2-xz-yz+xy=2$$

Adding it to the second one, we get

$$z^2+2xy=49 \tag{a}$$

whence $z^2=49-2xy$. Substitute this expression into the first equation. We get $(x+y)^2=49$, i.e. $x+y=\pm 7$. First put $x+y=7$.

Represent the second equation in the form

$$xy+z(x+y)=47$$

and substitute into it the expression $xy=\frac{49-z^2}{2}$, obtained from (a), and the value $x+y=7$. We get $z^2-14z+45=0$, whence $z_1=5$ and $z_2=9$. If $z=5$, then $xy=\frac{49-z^2}{2}=12$; but if $z=9$, then $xy=\frac{49-z^2}{2}=-16$. And so we have two systems

$$(1)\ \begin{cases} x+y=7 \\ xy=12 \end{cases} \quad \text{and} \quad (2)\ \begin{cases} x+y=7 \\ xy=-16 \end{cases}$$

each having two solutions. Thus, we obtain four solutions:

(1) $x=3,\quad y=4,\quad z=5$

(2) $x=4,\quad y=3,\quad z=5$

(3) $x=\frac{7+\sqrt{113}}{2},\quad y=\frac{7-\sqrt{113}}{2},\quad z=9$

(4) $x=\frac{7-\sqrt{113}}{2},\quad y=\frac{7+\sqrt{113}}{2},\quad z=9$

Now put $x+y=-7$ and find four more solutions by using the same method.

Answer:

(1) $x=3,\quad y=4,\quad z=5$

(2) $x=4,\quad y=3,\quad z=5$

(3) $x=\frac{7+\sqrt{113}}{2},\quad y=\frac{7-\sqrt{113}}{2},\quad z=9$

(4) $x=\frac{7-\sqrt{113}}{2},\quad y=\frac{7+\sqrt{113}}{2},\quad z=9$

(5) $x=-3,\quad y=-4,\quad z=-5$

(6) $x=-4.\quad y=-3,\quad z=-5$

(7) $x=\frac{-7+\sqrt{113}}{2}, \quad y=\frac{-7-\sqrt{113}}{2}, \quad z=-9$

(8) $x=\frac{-7-\sqrt{113}}{2}, \quad y=\frac{-7+\sqrt{113}}{2}, \quad z=-9$

246. Subtract first the second equation and then the third from the first one. We get

$$(a^3-b^3)+(a^2-b^2)x+(a-b)y=0 \tag{a}$$

$$(a^3-c^3)+(a^2-c^2)x+(a-c)y=0 \tag{b}$$

Reduce the equation (a) by $(a-b)$ and equation (b), by $(a-c)$. We have

$$(a^2+ab+b^2)+(a+b)x+y=0 \tag{c}$$

$$(a^2+ac+c^2)+(a+c)x+y=0 \tag{d}$$

Subtracting (d) from (c), we get

$$(ab-ac+b^2-c^2)+(b-c)x=0$$

Hence,

$$x=-\frac{ab-ac+b^2-c^2}{b-c}=-(a+b+c)$$

The unknown y is found from (c) or (d). Now find z from any of the given equations.

Answer: $x=-(a+b+c)$
$y=ab+bc+ca$
$z=-abc$

247. Putting $\frac{1}{\sqrt{x-1}}=u$ and $\frac{1}{\sqrt{y+\frac{1}{4}}}=v$, we get the following system:

$$\begin{cases} 12u+5v=5 \\ 8u+10v=6 \end{cases}$$

Its roots are:

$$u=\frac{1}{4}, \quad v=\frac{2}{5}, \quad \text{i.e.} \quad \frac{1}{\sqrt{x-1}}=\frac{1}{4}; \quad \frac{1}{\sqrt{y+\frac{1}{4}}}=\frac{2}{5}$$

Hence, $x=17$; $y=6$.

Answer: $x=17$; $y=6$.

248. By virtue of the second equation the first one may be rewritten in the form $10-2\sqrt{xy}=4$. Hence, $xy=9$. We get the system

$$\begin{cases} x+y=10 \\ xy=9 \end{cases}$$

Answer: (1) $x=9$, $y=1$; (2) $x=1$, $y=9$.

249. Put $\sqrt{\frac{3x}{x+y}}=z$. The first equation takes the form $z-2+\frac{1}{z}=0$. Hence, $z=1$, i.e. $\sqrt{\frac{3x}{x+y}}=1$. From this equation we find $y=2x$ and substitute it into the second one.

Answer: (1) $x=6$, $y=12$; (2) $x=-4.5$, $y=-9$.

250. The first equation is reduced to the form $\sqrt[3]{x^2+y^2}=2\sqrt[3]{17}$, whence

$$x^2+y^2=136 \tag{a}$$

Square the second equation to obtain $\sqrt{x^2-y^2}=18-x$, whence

$$y^2=36x-324 \tag{b}$$

Substitute this expression into (a). We get $x^2+36x-460=0$, whence $x=10$ and $x=-46$. Substituting into (b), we find y and obtain four pairs of solutions:

(1) $x=10$, $y=6$; (3) $x=-46$, $y=6\sqrt{55}\,i$

(2) $x=10$, $y=-6$; (4) $x=-46$, $y=-6\sqrt{55}\,i$

The third and fourth pairs of solutions do not suit, since the expressions $\sqrt{x+y}$ and $\sqrt{x-y}$, where the radicals must mean arithmetical values of the root (otherwise they are indefinite since the root has two values), make no sense at complex values of $x+y$ and $x-y$. The first and second pairs of solutions should be checked.

Answer: (1) $x=10$, $y=6$; (2) $x=10$, $y=-6$.

251. The system makes sense only if $a \geqslant 0$ (see the preceding explanation). Square the first equation:

$$\sqrt{x^2-y^2}=8a-x \tag{a}$$

Substituting this expression into the second equation, we get

$$\sqrt{x^2+y^2}=(\sqrt{41}+5)\,a-x \tag{b}$$

Square the equations (a) and (b):

$$y^2=-64a^2+16ax \tag{a'}$$

$$y^2=(\sqrt{41}+5)^2\,a^2-2\,(\sqrt{41}+5)\,ax \tag{b'}$$

Eliminating y from (a′) and (b′), we get

$$(130+10\sqrt{41})\,a^2=(26+2\sqrt{41})\,ax$$

whence $x=5a$. From (a′) we find $y=\pm 4a$ and then perform a check.

Answer: (1) $x=5a$, $y=4a$; (2) $x=5a$, $y=-4a$.

252. Square the first equation: $2x^2-2\sqrt{x^4-y^4}=y^2$. Substitute here the value $x^4-y^4=144a^4$ from the second equation. We get

$$y^2=2x^2-24a^2 \tag{a}$$

wherefrom we find y^4 and substitute it into the second of the given equations. We get

$$x^4-32a^2x^2+240a^4=0$$

Hence, $x=\pm\sqrt{20}\,a$ and $x=\pm\sqrt{12}\,a$. We find y from the equation (a). For each of the values $x=\pm\sqrt{20}\,a$ we have $y=\pm 4a$, and for each of the values $x=\pm\sqrt{12}\,a$ we have $y=0$. A check shows that out of the six pairs of obtained roots some are extraneous for $a>0$, others, for $a<0$. Let us take, for instance, the pair $x=\sqrt{20}\,a$, $y=4a$. Substituting it into the first equation, we find

$\sqrt{36a^2}-\sqrt{4a^2}=4a$, i.e. $6|a|-2|a|=4a$. This equality is an identity for $a\geqslant 0$, but it does not hold true for $a<0$.

Answer: For $a\geqslant 0$ the solutions are:

(1) $x=\sqrt{20}\,a,\quad y=4a$; (2) $x=-\sqrt{20}\,a,\quad y=4a$
(3) $x=\sqrt{12}\,a,\quad y=0$; (4) $x=-\sqrt{12}\,a,\quad y=0$
For $a<0$ the solutions are:
(5) $x=\sqrt{20}\,a,\quad y=-4a$; (6) $x=-\sqrt{20}\,a,\quad y=-4a$.

253. *First method.* From the second equation we find $x+y=14-\sqrt{xy}$. Squaring it, we get

$$x^2+y^2+2xy=196+xy-28\sqrt{xy}$$

whence

$$x^2+y^2+xy=196-28\sqrt{xy}$$

By virtue of the first equation we have $84=196-28\sqrt{xy}$, whence $\sqrt{xy}=4$, i.e. $xy=16$. Substituting $\sqrt{xy}=4$ into the second equation, we find $x+y=10$, and then solve the system

$$\begin{cases} x+y=10 \\ xy=16 \end{cases}$$

Second method. Factorize the left member of the first equation:

$$x^2+xy+y^2=(x+y)^2-(\sqrt{xy})^2=(x+y+\sqrt{xy})(x+y-\sqrt{xy})=84$$

Hence, by virtue of the second equation, we get

$$14(x+y-\sqrt{xy})=84$$

i.e. $x+y-\sqrt{xy}=6$. From the system

$$\begin{cases} x+y-\sqrt{xy}=6 \\ x+y+\sqrt{xy}=14 \end{cases}$$

we may find $x+y$ and $\sqrt{xy}$.

Answer: (1) $x=2,\quad y=8$
(2) $x=8,\quad y=2$

253a. From the first equation we find $y=\frac{x-m}{1+mx}$, from the second $y=-\frac{2+x}{1+x}$; equating these expressions, we get $\frac{x-m}{1+mx}=-\frac{2+x}{1+x}$; hence, we have the following equation:

$$(1+m)x^2+(2+m)x+(2-m)=0$$

This equation has real roots, provided

$$(2+m)^2-4(1+m)(2-m)\geqslant 0$$

Simplifying the left member, we get the expression $5m^2-4\geqslant 0$, whence $|m|\geqslant\frac{2}{\sqrt{5}}$. Under this condition x has real values, which means that $y=-\frac{2+x}{1+x}$ also has real values.

Answer: $|m|\geqslant\frac{2}{\sqrt{5}}$.

CHAPTER IV

LOGARITHMIC AND EXPONENTIAL EQUATIONS

Preliminaries

To solve equations containing logarithms to different bases (see, for example, Problems 267, 268, 309 to 313) it may turn out to be convenient to reduce all the logarithms to one base. So, let us introduce some relevant formulas supplied with necessary explanations.

1. The formula

$$\log_b a = \frac{1}{\log_a b} \qquad \text{(a)}$$

enables us to change the roles of the logarithmic base and the number.

Example.

$$\log_8 2 = \frac{1}{\log_2 8} = \frac{1}{3}$$

Explanation. According to the definition of the logarithm, $\log_2 8$ is the exponent indicating the power to which it is necessary to raise the base 2 to obtain the number 8. Thus, symbolically, $\log_2 8 = 3$ is equivalent to $2^3 = 8$. But the last equality may be written in a different way: $\sqrt[3]{8} = 2$, i.e. $8^{\frac{1}{3}} = 2$. Hence, $\log_8 2 = \frac{1}{3}$.

In general, the equality $a^x = b$ may be written as $b^{\frac{1}{x}} = a$. The former equality means that $\log_a b = x$, the latter, that $\log_b a = \frac{1}{x}$, wherefrom the formula (a) is derived.

2. The formula (a) is a particular case of the general formula

$$\log_a N = \frac{\log_b N}{\log_b a} \qquad \text{(b)}$$

which expresses the following important fact: if we know the logarithms of various numbers to the base b, we can find the logarithms of the same numbers to the base a; to this effect it is sufficient to divide the former by $\log_b a$ (i.e. by the logarithm of the new base to the old one). Instead of dividing $\log_b N$ by $\log_b a$ we may [by virtue of (a)] multiply it by $\log_a b$:

$$\log_a N = \log_a b \cdot \log_b N \qquad \text{(c)}$$

The number by which logarithms in one system are multiplied to give logarithms in a second system is called the *modulus of the second system* with respect to the first. That is to say, the factor $\log_a b$ is the modulus of the system of logarithms to the base a with respect to the system of logarithms to the base b.

Example. Having a table of common logarithms, we can compile a table of logarithms to the base 2. To this end it is sufficient to perform division by

$\log 2 = 0.3010$ or multiplication by $\log_2 10 = \frac{1}{0.3010} = 3.322$. Thus,

$$\log_2 3 = \frac{\log 3}{\log 2} = \frac{0.4771}{0.3010} = 1.585$$

Explanation. By the definition of the logarithm we have $2^{\log_2 3} = 3$. Take the logarithms of this equality to the base 10. We get $\log_2 3 \cdot \log 2 = \log 3$, whence $\log_2 3 = \frac{\log 3}{\log 2}$. Just in the same way we obtain the formula (b) from the identity $a^{\log_a N} = N$ by taking the logarithms to the base b.

In order not to confuse the notations, it is advisable to use the following method for a check: write the fraction $\frac{b}{a}$ instead of the expression $\log_a b$ (of course, these expressions are not equal to each other); treat the expressions $\log_b a$, $\log_a N$ and so on in a similar way. Then instead of the formulas (a), (b), (c) we get other formulas, which are also true. Thus, instead of (c) we get

$$\frac{N}{a} = \frac{b}{a} \cdot \frac{N}{b}$$

254. *First method*

$$x = 10 \cdot 10^{2\left(\frac{1}{2}\log 9 - \log 2\right)} = 10 \cdot 10^{\log 9 - 2\log 2} = 10 \cdot 10^{\log \frac{9}{4}}$$

By definition, $10^{\log \frac{9}{4}} = \frac{9}{4}$, therefore $x = 10 \cdot \frac{9}{4} = 22.5$.

Answer: $x = 22.5$.

Second method. Taking the logarithms, we have

$$\log x = \log 10 + \left(\frac{1}{2}\log 9 - \log 2\right)\log 100$$

or

$$\log x = \log 10 + \log 9 - 2\log 2 = \log \frac{10 \cdot 9}{2^2}$$

Answer: $x = 22.5$.

255. As in Problem 254 (second method), we have

$$\log x = \left(\frac{1}{2} - \frac{1}{4}\log 4\right)\log 100$$

$$\log x = 1 - \frac{1}{2}\log 4 = \log \frac{10}{\sqrt{4}}; \quad x = \frac{10}{\sqrt{4}}$$

Answer: $x = 5$.

256. Proceeding in the same way as in the previous problems, we have

$$\log x = \frac{1}{2}\left(2 + \frac{1}{2}\log 16\right)\log 10 = 1 + \frac{1}{4}\log 16 = \log\left(10\sqrt[4]{16}\right);$$

$$x = 10\sqrt[4]{16}$$

Answer: $x = 20$.

257. *First method*

$$x=7^{2-2\log_7 2}+\frac{1}{5^{\log_5 4}}=\frac{7^2}{7^{\log_7 4}}+\frac{1}{4}=\frac{49}{4}+\frac{1}{4}=\frac{25}{2}$$

(cf. solution of Problem **254** by the first method).

Second method

Let us denote $y=49^{1-\log_7 2}$, and $z=5^{-\log_5 4}$; then

$$x=y+z$$

Taking the logarithms, we find that $\log_7 y=(1-\log_7 2)\log_7 49$, or

$$\log_7 y=(\log_7 7-\log_7 2)\,2=2\log_7\frac{7}{2}=\log_7\frac{49}{4}$$

whence $y=\frac{49}{4}$; similarly, we find that $z=\frac{1}{4}$. Hence, $x=\frac{25}{2}$.

Answer: $x=\frac{25}{2}$.

258. We have $\log_4\log_3\log_2 x=\log_4 1$, whence $\log_3\log_2 x=1$; $\log_2 x=3$.

Answer: $x=8$.

259. Like in the preceding problem, we have

$$1+\log_b[1+\log_c(1+\log_p x)]=1$$
$$\log_b[1+\log_c(1+\log_p x)]=0$$

then

$$1+\log_c(1+\log_p x)=1;\quad \log_c(1+\log_p x)=0$$
$$1+\log_p x=1;\qquad \log_p x=0;\quad x=1$$

Answer: $x=1$.

260. The expression in braces must be a positive number since a negative number has no (real) logarithm to base 4. Therefore, having rewritten the given equation in the form

$$2\log_3[1+\log_2(1+3\log_2 x)]=4^{\frac{1}{2}}=\sqrt{4}$$

we should take only the positive value of $\sqrt{4}$, i.e. 2. Applying similar transformations for the second time, we then obtain

$$\log_3[1+\log_2(1+3\log_2 x)]=1,\quad 1+\log_2(1+3\log_2 x)=3,$$
$$\log_2(1+3\log_2 x)=2$$

hence, $1+3\log_2 x=4$, $\log_2 x=1$.

Answer: $x=2$.

261. Represent the given equation in the form $\log_2(x+14)(x+2)=6$, or $(x+14)(x+2)=2^6=64$, whence $x^2+16x-36=0$, $x_1=2$, $x_2=-18$. The second root does not suit, since the left member contains the expressions $\log_2(x+14)$ and $\log_2(x+2)$, which have no real value at a negative x.

Answer: $x=2$.

262. Represent the given equation in the form

$$\log_a[y(y+5)\cdot 0.02]=0$$

hence,

$$y\,(y+5)\cdot 0.02=1 \text{ or } y^2+5y-50=0$$

we get two roots: $y_1=5$, $y_2=-10$. The second root does not suit (see solution of the preceding problem).

Answer: $y=5$.

263. We have

$$\log(35-x^3)=3\log(5-x) \text{ or } \log(35-x^3)=\log(5-x)^3$$

hence,

$$35-x^3=(5-x)^3 \text{ or } x^2-5x+6=0$$

Answer: $x_1=2$, $x_2=3$.

264. Transforming the bracketed expression, we get

$$b-\frac{(3a-b)\,(a^2+ab)^{-1}}{b^{-2}}=\frac{b\,(a-b)^2}{a\,(a+b)}$$

Then the given equation takes the form

$$1+\log x=\frac{1}{3}\log\frac{b\,(a-b)^2}{a\,(a+b)}-\frac{4}{3}\log b+\frac{1}{3}\log[a\,(a+b)\,(a-b)]$$

Applying the theorem on the logarithm of a product (and of a fraction) to the right member, we obtain

$$1+\log x=\log(a-b)-\log b$$

Substituting log 10 for unity, rewrite the equation in the form

$$\log 10+\log x=\log(a-b)-\log b \quad \text{or} \quad \log(10x)=\log\frac{a-b}{b}$$

hence, $10x=\dfrac{a-b}{b}$.

Answer: $x=\dfrac{a-b}{10b}$.

265. The given equation may be represented in the form

$$\log\left(x-\frac{a}{\sqrt{1-a}}\right)=\log\sqrt{1+\frac{1}{a}}+\log\sqrt{\frac{a\,(1-a)}{1+a}}$$

wherefrom, taking antilogarithms, we find

$$x-\frac{a}{\sqrt{1-a}}=\sqrt{1+\frac{1}{a}}\sqrt{\frac{a\,(1-a)}{1+a}}$$

or

$$x-\frac{a}{\sqrt{1-a}}=\sqrt{1-a}$$

whence

$$x=\frac{1}{\sqrt{1-a}}$$

Answer: $x=\dfrac{1}{\sqrt{1-a}}$.

266. The given equation may be written in a different way:

$$\frac{1}{2}\log_x 5+\log_x 5+\log_x x-2.25=\left(\frac{1}{2}\log_x 5\right)^2$$

since $\log_x x = 1$, after simplifications we get

$$\log_x^2 5 - 6\log_x 5 + 5 = 0$$

Solving the quadratic equation (in the unknown $\log_x 5$), we find two roots: $\log_x 5 = 5$ and $\log_x 5 = 1$.

Answer: $x_1=\sqrt[5]{5}$; $x_2=5$.

267. *First method.* Putting $\log_{16} x = z$, we have $x = 16^z$, hence,

$$\log_4 x = z\log_4 16 = 2z \text{ and } \log_2 x = z\log_2 16 = 4z$$

The given equation takes the form $z + 2z + 4z = 7$, i.e. $z = 1$.

Second method. Reduce all the logarithms to the base 2 by the formula (b) (page 142). We find $\log_4 x=\frac{\log_2 x}{\log_2 4}=\frac{\log_2 x}{2}$; similarly, $\log_{16} x=\frac{\log_2 x}{4}$. We get the equation $\frac{1}{4}\log_2 x+\frac{1}{2}\log_2 x+\log_2 x=7$, whence $\log_2 x=4$.

Answer: $x=16$.

268. Solved in the same way as the preceding problem.

Answer: $x=a$.

269. Rewrite the given equation in the form

$$\left(\frac{3}{7}\right)^{3x-7}=\left(\frac{3}{7}\right)^{3-7x}$$

whence $3x-7=3-7x$.

Answer: $x=1$.

270. Represent the given equation in the form

$$7\cdot 3^{x+1}-3^{x+4}=5^{x+2}-5^{x+3}$$

Factoring out 3^x and 5^x, we have

$$3^x(7\cdot 3-3^4)=5^x(5^2-5^3) \quad \text{or} \quad \left(\frac{3}{5}\right)^x=\frac{5}{3}$$

whence $x=-1$.

Answer: $x=-1$.

271. Rewrite the given equation in the form

$$2^{-3}\cdot 2^{4x-6}=\frac{2^{-\frac{x}{2}}}{2^{-3x}} \quad \text{or} \quad 2^{4x-9}=2^{\frac{5}{2}x}$$

Hence,

$$4x-9=\frac{5}{2}x$$

Answer: $x=6$.

272. The given equation may be written as

$$2^{-x^2}\cdot 2^{2x+2}=2^{-6} \quad \text{or} \quad 2^{-x^2+2x+2}=2^{-6}$$

Consequently, $-x^2+2x+2=-6$.

Answer: $x_1=4$; $x_2=-2$.

273. Represent the given equation in the form

$$2^{\frac{5(x+5)}{x-7}}=2^{-2}\cdot 2^{\frac{7(x+17)}{x-3}}$$

whence

$$\frac{5(x+5)}{x-7}=-2+\frac{7(x+17)}{x-3}$$

Answer: $x=10$.

274. Since $\frac{\log 4}{\log 8}=\frac{2\log 2}{3\log 2}=\frac{2}{3}$, the given equation may be rewritten in the following way:

$$\left(\frac{2}{3}\right)^{2x}\left(\frac{2}{3}\right)^{(1-x)3}=\frac{2}{3}$$

Hence,

$$2x+3(1-x)=1$$

Answer: $x=2$.

275. Represent the given equation in the form

$$2^{\left(1+\frac{\sqrt{x}+3}{2\sqrt{x}}\right)\frac{2}{\sqrt{x}-1}}=2^2,$$

equating the exponents, we find

$$\frac{3\sqrt{x}+3}{\sqrt{x}(\sqrt{x}-1)}=2, \quad \text{or} \quad 2x-5\sqrt{x}-3=0$$

Let us denote $\sqrt{x}$ by z; then we have

$$2z^2-5z-3=0, \quad \text{whence } z_1=3,\ z_2=-\frac{1}{2}$$

But the second root does not suit, because the quantity z (which represents the arithmetical value of the radical $\sqrt{x}$) must be positive. Thus, we have $z=\sqrt{x}$; hence $x=9$.

Answer: $x=9$.

276. The given equation may be represented in the form

$$2^{1+\frac{\sqrt{x}+3}{2\sqrt{x}}}=2^{\frac{4}{\sqrt{x}-1}}$$

Hence,

$$1+\frac{\sqrt{x}+3}{2\sqrt{x}}=\frac{4}{\sqrt{x}-1}$$

therefrom $3x-8\sqrt{x}-3=0$. Putting $\sqrt{x}=z$, we have $3z^2-8z-3=0$; $z_1=3$; $z_2=-\frac{1}{3}$; the second root $z_2=-\frac{1}{3}$ does not suit (see solution of Problem 275). Consequently, $x=9$.

Answer: $x=9$.

277. The given equation may be written as

$$a^{\frac{3}{x^2-1}+\frac{1}{2x-2}-\frac{1}{4}}=a^0$$

Consequently,

$$\frac{3}{x^2-1}+\frac{1}{2x-2}-\frac{1}{4}=0$$

After simplifications we get $x^2-2x-15=0$.

Answer: $x_1=5$; $x_2=-3$.

278. Using the formula (a) (page 142), we obtain

$$\frac{3}{\log_x x+2\log_x a}+\frac{1}{2\left(\log_x x-\frac{1}{2}\log_x a\right)}=2$$

or

$$\frac{3}{1+2\log_x a}+\frac{1}{2-\log_x a}=2$$

Solving for $\log_x a$, we get

$$\log_x a=\frac{7\pm\sqrt{49-48}}{8}=\frac{7\pm 1}{8}$$

Answer: $x_1=a$; $x_2=a^{\frac{4}{3}}$.

279. By formula (b) (page 142) we find

$$\log_x 2=\frac{\log_4 2}{\log_4 x}=\frac{1}{2\log_4 x}$$

Then the given equation takes the form $\log_4(x+12)=2\log_4 x$, whence $x+12=x^2$. We take only the positive root $x=4$; at negative x the expression $\log_x 2$ has no real value.

Answer: $x=4$.

280. Write the given equation in the form:

$$(\log_x 5+2)\log_5^2 x=1$$

Since $\log_x 5=\frac{1}{\log_5 x}$, we get the equation

$$\left(\frac{1}{\log_5 x}+2\right)\log_5^2 x=1$$

Solving it for $\log_5 x$, we find

$$(\log_5 x)_1=\frac{1}{2} \quad\text{and}\quad (\log_5 x)_2=-1$$

Answer: $x_1=\sqrt{5}$; $x_2=\frac{1}{5}$.

281. The left member of the equation is the sum of $x + 1$ terms of a geometric progression, and therefore we have (for $a \neq 1$)

$$\frac{1-a^{x+1}}{1-a}=(1+a)(1+a^2)(1+a^4)(1+a^8)$$

or

$$1-a^{x+1}=(1-a)(1+a)(1+a^2)(1+a^4)(1+a^8)$$

or

$$1-a^{x+1}=1-a^{16}$$

whence $a^{x+1} = a^{16}$; $x + 1 = 16$; $x = 15$. At $a = 1$ the general formula for the sum of the terms of a geometric progression is not applicable. In this case the left member of the given equation is the sum of $x + 1$ addends, each being equal to 1, and so, the equation takes the form $x + 1 = 16$, hence, $x = 15$.

Answer: $x = 15$.

282. Rewrite the given equation in the form

$$5^{2+4+6+\ldots+2x}=5^{56}$$

whence

$$2+4+6+\ldots+2x=56 \text{ or } 1+2+3+\ldots+x=28$$

The left member of the equation is the sum of the terms of an arithmetic progression. Therefore we get the equation

$$\frac{(1+x)\,x}{2}=28$$

whence $x_1 = 7$, $x_2 = -8$. The second root does not suit, since the number x must be a positive integer.

Answer: $x = 7$.

283. Rewrite the given equation in the form

$$2^{2x}2^{-4}-17\cdot 2^{x}2^{-4}+1=0$$

Denoting 2^x by z, we get

$$z^2-17z+16=0; \quad z_1=16; \quad z_2=1$$

whence $x_1 = 4$; $x_2 = 0$.

Answer: $x_1 = 4$; $x_2 = 0$.

284. As in the preceding problem, putting $4^x = z$, we have $2z^2 - 17z + 8 = 0$.

Answer: $x_1=\frac{3}{2}$; $x_2=-\frac{1}{2}$.

285. Putting $9^{\frac{1}{x}}=z$, we obtain the equation

$$3z^2-10z+3=0$$

Answer: $x_1 = 2$; $x_2 = -2$.

286. Taking the logarithms of the given equation (to the base 10), we obtain

$$\frac{\log x+7}{4}\log x=\log x+1 \quad \text{or} \quad \log^2 x+3\log x-4=0$$

whence $\log x_1 = 1$; $\log x_2 = -4$.

Answer: $x_1 = 10$; $x_2 = 0.0001$.

287. Transform the given equation so that either of its members represents the logarithm of a certain expression. For this purpose substitute log 10 for unity in the left member of the equation. Now the given equation may be written in the form

$$\log \frac{4^{-1}2^{\sqrt{x}}-1}{10}=\log \frac{\sqrt{2^{\sqrt{x}-2}}+2}{2^2}$$

Since the logarithms are equal, the numbers are also equal

$$\frac{4^{-1}2^{\sqrt{x}}-1}{10}=\frac{\sqrt{2^{\sqrt{x}-2}}+2}{4}$$

After simplifications we get the equation

$$2^{\sqrt{x}}-5\cdot 2^{\frac{\sqrt{x}}{2}}-24=0$$

Since $2^{\sqrt{x}}=(2^{\frac{\sqrt{x}}{2}})^2$, then putting $2^{\frac{\sqrt{x}}{2}}=z$, we have the equation $z^2-5z-24=0$, whose roots are $z_1=8$ and $z_2=-3$. Taking $z_1=8$, we get the equation $2^{\frac{\sqrt{x}}{2}}=8$, whence $\frac{\sqrt{x}}{2}=3$, i.e. $x=36$.

The second root $z=-3$ leads to the equation $2^{\frac{\sqrt{x}}{2}}=-3$ which has no solutions since no power of the positive number 2 can be a negative number.

Answer: $x=36$.

288. Find successively (see solution of the preceding problem):

$$2\log\frac{2}{10}+\log(5^{\sqrt{x}}+1)=\log\left(\frac{5}{5^{\sqrt{x}}}+5\right)$$

$$\log\left[\left(\frac{1}{5}\right)^2(5^{\sqrt{x}}+1)\right]=\log\left(\frac{5(1+5^{\sqrt{x}})}{5^{\sqrt{x}}}\right)$$

hence,

$$\frac{1}{25}(5^{\sqrt{x}}+1)=\frac{5(1+5^{\sqrt{x}})}{5^{\sqrt{x}}} \qquad \text{(A)}$$

After simplification we get

$$5^{2\sqrt{x}}-124\cdot 5^{\sqrt{x}}-125=0$$

whence $5^{\sqrt{x}}=125$, or $5^{\sqrt{x}}=-1$. The second equation has no solution; the first one gives $\sqrt{x}=3$; $x=9$.

The equation (A) may be solved in a different way. It may be reduced by $5^{\sqrt{x}}+1\neq 0$, and then we get $\frac{1}{25}=\frac{5}{5^{\sqrt{x}}}$; hence, $5^{\sqrt{x}}=125$ and $x=9$.

Answer: $x=9$.

289. Represent the given equation in the form

$$5^{\log x}+5^{\log x-1}=3^{\log x+1}+3^{\log x-1}$$

Factoring out $5^{\log x}$ and $3^{\log x}$, we have

$$5^{\log x}(1+5^{-1})=3^{\log x}(3+3^{-1})$$

or

$$\frac{5^{\log x}}{3^{\log x}}=\frac{25}{9};\ \left(\frac{5}{3}\right)^{\log x}=\left(\frac{5}{3}\right)^{2}$$

whence $\log x=2$.

Answer: $x=100$.

290. Taking the logarithms to the base 10, we get

$$2\log^4 x-1.5\log^2 x=\frac{1}{2}$$

This biquadratic equation (in the unknown $\log x$) has two real roots: $\log x = 1$ and $\log x = -1$; hence, $x_1 = 10$, $x_2 = 0.1$.

Answer: $x_1 = 10$; $x_2 = 0.1$.

291. Taking antilogarithms, we obtain

$$64\sqrt[24]{2^{x^2-40x}}=1, \quad \text{or} \quad 2^{x^2-40x}=\left(\frac{1}{64}\right)^{24},$$

i.e. $2^{x^2-40x}=2^{-6\cdot 24}$; hence, $x^2-40x+144=0$.

Answer: $x_1=36$; $x_2=4$.

292. By the definition of the logarithm, the given equation is equivalent to $9-2^x=2^{3-x}$, or $9-2^x=\frac{2^3}{2^x}$, whence $2^{2x}-9\cdot 2^x+8=0$. Solving this equation (quadratic equation in the unknown 2^x), we find

$$x_1=3; \quad x_2=0$$

Answer: $x_1=3$; $x_2=0$.

293. As in Problem 288, we get

$$2(4^{x-2}+9)=10(2^{x-2}+1)$$

Noting that

$$2^{x-2}=2^x\cdot 2^{-2}=\frac{1}{4}\cdot 2^x, \quad \text{and} \quad 4^{x-2}=4^x\cdot 4^{-2}=\frac{1}{16}\cdot 4^x$$

we obtain the equation

$$2^{2x}-20\cdot 2^x+64=0$$

whence, like in the preceding problem, we find $x_1=4$; $x_2=2$.

Answer: $x_1=4$; $x_2=2$.

294. It is convenient to transpose the last term to the right. Then, as in Problem 288, we get $4\cdot 3^{1+\frac{1}{2x}}=3^{\frac{1}{x}}+27$. Noting that $3^{1+\frac{1}{2x}}=3\cdot 3^{\frac{1}{2x}}$, we get the equation

$$12\cdot 3^{\frac{1}{2x}}=3^{\frac{1}{x}}+27$$

Putting $3^{\frac{1}{2x}}=z$, we have $3^{\frac{1}{x}}=(3^{\frac{1}{2x}})^2$, and so, we get the equation $z^2-12z+27=0$, with $z_1=9$; $z_2=3$ being its roots.

Answer: $x_1=\frac{1}{4}$; $x_2=\frac{1}{2}$.

295. Taking antilogarithms (cf. solution of Problem 288), we have

$$\frac{3^{\sqrt{4x+1}}-2^{4-\sqrt{4x+1}}}{100}=\frac{\sqrt[4]{16}}{4^{\sqrt{x+0.25}}}$$

The equation may be represented in the form

$$\frac{1}{100}\left(3^{\sqrt{4x+1}}-\frac{16}{2^{\sqrt{4x+1}}}\right)=\frac{2}{2^{\sqrt{4x+1}}}$$

Getting rid of the denominator, we obtain

$$6^{\sqrt{4x+1}}-16=200, \text{ i.e. } 6^{\sqrt{4x+1}}=6^3,$$

whence $x=2$.

Answer: $x=2$.

296. Represent the given equation in the form

$$4\log 2+2\log(x-3)=\log(7x+1)+\log(x-6)+\log 3$$

whence, taking antilogarithms, we find

$$2^4(x-3)^2=3(7x+1)(x-6)$$

The roots of this quadratic equation are $x_1=9$; $x_2=-3.6$. The second root does not suit, since it yields $x-3=-6.6$, which means that the expression $\log(x-3)$ has no real value [the same can be stated about the expressions $\log(7x+1)$ and $\log(x-6)$].

Answer: $x=9$.

297. Represent the right member in the form

$$-\log_5(0.2-0.2\cdot5^{x-3})=-\log_5 0.2-\log_5(1-5^{x-3})$$

Represent the addend $(x-3)$ in the form $\log_5 5^{x-3}$. Transposing the terms, we get the equation

$$\log_5 120+\log_5 5^{x-3}+\log_5 0.2=2\log_5(1-5^{x-3})-\log_5(1-5^{x-3})$$

or

$$120\cdot0.2\cdot5^{x-3}=1-5^{x-3}$$

Answer: $x=1$.

298. The given equations may be represented in the form

$$\begin{cases} 2^{6x+3}=2^{4y+4} \\ 5^{1+x-y}=5^{\frac{4y+2}{2}} \end{cases}$$

Equating the exponents, we get the following system

$$\begin{cases} 6x-4y=1 \\ x-3y=0 \end{cases}$$

Answer: $x=\frac{3}{14}$; $y=\frac{1}{14}$.

299. Taking antilogarithms of the first equation, we get the following system of equations:

$$\begin{cases} xy=1 \\ x+y=\frac{10}{3} \end{cases}$$

Answer: $x_1=3$; $y_1=\frac{1}{3}$; $x_2=\frac{1}{3}$; $y_2=3$.

300. In algebra, consideration is usually given only to the logarithms of positive numbers to positive bases, otherwise a number may have no (real) logarithm. Therefore, we consider the known quantities a and b (logarithmic bases) to be positive; the unknown quantities x, y ("numbers") must also be positive.

Taking antilogarithms, we find

$$xy=a^2, \quad \frac{x}{y}=b^4$$

The system has two solutions:

$$(1)\ x=ab^2, \qquad y=\frac{a}{b^2}$$

$$(2)\ x=-ab^2, \qquad y=-\frac{a}{b^2}$$

But the second solution does not suit, since at positive values of a and b, it yields negative values of x and y.

Answer: $x=ab^2$; $y=\frac{a}{b^2}$.

301. Taking antilogarithms, we get the system

$$\frac{x^2+y^2}{10}=13, \qquad \frac{x+y}{x-y}=8$$

from the second equation we find $y=\frac{7}{9}x$; substituting it into the first equation, we have two solutions:

$$(1)\ x_1=9, \quad y_1=7; \qquad (2)\ x_2=-9, \quad y_2=-7$$

The second solution does not fit, since it yields $x+y<0$ and $x-y<0$ (see solution of Problem 300).

Answer: $x=9$; $y=7$.

302. Taking antilogarithms, we have

$$\begin{cases} x-y=xy \\ x+y=1 \end{cases}$$

This system has two solutions:

$$x_1=\frac{-1+\sqrt{5}}{2}, \qquad y_1=\frac{3-\sqrt{5}}{2}$$

$$x_2=\frac{-1-\sqrt{5}}{2}, \qquad y_2=\frac{3+\sqrt{5}}{2}$$

The first solution yields

$$x-y=xy=-2+\sqrt{5}>0$$

The second one gives us

$$x-y=xy=-2-\sqrt{5}<0$$

The second solution does not suit, since the base of the logarithms xy must be positive (see Problem 300).

Answer: $x=\dfrac{-1+\sqrt{5}}{2}$; $y=\dfrac{3-\sqrt{5}}{2}$.

303. Taking antilogarithms, we get the following system

$$1+\frac{x}{y}=\frac{a^2}{y}; \qquad xy=b^4$$

or

$$\begin{cases} x+y=a^2 \\ xy=b^4 \end{cases}$$

This system has two solutions:

$$(1)\quad x_1=\frac{a^2+\sqrt{a^4-4b^4}}{2}, \qquad y_1=\frac{a^2-\sqrt{a^4-4b^4}}{2}$$

$$(2)\quad x_2=\frac{a^2-\sqrt{a^4-4b^4}}{2}, \qquad y_2=\frac{a^2+\sqrt{a^4-4b^4}}{2}$$

Considering the given quantities a and b to be positive (as the logarithmic bases), we must distinguish between the following two cases:

(1) $a^4<4b^4$, i.e. $a<\sqrt{2}b$, and (2) $a^4\geqslant 4b^4$, i.e. $a\geqslant\sqrt{2}b$. In the first case the system has no solution, since x and y are imaginary numbers. In the second case x and y are not only real, but also positive, since both the sum $x+y=a^2$ and the product $xy=b^4$ are positive.

Answer: $x=\dfrac{a^2+\sqrt{a^4-4b^4}}{2}$; $y=\dfrac{a^2-\sqrt{a^4-4b^4}}{2}$.

304. Taking antilogarithms of the first equation, we obtain the system

$$\begin{cases} 4xy=9a^2 \\ x+y=5a \end{cases}$$

Both solutions are suitable.

Answer: (1) $x_1=\dfrac{a}{2}$, $y_1=\dfrac{9}{2}a$; (2) $x_2=\dfrac{9}{2}a$, $y_2=\dfrac{a}{2}$.

305. Since in the second equation the unknowns x and y are preceded by the logarithm symbols, both of them are positive (if a solution exists). As far as the quantity a is concerned, it may be negative as well (since the logarithm symbol is followed by the positive number a^2). But in this case it should be written $\log(a^2)=2\log|a|$ instead of the equality $\log(a^2)=2\log a$. For the sake of brevity let us denote $\log x=X$; $\log y=Y$; $\log|a|=A$. Taking logarithms of the first equation in the given system, we get the following system

$$X+Y=2A,\ X^2+Y^2=10A^2$$

Squaring the first equation and subtracting the second from it, we get $XY = -3A^2$. Thus, we have an equivalent system

$$X + Y = 2A, \quad XY = -3A^2.$$

Consequently, X and Y are the roots of the equation $z^2 - 2Az - 3A^2 = 0$. Hence, one solution is $X = 3A$, $Y = -A$, i.e. $x = |a|^3$, $y = \frac{1}{|a|}$. The other solution is $x = \frac{1}{|a|}$, $y = |a|^3$.

A check shows that both solutions suit.

Answer: $x_1 = |a|^3$, $y_1 = \frac{1}{|a|}$; $x_2 = \frac{1}{|a|}$, $y_2 = |a|^3$.

306. From the second equation we have $y - x = (\sqrt{2})^4 = 4$. Hence, $y = x + 4$. Substituting it into the first equation, we get $3^x \cdot 2^{x+4} = 576$ or $6^x \cdot 2^4 = 576$.

Answer: $x = 2$; $y = 6$.

307. The given system may be written as

$$\begin{cases} xy = a \\ \left(\frac{x}{y}\right)^2 = b \end{cases}$$

Since x and y must be positive, we get the following system

$$\begin{cases} xy = a \\ \frac{x}{y} = \sqrt{b} \end{cases}$$

Answer: $x = \sqrt{a}\sqrt[4]{b}$; $y = \frac{\sqrt{a}}{\sqrt[4]{b}}$

308. The given system may be written in the form

$$\log_a x + \frac{1}{2}\log_a y = \frac{3}{2}, \qquad \frac{1}{2}\log_b x + \log_b y = \frac{3}{2}$$

whence

$$x\sqrt{y} = a^{\frac{3}{2}}, \quad \sqrt{xy} = b^{\frac{3}{2}}$$

Multiplying them, we have $x^{\frac{3}{2}}y^{\frac{3}{2}} = a^{\frac{3}{2}}b^{\frac{3}{2}}$ or $xy = ab$. Divide the last equation by each of the previous ones.

Answer: $x = \frac{a^2}{b}$; $y = \frac{b^2}{a}$.

309. The solution is similar to the preceding one.

Answer: $x = a\sqrt[3]{b^2}$; $y = \frac{a}{b\sqrt[3]{b}}$.

310. Using the formula (a) (page 142), write the first equation as

$$\log_v u + \frac{1}{\log_v u} = 2, \text{ whence } \log_v u = 1,$$

i.e. $u = v$. Substituting it into the second equation, we have $u^2 + u - 12 = 0$. Only positive solution is acceptable (see solution of Problem 300).

Answer: $u = v = 3$.

311. Put $\sqrt[x]{a}=u$; then

$$\sqrt{a}=u^{\frac{x}{2}}$$

and

$$\log_{\sqrt[x]{a}} \sqrt{a}=\log_u u^{\frac{x}{2}}=\frac{x}{2},$$

similarly,

$$\log_{\sqrt[y]{b}} \sqrt{b}=\frac{y}{2}$$

Consequently, the second equation may be written as

$$\frac{x}{2}+\frac{y}{2}=\frac{a}{\sqrt{3}}$$

We get the following system:

$$\begin{cases} x^2+xy+y^2=a^2 & (1)\\ x+y=\dfrac{2a}{\sqrt{3}} & (2)\end{cases}$$

which is equivalent to the given one. Squaring the equation (2), we get

$$x^2+2xy+y^2=\frac{4a^2}{3} \qquad (2a)$$

Subtracting (1) from (2a), we find

$$xy=\frac{a^2}{3}$$

And so we obtain the system

$$\begin{cases} x+y=\dfrac{2a}{\sqrt{3}} & (2)\\ xy=\dfrac{a^2}{3} & (3)\end{cases}$$

which has only one solution

$$x=y=\frac{a}{\sqrt{3}}$$

Note. When squaring an equation there is a probability of obtaining extraneous solutions. It is just the case here: equation (2a) has extraneous solutions as compared with equation (2). For instance, the values $x=y=-\frac{a}{\sqrt{3}}$

satisfy equation (2a), but do not satisfy equation (2). In other words, the equation $x^2+2xy+y^2=\frac{4a^2}{3}$ is not equivalent to the equation $x+y=\frac{2a}{\sqrt{3}}$; it is equivalent to two equations: $x+y=\frac{2a}{\sqrt{3}}$ and $x+y=-\frac{2a}{\sqrt{3}}$. Nevertheless, the given system is equivalent to the system of equations $x+y=$ $=\frac{2a}{\sqrt{3}}$, $xy=\frac{a^2}{3}$, since the latter contains the equation $x+y=\frac{2a}{\sqrt{3}}$ — the fact which eliminates the possibility of equality $x+y=-\frac{2a}{\sqrt{3}}$ for $a\neq 0$ $\left(\text{at } a=0 \text{ the equations } x+y=\frac{2a}{\sqrt{3}} \text{ and } x+y=-\frac{2a}{\sqrt{3}} \text{ coincide}\right)$.

But had we taken instead of the system (2)-(3) the system (1)-(3), i.e. the system

$$\begin{cases} x^2+xy+y^2=a^2 & (1) \\ xy=\dfrac{a^2}{3} & (3) \end{cases}$$

it would not have been equivalent to the given one. Indeed, in addition to the solution $x=y=\frac{a}{\sqrt{3}}$, it would have had another solution $x=y=-\frac{a}{\sqrt{3}}$.

Therefore, when squaring one or several equations, it is always necessary either to clear out the problem of equivalency, or to check by substitution the suitability of the solutions.

Answer: $x=y=\frac{a}{\sqrt{3}}$.

312. Taking into consideration the formula (b) on page 142, we have $\log_4 x=\frac{1}{2}\log_2 x$; therefore, the first equation is reduced to the form $x=y^2$. Now we solve the system

$$\begin{cases} x=y^2 \\ x^2-5y^2+4=0 \end{cases}$$

Answer: $x_1=4$, $y_1=2$; $x_2=1$, $y_2=1$.

313. With the aid of the formula (b) on page 142 we may write the given system as

$$\begin{cases} \log_2 x+\frac{1}{2}\log_2 y+\frac{1}{2}\log_2 z=2 \\ \log_3 y+\frac{1}{2}\log_3 z+\frac{1}{2}\log_3 x=2 \\ \log_4 z+\frac{1}{2}\log_4 x+\frac{1}{2}\log_4 y=2 \end{cases}$$

Taking antilogarithms, we find

$$\begin{cases} x\sqrt{yz}=4 \\ y\sqrt{zx}=9 \\ z\sqrt{xy}=16 \end{cases} \qquad \text{(a)}$$

Multiplying all the equations (a), we obtain

$$(xyz)^2 = 4\cdot 9\cdot 16$$

whence

$$xyz = 24 \tag{b}$$

(we take the arithmetical value of the root, since by virtue of the given equations x, y, z must be positive). Square each of the equations (a) and then divide them by (b).

Answer: $x=\frac{2}{3}$; $y=\frac{27}{8}$; $z=\frac{32}{3}$.

314. From the first equation we find $x+y=2^{x-y}\,3^{\frac{x-y}{2}}$, from the second, $x+y=3\cdot 2^{x-y}$, consequently,

$$3^{\frac{x-y}{2}}=3 \quad \text{or} \quad \frac{x-y}{2}=1$$

Hence, $x+y=3\cdot 2^2=12$.

Answer: $x=7$; $y=5$.

315. The given system is reduced to the following one:

$$\frac{x+y}{10}=\frac{7}{x}, \quad x^2-y^2=40$$

Dividing the second equation by the first one, we get $x-y=\frac{4x}{7}$. Solving the system

$$x+y=\frac{70}{x} \quad \text{and} \quad x-y=\frac{4x}{7}$$

we have $x_1 = 7$, $y_1 = 3$; $x_2 = -7$, $y_2 = -3$. The roots x_2, y_2 do not satisfy the second equation of the given system, since the numbers $x_2 + y_2$ and $x_2 - y_2$ are negative.

Answer: $x = 7$; $y = 3$.

316. Represent the given system in the form

$$2^{\frac{2x}{y}}=2^{5+\frac{3y}{x}}, \quad 3^{\frac{x}{y}}=3^{1+\frac{2-2y}{y}}$$

wherefrom we get

$$\frac{2x}{y}=5+\frac{3y}{x}, \quad \frac{x}{y}=1+\frac{2-2y}{y}$$

Put $\frac{x}{y}=t$; then from the first equation we have $2t^2-5t-3=0$; $t_1=3$, $t_2=-\frac{1}{2}$, i.e. $\frac{x}{y}=3$ or $\frac{x}{y}=-\frac{1}{2}$. Hence we find the expressions $x=3y$ and $x=-\frac{1}{2}y$; substituting them into the second equation, we find

$$x_1=-2, \quad y_1=4; \quad x_2=\frac{3}{2}, \; y_2=\frac{1}{2}$$

Answer: $x_1=-2$, $y_1=4$; $x_2=\frac{3}{2}$, $y_2=\frac{1}{2}$.

317. The given system is reduced to the following one:

$$\begin{cases} \frac{2x}{y}-\frac{3y}{x}=5 \\ x+y=2 \end{cases}$$

From the first equation (see solution of Problem 316) we find $\frac{x}{y}=3$ or $\frac{x}{y}=-\frac{1}{2}$. The second equation gives $x_1=\frac{3}{2}$, $y_1=\frac{1}{2}$; $x_2=-2$, $y_2=4$. The roots x_2, y_2 are rejected.

Answer: $x=\frac{3}{2}$, $y=\frac{1}{2}$.

318. The given system is reduced as follows:

$$\begin{cases} \sqrt{xy}=4-\sqrt{x} \\ 2\sqrt{xy}=3+\sqrt{y} \end{cases}$$

Putting $\sqrt{x}=u$; $\sqrt{y}=v$, we get $uv=4-u$; $2uv=3+v$.

Answer: $x_1=4$, $y_1=1$; $x_2=1$, $y_2=9$.

319. Rewrite the given system in the form

$$ay=x^p, \quad bx=y^q$$

Since x and y must be positive (as the logarithmic bases), the original system is only solvable at positive values of a and b. From the first equation we find $y=\frac{x^p}{a}$; substituting it into the second equation, we get $x^{pq}=a^qbx$. Rejecting the root $x=0$ (since x must be positive), we obtain the equation $x^{pq-1}=a^qb$. If $pq=1$, then this equation either has no solutions (for $a^qb\neq 1$), or is an identity (at $a^qb=1$). In the latter case the original system has an infinite number of solutions (x is an arbitrary number, and $y=\frac{x^p}{a}$; or y is an arbitrary number, and $x=\frac{y^q}{b}$). If $pq\neq 1$, then we get the following solution:

$$x=\sqrt[pq-1]{a^qb}, \quad y=\sqrt[pq-1]{b^pa}$$

Answer: $x=\sqrt[pq-1]{a^qb}$, $y=\sqrt[pq-1]{b^pa}$ $(pq\neq 1)$.

CHAPTER V

PROGRESSIONS

Arithmetic Progression

320. By hypothesis, $a_1=5$, $d=4$. Substituting these values into (3) and performing some transformations, we get the equation

$$2n^2+3n-10\,877=0$$

Its roots are: $n_1=73$ and $n_2=-74.5$, only the former being suitable.

Answer: 73 terms.

321. By hypothesis,

$$a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) = 26$$
$$a_1 (a_1 + d) (a_1 + 2d) (a_1 + 3d) = 880$$

The first equation gives $4a_1 + 6d = 26$, whence $a_1 = \frac{13-3d}{2}$. Substituting it into the second equation and simplifying the parenthesized expressions, we get

$$\frac{13-3d}{2} \cdot \frac{13-d}{2} \cdot \frac{13+d}{2} \cdot \frac{13+3d}{2} = 880$$

Getting rid of the denominator and multiplying the numerators (it is most convenient to multiply the first numerator by the fourth and the second by the third), we find:

$$9d^4 - 1690d^2 + 14\,481 = 0$$

Denoting the roots of this biquadratic equation by d', d'', d''', d'''', we find $d' = 3$; $d'' = -3$; $d''' = \frac{\sqrt{1609}}{3}$ and $d'''' = -\frac{\sqrt{1609}}{3}$; from the equation $a_1 = \frac{13-3d}{2}$ we find the corresponding values of the first term:

$$a_1' = 2; \quad a_1'' = 11; \quad a_1''' = \frac{13-\sqrt{1609}}{2}; \quad a_1'''' = \frac{13+\sqrt{1609}}{2}$$

Answer: the problem has four solutions:

(1) $2;\ 5;\ 8;\ 11;\ 14;\ \ldots$

(2) $11;\ 8;\ 5;\ 2;\ -1;\ \ldots$

(3) $\frac{13-\sqrt{1609}}{2};\ \frac{39-\sqrt{1609}}{6};\ \frac{39+\sqrt{1609}}{6};\ \frac{13+\sqrt{1609}}{2};\ \ldots$

(4) $\frac{13+\sqrt{1609}}{2};\ \frac{39+\sqrt{1609}}{6};\ \frac{39-\sqrt{1609}}{6};\ \frac{13-\sqrt{1609}}{2};\ \ldots$

322. Denoting a_p and a_q by a_1 and d, we get (by hypothesis) the following system:

$$\begin{cases} a_1 + d(p-1) = q \\ a_1 + d(q-1) = p \end{cases}$$

Hence, $d = -1$ and $a_1 = p + q - 1$. By the formula (1) we find:

$$a_n = (p + q - 1) - (n - 1) = p + q - n$$

Answer: $a_n = p + q - n$.

323. Natural two-digit numbers form an arithmetic progression with common difference $d = 1$; its first term $a_1 = 10$, and the last one $a_n = 99$. By the formula (1) we find the number of terms $n = 90$. The formula (2) gives:

$$S_n = \frac{(10+99)\cdot 90}{2} = 4905$$

Answer: 4905.

324. Let us denote the odd numbers by n, $(n+2)$, $(n+4)$, $(n+6)$. Then the even numbers contained between them will be $(n+1)$, $(n+3)$, $(n+5)$. By hypothesis,

$$n^2 + (n+2)^2 + (n+4)^2 + (n+6)^2 = (n+1)^2 + (n+3)^2 + (n+5)^2 + 48$$

or

$$n^2 + [(n+2)^2 - (n+1)^2] + [(n+4)^2 - (n+3)^2] + [(n+6)^2 - (n+5)^2] - 48 = 0$$

whence

$$n^2 + (2n+3) + (2n+7) + (2n+11) - 48 = 0$$

or

$$n^2 + 6n - 27 = 0$$

Hence, $n = 3$ or $n = -9$.

Answer: (1) 3; 5; 7; 9 or (2) -9; -7; -5; -3.

325. The terms $a_2; a_4; a_6; \ldots; a_{20}$ constitute an arithmetic progression with common difference $2d$ and the number of terms 10. Using the formula (3) (in which a_1 should be replaced by a_2 and d, by $2d$), we find

$$\frac{(2a_2 + 2d \cdot 9)\,10}{2} = 250.$$

i.e.

$$a_2 + 9d = 25$$

Substituting $a_2 = a_1 + d$, we have

$$a_1 + 10d = 25 \qquad \text{(a)}$$

In the same way, proceeding from the progression $a_1; a_3; a_5; \ldots; a_{19}$, we find in the same way

$$a_1 + 9d = 22 \qquad \text{(b)}$$

From (a) and (b) we may find a_1 and d, and then all the terms of the progression. But since it is required to find the medium terms only, i.e. $a_{10} = a_1 + 9d$, and $a_{11} = a_1 + 10d$, then from (a) and (b) we immediately have: $a_{10} = 22$ and $a_{11} = 25$.

Answer: the medium terms are equal to 22 and 25, respectively.

326. Put $b_1 = (a+x)^2$, $b_2 = (a^2 + x^2)$, $b_3 = (a-x)^2$. We find $b_2 - b_1 = b_3 - b_2 = -2ax$. Hence, the terms b_1, b_2, b_3 constitute an arithmetic progression with common difference $d = -2ax$. By the formula (3) we have

$$S_n = \frac{[2(a+x)^2 - 2ax(n-1)]\,n}{2} = [a^2 + (3-n)\,ax + x^2]\,n$$

Answer: $S_n = [a^2 + (3-n)\,ax + x^2]\,n$.

327. By the formula (3) we have

$$S_1 = \frac{2a_1 + d(n_1 - 1)}{2} \cdot n_1$$

$$S_2 = \frac{2a_1 + d(n_2 - 1)}{2} \cdot n_2$$

$$S_3 = \frac{2a_1 + d(n_3 - 1)}{2} \cdot n_3$$

or

$$\frac{S_1}{n_1}=a_1+\frac{d}{2}(n_1-1)$$

$$\frac{S_2}{n_2}=a_1+\frac{d}{2}(n_2-1)$$

$$\frac{S_3}{n_3}=a_1+\frac{d}{2}(n_3-1)$$

Multiplying the obtained equalities by (n_2-n_3), (n_3-n_1) and (n_1-n_2), respectively, and adding the products, we find:

$$\frac{S_1}{n_1}(n_2-n_3)+\frac{S_2}{n_2}(n_3-n_1)+\frac{S_3}{n_3}(n_1-n_2)=$$
$$=a_1[(n_2-n_3)+(n_3-n_1)+(n_1-n_2)]+$$
$$+\frac{d}{2}[(n_1-1)(n_2-n_3)+(n_2-1)(n_3-n_1)+(n_3-1)(n_1-n_2)]$$

The bracketed expressions are identically equal to zero, consequently,

$$\frac{S_1}{n_1}(n_2-n_3)+\frac{S_2}{n_2}(n_3-n_1)+\frac{S_3}{n_3}(n_1-n_2)=0$$

which completes the proof.

328. By hypothesis, $S_{10}=5S_5$. Expressing S_5 and S_{10} by the formula (3) and taking into account that $a_1=1$, we find

$$\frac{(2+9d)\,10}{2}=5\cdot\frac{(2+4d)\,5}{2}$$

whence $d=-3$.

Answer: $+1$; -2; -5; -8; ...

329. By hypothesis,

$$S_n=3n^2 \text{ or } \frac{[2a_1+d(n-1)]\,n}{2}=3n^2$$

Since $n\neq 0$, then, reducing this equation by n, we get $2a_1+dn-d=6n$ or

$$2a_1-d=(6-d)\,n \tag{a}$$

By hypothesis, the equality (a) must be satisfied at any n, but the left member of (a) contains no n, whereas the right member varies with n, provided the factor $6-d$ is non-zero. Only in the case $6-d=0$ the right member is independent of n (is equal to zero), therefore we must have $d=6$. Then from (a) we find $2a_1-d=0$, i.e. $a_1=\frac{d}{2}=3$.

Answer: 3; 9; 15; 21; ...

330. The numbers which are not exactly divisible by 4, yielding the remainder 1, have the form $4k+1$ (k—any natural number). They form an arithmetic progression with common difference 4. The first two-digit number of this form is 13 (it is obtained at $k=3$); the last one is 97. By the formula (1), where $a_1=13$, $a_n=97$ and $d=4$, we find $n=22$. The formula (3) yields the required sum.

To determine for what values of k the numbers of the form $4k+1$ will be two-digit, we may also make use of the following system of inequalities:

$$\begin{cases} 4k+1 \geqslant 10 \\ 4k+1 < 100 \end{cases}$$

wherefrom we find $2\frac{1}{4} \leqslant k < 24\frac{3}{4}$; hence, k may have values equal to 3; 4; 5; . . .; 24, the total number of which $n = (24-3)+1 = 22$.
Answer: 1210.

Geometric Progression

331. *The geometric mean* of two (positive) numbers a and b is a positive number x determined from the proportion $a : x = x : b$. To insert three geometric means between the numbers 1 and 256 means to find the three numbers u_2, u_3, u_4 which satisfy the conditions:

$$1 : u_2 = u_2 : u_3 = u_3 : u_4 = u_4 : 256$$

Hence, the numbers $u_1 = 1$, u_2, u_3, u_4 and $u_5 = 256$ form a geometric progression. By the formula of the nth term of the progression, $256 = 1 \cdot q^4$. This equation has one positive root $q = \sqrt[4]{256} = 4$ (-4; $+4i$; $-4i$ are discarded, since they are not suitable). Now, by the same formula, we find: $u_2 = 4$; $u_3 = 16$; $u_4 = 64$.
Answer: 4; 16; 64.

332. By hypothesis, $u_1 + u_3 = 52$ and $u_2^2 = 100$, or $u_2 = \pm 10$. By the property of the geometric progression, $u_1 u_3 = u_2^2 = 100$; hence, u_1 and u_3 are the roots of the equation $u^2 - 52u + 100 = 0$, whence $u_1' = 50$ and $u_3' = 2$, or $u_1'' = 2$ and $u_3'' = 50$.
Answer: (1) 50; 10; 2, or (2) 50; -10; 2, or the same numbers following in the reverse order.

333. By hypothesis: (1) $u_3 - u_1 = 9$ and (2) $u_5 - u_3 = 36$. Using the formula $u_n = u_1 q^{n-1}$, rewrite these equations in the form: (1) $u_1 q^2 - u_1 = 9$; (2) $u_1 q^4 - u_1 q^2 = 36$. Dividing (2) by (1), we get $q^2 = 4$, hence, $q = \pm 2$; from (1) we find: $u_1 = 3$.

Answer: (1) 3; 6; 12; 24; 48; . . .
(2) 3; -6; 12; -24; 48; . . .

334. By hypothesis, $u_1 + u_4 = 27$ and $u_2 u_3 = 72$; but since $\frac{u_2}{u_1} = \frac{u_4}{u_3}$, or $u_2 u_3 = u_1 u_4$, we have a system of two equations:

$$(1)\ u_1 + u_4 = 27 \text{ and } (2)\ u_1 u_4 = 72$$

whence $u_1 = 3$ and $u_4 = 24$, or $u_1 = 24$ and $u_4 = 3$. From the formula $u_4 = u_1 q^3$ we find $q = 2$, or $q = 1/2$.
Answer: 3; 6; 12; 24, or in the reverse order: 24; 12; 6; 3.

335. By hypothesis: (1) $u_1 + u_4 = 35$ and (2) $u_2 + u_3 = 30$. As in Problem 333, for determining q we get the following equation

$$\frac{1+q^3}{q(1+q)} = \frac{35}{30}$$

or, when reduced,

$$\frac{1-q+q^2}{q}=\frac{7}{6}$$

We find:

$$(1)\ q=3/2;\ u_1=8;\ (2)\ q=2/3;\ u_1=27.$$

We get two progressions:

(1) 8; 12; 18; 27; 40.5; . . .

(2) 27; 18; 12; 8; $5\frac{1}{3}$; . . .

whose first four terms are equal, but follow in the reverse order.

Answer: 8; 12; 18; 27.

336. In the second of the given sums we replace each term by the preceding one multiplied by q (by the definition of the geometric progression). We obtain

$$u_1q+u_2q+u_3q+u_4q+u_5q=62$$

or

$$q(u_1+u_2+u_3+u_4+u_5)=62$$

By hypothesis, the parenthesized expression is equal to 31; hence, $q=2$. Using the formula $S_n=\frac{u_1(q^n-1)}{q-1}$, we have $31=\frac{u_1(2^5-1)}{2-1}$, wherefrom $u_1=1$.

Answer: 1; 2; 4; 8; . . .

337. By hypothesis, we have:

(1) $u_2+u_3+u_4+u_5=19.5$; (2) $u_1+u_2+u_3+u_4=13$

The problem is similar to the preceding one.

Answer: $u_1=1.6$ and $u_5=8.1$.

338. The terms u_4 and u_6 are equidistant from the beginning and the end of the given sequence; therefore $u_4u_6=u_1u_9$. Since, by hypothesis, $u_1u_9=2304$, then $u_4u_6=2304$; besides, also by hypothesis, $u_4+u_6=120$. From these two equations we find $u_4'=24$; $u_6'=96$, and $u_4''=96$; $u_6''=24$. Let us take the first solution. By the formula $u_n=u_1q^{n-1}$ we have:

$$(1)\ 24=u_1q^3;\qquad (2)\ 96=u_1q^5$$

Dividing (2) by (1), we find $q^2=4$, whence $q=2$ or $q=-2$. In the first case the equation (1) yields $u_1=3$, in the second, $u_1=-3$. In the first case the nine terms of the progression are:

3; 6; 12; 24; 48; 96; 192; 384; 768;

in the second:

−3; 6; −12; 24; −48; 96; −192; 384; −768

Taking the solution $u_4''=96$; $u_6''=24$, we find the same two sequences of the terms, but in the reverse order.

Answer:
(1) $u_1=3$; $q=2$
(2) $u_1=-3$; $q=-2$
(3) $u_1=768$; $q=1/2$
(4) $u_1=-768$; $q=-1/2$

339. By hypothesis: (1) $u_1+u_2+u_3=126$ and (2) $u_1u_2u_3=13\,824$. Since u_2 is the geometric mean of u_1 and u_3, we have $u_1u_3=u_2^2$; therefore instead of (2) we may write $u_2^3=13\,824$, whence $u_2=\sqrt[3]{13\,824}$. In the given case, by factoring 13 824, it is easy to find that $u_2=24$. Substituting it into (1) and (2),

we get the following system of equations: $u_1 + u_3 = 102$; $u_1u_3 = 576$. Solving it, we find: $u_1 = 6$; $u_3 = 96$, and $u_1 = 96$; $u_3 = 6$. Thus, we get two progressions: 6; 24; 96, and 96; 24; 6, which differ only in the order of the terms.

Answer: 6; 24; 96.

340. It follows (by hypothesis) that the sum of the even terms is twice that of the odd terms, i.e.

$$\frac{u_2+u_4+u_6+\ldots+u_{2n}}{u_1+u_3+u_5+\ldots+u_{2n-1}}=2$$

Replacing the terms u_2; u_4; u_6; . . .; u_{2n} by the expressions $u_2 = u_1q$; $u_4 = u_3q$; . . .; $u_{2n} = u_{2n-1}q$, we find $q = 2$.

Answer: $q = 2$.

Infinitely Decreasing Geometric Progression

341. To prove the fact that the given numbers constitute a decreasing geometric progression we have to check whether the ratios $\frac{u_2}{u_1}$ and $\frac{u_3}{u_2}$ are equal and whether each of them is less than unity. We have:

$$(1)\quad \frac{u_2}{u_1}=\frac{1}{2-\sqrt{2}}:\frac{\sqrt{2}+1}{\sqrt{2}-1}=\frac{1}{\sqrt{2}\,(\sqrt{2}-1)}:\frac{\sqrt{2}+1}{\sqrt{2}-1}=\frac{1}{2+\sqrt{2}}$$

$$(2)\quad \frac{u_3}{u_2}=\frac{1}{2}:\frac{1}{2-\sqrt{2}}=\frac{2-\sqrt{2}}{2}=\frac{(2-\sqrt{2})\,(2+\sqrt{2})}{2\,(2+\sqrt{2})}=\frac{1}{2+\sqrt{2}}$$

Since $\frac{u_2}{u_1}=\frac{u_3}{u_2}=q=\frac{1}{2+\sqrt{2}}<1$, the given numbers form a decreasing geometric progression. By the formula of its sum we find

$$S=\frac{\sqrt{2}+1}{(\sqrt{2}-1)\left(1-\frac{1}{2+\sqrt{2}}\right)}=\frac{(\sqrt{2}+1)\,(2+\sqrt{2})}{(\sqrt{2}-1)\,(\sqrt{2}+1)}=4+3\sqrt{2}$$

Answer: $S=4+3\sqrt{2}$.

342. As in the preceding problem, we find that the bracketed expression is equal to $\frac{3\,(\sqrt{3}-2)}{\sqrt{3}-1}$. The whole expression then takes the form

$$(4\sqrt{3}+8)\cdot\frac{3\,(\sqrt{3}-2)}{\sqrt{3}-1}=-\frac{12}{\sqrt{3}-1}=-6\,(\sqrt{3}+1)$$

Answer: $-6\,(\sqrt{3}+1)$.

343. By hypothesis,

$$u_1=4 \text{ and } u_3-u_5=\frac{32}{81}$$

By the formula $u_n=u_1q^{n-1}$ we get from the second equality

$$u_1q^2-u_1q^4=\frac{32}{81}$$

Taking into account that $u_1 = 4$, we get the biquadratic equation $81q^4 - 81q^2 + 8 = 0$, whose roots are: $q_{1,2} = \pm \frac{2\sqrt{2}}{3}$ and $q_{3,4} = \pm \frac{1}{3}$. The negative roots are not suitable, since, by hypothesis, all the terms are positive; both positive roots are suitable, since they are less than unity. And so, we get two infinitely decreasing progressions.

Answer: $S' = 12(3 + 2\sqrt{2})$ and $S'' = 6$.

344. By hypothesis,

$$u_1 + u_4 = 54 \text{ and } u_2 + u_3 = 36$$

Using the formula $u_n = u_1q^{n-1}$, we get a system of two equations:

$$\begin{cases} u_1 + u_1q^3 = 54 \\ u_1q + u_1q^2 = 36 \end{cases} \text{ or } \begin{cases} u_1(1+q)(1-q+q^2) = 54 & (1) \\ u_1q(1+q) = 36 & (2) \end{cases}$$

Dividing (1) by (2), we obtain the equation

$$\frac{1-q+q^2}{q} = \frac{3}{2}$$

whence $q_1 = 2$ and $q_2 = 1/2$. The suitable root is $q_2 = \frac{1}{2} < 1$. We find from (2) $u_1 = 48$.

Answer: $S = 96$.

345. *First method*. By hypothesis,

(1) $u_1 + u_3 + u_5 + \ldots = 36$
(2) $u_2 + u_4 + u_6 + \ldots = 12$

The terms of the first and the second sums also constitute infinitely decreasing progressions with one and the same common ratio q^2. The first term of the first progression is equal to u_1, the first term of the second progression, to u_2, i.e. to u_1q. Expressing the sums of the first and second progressions by the formula for the sum of an infinitely decreasing progression (where instead of q we take q^2, and instead of u_1 in the second case we take u_1q), we obtain (1) $\frac{u_1}{1-q^2} = 36$ and (2) $\frac{u_1q}{1-q^2} = 12$. Dividing (2) by (1), we get $q = 1/3$, and from the first equation we find $u_1 = 32$.

Second method. Since $u_2 = u_1q$, $u_4 = u_3q$ and so on, then instead of $u_2 + u_4 + u_6 + \ldots = 12$ we get $u_1q + u_3q + u_5q + \ldots = 12$, or

$$q(u_1 + u_3 + u_5 + \ldots) = 12 \qquad (1)$$

Dividing the expression $u_1 + u_3 + u_5 + \ldots = 36$ by (1), we find $q = \frac{1}{3}$. On the other hand, the sum of all the terms of the progression is $12 + 36 = 48$. By the formula for the sum of an infinitely decreasing progression we have

$$48 = \frac{u_1}{1 - \frac{1}{3}}$$

whence $u_1 = 32$.

Answer: $32; \frac{32}{3}; \frac{32}{9}; \ldots$

346. By hypothesis,

$$u_1 + u_2 + u_3 + \ldots = 56; \quad u_1^2 + u_2^2 + u_3^2 + \ldots = 448$$

The addends of the second sum also form an infinitely decreasing geometric progression with the first term u_1^2 and common ratio q^2. Expressing the sums of these progressions, we get

$$\frac{u_1}{1-q} = 56, \quad \frac{u_1^2}{1-q^2} = 448$$

or

$$u_1 = 56\,(1-q) \tag{1}$$

$$u_1^2 = 448\,(1-q^2) \tag{2}$$

Dividing (2) by (1), we find

$$u_1 = 8\,(1+q) \tag{3}$$

Eliminating u_1 from the equations (1) and (3), we get

$$8\,(1+q) = 56\,(1-q)$$

whence $q = \frac{3}{4}$. From (1) we find $u_1 = 14$.

Answer: $u_1 = 14$, $q = \frac{3}{4}$.

347. Solved in the same way as the preceding problem. For determining u_1 and q we get the following system of equations:

$$\begin{cases} \dfrac{u_1}{1-q} = 3 & (1) \\ \dfrac{u_1^3}{1-q^3} = \dfrac{108}{13} & (2) \end{cases}$$

Eliminating u_1 from these equations, we get the equation $3q^2 - 10q + 3 = 0$. Out of its two roots only one, namely $q = \frac{1}{3}$, is suitable (the other, $q = 3$, being more than unity). From the equation (1) we find $u_1 = 2$.

Answer: 2; $\frac{2}{3}$; $\frac{2}{9}$; . . .

348. Solved in the same way as Problems 346 and 347. To determine u_1 and q we obtain the following system of equations:

$$\begin{cases} u_1 q = 6 & (1) \\ \dfrac{u_1}{1-q} = \dfrac{1}{8} \bullet \dfrac{u_1^2}{1-q^2} & (2) \end{cases}$$

Equation (2) is equivalent to the equation $u_1 = 8\,(1+q)$. Eliminating u_1 from the system $u_1 q = 6$, $u_1 = 8\,(1+q)$, we get the equation $4q^2 + 4q - 3 = 0$ with two roots: $q_1 = -3/2$, $q_2 = 1/2$, but the first root is not suitable, since its absolute value is more than unity. From (1) we find $u_1 = 12$.

Answer: 12; 6; 3; . . .

Arithmetic and Geometric Progressions

349. By hypothesis, we find:

$$d = 16 - 14 = 2; \quad a_1 = 14 - d = 12$$

$$a_1 + a_2 + a_3 = 12 + 14 + 16 = 42$$

Consequently, in the required geometric progression (1) $q = 2$ and (2) $u_1 + u_1q + u_1q^2 = 42$, whence $u_1 = 6$.

Answer: 6; 12; 24; . . .

350. The first three terms of the geometric progression are: 3; $3q$; $3q^2$. By hypothesis, $a_1 = 3$; $a_2 = 3q + 6$; since $a_3 - a_2 = a_2 - a_1$, we have $a_3 = 2a_2 - a_1 = 6q + 9$. By hypothesis, this third term is equal to the third term of the geometric progression, i.e. $3q^2$. Thus, we get the equation $6q + 9 = 3q^2$, whose roots are: $q = 3$ and $q = -1$. In the first case the geometric progression is 3; 9; 27; . . . and the arithmetic one is 3; 15; 27; . . . . In the second case we get two sequences of numbers: 3; -3; 3; -3; . . . and 3; 3; 3; . . . which may be considered respectively as a geometric progression with common ratio $q = -1$, as well as an arithmetic progression with common difference $d = 0$.

Answer: (1) 3; 15; 27; . . .; 3; 9; 27; . . .
(2) 3; 3; 3; . . .; 3; -3; 3; -3; . . .

351. The problem is similar to the preceding one. By hypothesis, $a_1 = u_1 = 5$; consequently, $u_3 = 5q^2$; $u_5 = 5q^4$. Then, also by hypothesis, $a_4 = u_3 = 5q^2$; $a_{16} = u_5 = 5q^4$. Hence: (1) $5q^2 = 5 + 3d$, (2) $5q^4 = 5 + 15d$. Eliminating d, we get the equation $q^4 - 5q^2 + 4 = 0$, whence $q^2 = 4$ or $q^2 = 1$. Since $a_4 = 5q^2$, the fourth term of the arithmetic progression is equal to 20 in the first case, and to 5 in the second.

Note. In either of these cases we obtain two different geometric progressions, the arithmetic progressions being the same. Namely, in the first case we have the following geometric progressions: 5; 10; 20; . . . and 5; -10; 20; . . ., the arithmetic progression $\left(\text{with common difference } d = \frac{5q^2 - 5}{3} = 5\right)$ being 5; 10; 15; 20; . . . . In the second case we get 5; 5; 5; . . . and 5; -5; 5; -5; . . . the corresponding arithmetic progression containing equal terms: 5; 5; 5; . . . .

Answer: 20 or 5.

352. By hypothesis, $a_1 = u_1$; $a_2 = u_1q$; $a_7 = u_1q^2$, wherefrom we find: (1) $d = a_2 - a_1 = u_1(q - 1)$ and (2) $6d = a_7 - a_1 = u_1(q^2 - 1)$. Eliminating d, we get $u_1(q^2 - 1) = 6u_1(q - 1)$. Since $u_1 \neq 0$, then $q^2 - 1 = 6(q - 1)$, whence $q = 5$ or $q = 1$. From the condition $u_1 + u_1q + u_1q^2 = 93$ we find $u_1 = 3$ and $u_1 = 31$, respectively.

Answer: (1) 3; 15; 75, (2) 31; 31; 31.

353. By the formula (2) on page 32 we find $a_7 = 729$; consequently, in the geometric progression we have: $u_1 = a_1 = 1$; $u_7 = a_7 = 729$. It is required to find the medium term which is the fourth one both from the beginning and from the end. Hence, the first term u_1, the required medium term u_4 and the last one u_7 form a continued proportion: $u_1 : u_4 = u_4 : u_7$. Hence, $u_4^2 = u_1u_7$ and $u_4^2 = 729$.

Answer: $u_4 = \pm 27$.

354. By hypothesis, $a_1 + a_2 + a_3 = 15$. Since $a_2 - a_1 = a_3 - a_2$, then $2a_2 = a_1 + a_3$, and from the given condition we have $2a_2 + a_2 = 15$. Hence, $a_2 = 5$. Then $a_1 = 5 - d$; $a_2 = 5$; $a_3 = 5 + d$ and, by hypothesis, $u_1 = a_1 + 1 = 6 - d$; $u_2 = a_2 + 4 = 9$; $u_3 = a_3 + 19 = 24 + d$. Since $u_2^2 = u_1u_3$,

we have

$$9^2 = (6 - d)(24 + d)$$

whence we find $d = 3$, $a_1 = 2$ or $d = -21$, $a_1 = 26$.

Answer: (1) 2; 5; 8. (2) 26; 5; −16.

355. By hypothesis, $a_1 = u_1 + 1$; $a_2 = u_2 + 6$; $a_3 = u_3 + 3$; hence $a_1 + a_2 + a_3 = (u_1 + u_2 + u_3) + (1 + 6 + 3)$, or, by virtue of the condition $u_1 + u_2 + u_3 = 26$, we get

$$a_1 + a_2 + a_3 = 26 + 10 = 36$$

Then proceed in the same way as in the preceding problem.

Answer: 2; 6; 18 or 18; 6; 2.

356. Suppose the required numbers are: u_1; u_1q; u_1q^2; then, by hypothesis, the numbers u_1, u_1q and $(u_1q^2 - 64)$ constitute an arithmetic progression and, consequently,

$$u_1q - u_1 = (u_1q^2 - 64) - u_1q \qquad (1)$$

Furthermore, by hypothesis, the numbers u_1; $(u_1q - 8)$; $(u_1q^2 - 64)$ form a geometric progression and, consequently,

$$(u_1q - 8) : u_1 = (u_1q^2 - 64) : (u_1q - 8) \qquad (2)$$

After simplifications the system of equations (1) and (2) takes the form

$$u_1(q^2 - 2q + 1) = 64, \quad u_1(q - 4) = 4$$

whence $q = 13$ and $u_1 = \frac{4}{9}$ or $q = 5$ and $u_1 = 4$.

Answer: (1) $\frac{4}{9}$; $\frac{52}{9}$; $\frac{676}{9}$. (2) 4; 20; 100.

357. Suppose the required numbers are: u_1, u_2 and u_3. If these numbers are the terms of a geometric progression, then

$$u_2^2 = u_1u_3 \qquad (1)$$

and if they are the terms of an arithmetic progression, then

$$2u_2 = u_1 + u_3 \qquad (2)$$

Eliminating u_2 from (1) and (2), we find $(u_1 + u_3)^2 = 4u_1u_3$ or $(u_1 - u_3)^2 = 0$, whence $u_1 = u_3$, and from (2) we find $u_2 = u_1$. Hence, $u_1 = u_2 = u_3$.

Answer: possible, if the three numbers are equal.

CHAPTER VI

COMBINATORICS AND NEWTON'S BINOMIAL THEOREM

Notation:

A_m^n = total number of permutations of m elements taken n at a time

P_n = total number of permutations of n elemetns

C_m^n = total number of combinations of m elements taken n at a time

$T_{k+1} = C_m^k a^k x^{m-k}$ is $(k+1)$th term of the expansion of the binomial $(x+a)^m$

358. By hypothesis,

$$\frac{P_n}{P_{n+2}}=\frac{0.1}{3} \quad \text{or} \quad \frac{1\cdot2\cdot3\ldots\ n}{1\cdot2\cdot3\ \ldots\ n\,(n+1)\,(n+2)}=\frac{1}{30}$$

whence $(n+1)\ (n+2)=30$. The roots of this equation are: $n_1=4$, $n_2=-7$. The second root does not suit.

Answer: $n=4$.

359. By hypothesis, $5C_n^3=C_{n+2}^4$ or

$$\frac{5n\,(n-1)\,(n-2)}{1\cdot2\cdot3}=\frac{(n+2)\,(n+1)\,n\,(n-1)}{1\cdot2\cdot3\cdot4}$$

whence

$$5\,(n-2)=\frac{(n+2)\,(n+1)}{4}$$

Answer: $n_1=14$; $n_2=3$.

360. The required term

$$T_9=(-1)^8C_{16}^8\left(\frac{a}{x}\right)^8(x^{\frac{1}{2}})^8=\frac{16\cdot15\cdot14\cdot13\cdot12\cdot11\cdot10\cdot9}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}\cdot\frac{a^8}{x^4}$$

Answer: $12\,870\ \frac{a^8}{x^4}$.

361. We have

$$T_{n+1}=C_{12}^n\left(\frac{2}{3}\sqrt{a}\right)^n\left(\frac{3}{4}\sqrt[3]{a^2}\right)^{12-n}$$

This expression contains a to the power $\frac{n}{2}+\frac{2\,(12-n)}{3}$. By hypothesis, $\frac{n}{2}+\frac{2\,(12-n)}{3}=7$, whence $n=6$, i.e. $n+1=7$.

Answer: the seventh term.

362. We have

$$T_{n+1}=C_{21}^n\left(\sqrt{\frac{b}{\sqrt[3]{a}}}\right)^n\left(\sqrt[3]{\frac{a}{\sqrt{b}}}\right)^{21-n}=C_{21}^n a^{\frac{21-n}{3}-\frac{n}{6}}\cdot b^{\frac{n}{2}-\frac{21-n}{6}}$$

By hypothesis, $\frac{n}{2}-\frac{21-n}{6}=\frac{21-n}{3}-\frac{n}{6}$, whence $n=9$.

Answer: the tenth term.

363. After simplification we get $(a^{\frac{1}{3}}-a^{-\frac{1}{2}})^{10}$. We have:

$$T_{n+1}=(-1)^n\,C_{10}^n\,(a^{-\frac{1}{2}})^n\,(a^{\frac{1}{3}})^{10-n}=(-1)^n\,C_{10}^n a^{\frac{10-n}{3}-\frac{n}{2}}$$

By hypothesis, $\frac{10-n}{3}-\frac{n}{2}=0$, whence $n=4$.

Answer: $T_5=210$.

364. Let x be the exponent of the first binomial. Then the sum of the binomial coefficients is 2^x. The sum of the binomial coefficients of the second bino-

mial is 2^{x+3}. We get the equation

$$2^x + 2^{x+3} = 144; \quad 2^x(1+8) = 144; \quad 2^x = 2^4; \quad x = 4$$

Answer: 4 and 7.

365. We have $\frac{m(m-1)}{1\cdot 2} = 105$, whence $m = 15$; then

$$T_{13} = (-1)^{12} C_{15}^{12} \left(\frac{1}{\sqrt{3x}}\right)^{12} (9x)^3 = \frac{455}{x^3}$$

Answer: $\frac{455}{x^3}$.

366. By hypothesis, $C_m^3 = C_m^{12}$, hence, $m = 15$. Then we have

$$T_{n+1} = C_{15}^n \left(\frac{a}{x}\right)^n (x^2)^{15-n} = C_{15}^n a^n x^{30-3n}$$

By hypothesis, $30 - 3n = 0$, $n = 10$.

Answer: $T_{11} = 3003a^{10}$.

367. By hypothesis, $\frac{C_m^4}{C_m^2} = \frac{14}{3}$, i.e. $m^2 - 5m - 50 = 0$, whence $m = 10$ (the root $m = -5$ is not suitable). The medium term

$$T_6 = C_{10}^5 (-1)^5 \left(\sqrt[5]{\frac{a^{-2}}{\sqrt{a}}}\right)^5 \left(a^{-}\sqrt[2]{a}\right)^5 = -252$$

Answer: the medium (sixth) term is -252.

368. By hypothesis, $1 + m + \frac{m(m-1)}{1\cdot 2} = 46$. Then solved in the same way as Problem 367.

Answer: the required (seventh) term $T_7 = 84$.

369. By hypothesis, $2^m = 128$, whence $m = 7$. We have

$$T_{n+1} = C_7^n x^{-\frac{n}{3}} x^{\frac{3}{2}(7-n)}$$

By hypothesis, $-\frac{n}{3} + \frac{3}{2}(7-n) = 5$, whence $n = 3$.

Answer: the required (fourth) term $T_4 = 35x^5$.

370. We have $u_6 = u_1 q^5 = \frac{1}{i}(1+i)^5$. The multiplicand $\frac{1}{i}$ is equal to $\frac{i}{i^2} = \frac{i}{-1} = -i$. According to the binomial theorem the factor $(1+i)^5$ is equal to $1 + 5i + 10i^2 + 10i^3 + 5i^4 + i^5$. Hence,

$$u_6 = -i - 5i^2 - 10i^3 - 10i^4 - 5i^5 - i^6$$

Now replace the powers of imaginary unit by their expressions:

$$i^2 = -1; \quad i^3 = i^2 i = -i; \quad i^4 = i^3 i = -ii = +1;$$
$$i^5 = i^4 i = i; \quad i^6 = -1$$

Note. In the given example, where the base is $1 + i$ (or, in general, when the base is a binomial of the form $a \pm ai$), involution may be done in a much simpler way. Namely, square $1 + i$. We get $(1 + i)^2 = 2i$, hence $(1 + i)^5 = (1 + i)^4 \cdot (1 + i) = (2i)^2 \cdot (1 + i) = -4\,(1 + i)$.

Answer: $u_6 = -4 + 4i$.

371. We have $u_7 = i\left(1 + \frac{1}{i}\right)^6$. Since $\frac{1}{i} = -i$, $u_7 = i\,(1 - i)^6$. Then we may proceed in the same way as in the preceding problem. We may find the modulus and argument of the product of six factors, each being equal to $1 - i$. The modulus of the quantity $1 - i$ is $\sqrt{2}$, the argument being equal to $-45°$. Hence, the modulus of the product is equal to $(\sqrt{2})^6 = 8$, and the argument, to $6\,(-45°) = -270°$. Consequently,

$$(1 - i)^6 = 8\,[\cos(-270°) + i \sin(-270°)] = 8i$$

Answer: $u_7 = -8$.

372. By hypothesis, the numbers C_n^1; C_n^2; C_n^3 form an arithmetic progression. Hence, $C_n^1 + C_n^3 = 2C_n^2$, i.e.

$$n + \frac{n(n-1)(n-2)}{6} = 2 \cdot \frac{n(n-1)}{2}$$

Since $n \neq 0$, both numbers of the equality can be divided by n. We get the equation $n^2 - 9n + 14 = 0$ with the roots $n_1 = 7$ and $n_2 = 2$. The second root is not suitable, since at $n = 2$ the expansion of the binomial has only three terms, whereas, by hypothesis, there is a fourth term.

Answer: $n = 7$.

373. Solved in the same way as the preceding problem. On reducing by $\frac{n(n-1)(n-2)(n-3)}{4!}$ (this number is non-zero, since, by hypothesis, $n \geqslant 6$) we get $n^2 - 21n + 98 = 0$.

Answer: $n = 14$ or $n = 7$.

374. Rewrite the first addend in parentheses in the form $a^{\frac{4}{5} - \frac{x-1}{x}} = a^{\frac{5-x}{5x}}$; the second, in the form $a \cdot a^{\frac{x-1}{x+1}} = a^{\frac{2x}{x+1}}$. The fourth term of the expansion is $56a^{\frac{5-x}{x} + \frac{6x}{x+1}}$. By hypothesis, $56a^{\frac{5-x}{x} + \frac{6x}{x+1}} = 56a^{5.5}$. Consequently,

$$\frac{5-x}{x} + \frac{6x}{x+1} = 5.5$$

Answer: $x = 2$ or $x = -5$.

375. Represent the given expression in the form

$$[2^{\frac{x-1}{x}} + 2^{\frac{2(3-x)}{4-x}}]^6$$

By hypothesis,

$$15 \cdot 2^{\frac{4(x-1)}{x}} 2^{\frac{4(3-x)}{4-x}} = 240,$$

i.e.

$$2^{\frac{4(x-1)}{x}+\frac{4(3-x)}{4-x}}=2^4$$

Hence,

$$\frac{4(x-1)}{x}+\frac{4(3-x)}{4-x}=4$$

Answer: $x=2$.

376. The seventh term T_7 of the expansion of the binomial $(2^{\frac{1}{3}}+3^{-\frac{1}{3}})^x$ is

$$T_7=C_x^6\,(2^{\frac{1}{3}})^{x-6}\,(3^{-\frac{1}{3}})^6$$

and the seventh term from the end is

$$T_7'=C_x^6\,(2^{\frac{1}{3}})^6\,(3^{-\frac{1}{3}})^{x-6}$$

onsequently,

$$T_7:T_7'=(2^{\frac{1}{3}})^{(x-6)-6}\,(3^{-\frac{1}{3}})^{6-(x-6)}=2^{\frac{x-12}{3}}3^{\frac{x-12}{3}}=6^{\frac{x-12}{3}}$$

By hypothesis, $6^{\frac{x-12}{3}}=\frac{1}{6}$, i.e. $6^{\frac{x-12}{3}}=6^{-1}$. Hence, $\frac{x-12}{3}=-1$.

Answer: $x=9$.

377. By hypothesis, $C_5^2x^3(x^{\log x})^2=10^6$, i.e. $10x^{3+2\log x}=10^6$ or $x^{3+2\log x}=10^5$. Taking logarithms of this equation, we get $(3+2\log x)\log x=5$. Solving the last equation, we get

$$(\log x_1)=1 \text{ and } (\log x_2)=-\frac{5}{2}$$

Answer: $x_1=10$; $x_2=10^{-\frac{5}{2}}=\frac{1}{100\sqrt{10}}$

378. By hypothesis,

$$C_6^3(\sqrt{x})^{\frac{3}{\log x+1}}(\sqrt[12]{x})^3=200, \text{ i.e. } 20x^{\frac{\log x+7}{4(\log x+1)}}=200$$

Dividing both members of this equation by 20 and then taking logarithms, we get after simplifications

$$(\log x)^2+3\log x-4=0$$

Hence, $(\log x_1)=1$, $(\log x_2)=-4$.

Answer: $x_1=10$; $x_2=0.0001$.

379. Solved in the same way as the preceding problem. We get the equation $x^{\log x-2}=1000$. Taking logarithms of the obtained equality, we find $(\log x_1)=3$ and $(\log x_2)=-1$.

Answer: $x_1=1000$; $x_2=0.1$.

380. Solved in the same way as the two previous problems.

Answer: $x_1=10$; $x_2=\frac{1}{\sqrt[5]{10}}$.

381. *Answer:* $x_1=100$; $x_2=\frac{1}{\sqrt[5]{100}}$.

382. *Answer:* $x_1=1000$; $x^2=\frac{1}{\sqrt{10}}$.

383. By hypothesis,

$$T_{k+2}-T_{k+1}=30 \tag{a}$$

where

$$T_{k+1}=C_{12}^{k}x^{-\frac{k}{2}}x^{\frac{12-k}{6}}=C_{12}^{k}x^{\frac{6-2k}{3}} \quad \text{and} \quad T_{k+2}=C_{12}^{k+1}x^{\frac{4-2k}{3}}$$

By hypothesis, the exponent $\frac{6-2k}{3}$ is twice as large as the exponent $\frac{4-2k}{3}$, i.e. $\frac{6-2k}{3}=2\cdot\frac{4-2k}{3}$, whence $k=1$. Then, after simplifications, the equation (a) takes the form:

$$2x^{\frac{4}{3}}-11x^{\frac{2}{3}}+5=0$$

Use the substitution $x^{\frac{2}{3}}=y$.

Answer: $x_1=5\sqrt{5}$; $x_2=\frac{\sqrt{2}}{4}$.

384. By hypothesis, $5C_m^1=C_m^3$, consequently, we have the equation

$$5m=\frac{m(m-1)(m-2)}{1\cdot2\cdot3}$$

Since $m\neq0$, both members of the equation can be divided by m. We get $m_1=7$ and $m_2=-4$. The second root is not suitable, because m should be a positive integer.

By hypothesis, $T_4=7\cdot20$; hence,

$$C_7^3\,(2^{-\frac{x}{3}})^3\,(2^{\frac{x-1}{2}})^4=140$$

Answer: $x=4$.

385. By hypothesis, we have $C_m^2-m=20$; $\frac{m(m-1)}{1\cdot2}-m=20$. Out of the two roots $m_1=8$ and $m_2=-5$ only the first one is suitable, since the exponent of the binomial is assumed to be a positive integer. Rewrite the binomial in the form $(2^{\frac{x}{2}-\frac{3}{16}}+2^{\frac{5}{16}-\frac{x}{2}})^8$. By hypothesis,

$$T_4-T_6=56$$

or

$$C_8^3\cdot2^{3\left(\frac{5}{16}-\frac{x}{2}\right)}2^{5\left(\frac{x}{2}-\frac{3}{16}\right)}-C_8^5 2^{5\left(\frac{5}{16}-\frac{x}{2}\right)}2^{3\left(\frac{x}{2}-\frac{3}{16}\right)}=56$$

After simplifications we get $56\cdot 2^x - 56\cdot\frac{2}{2^x} = 56$. Putting $2^x = y$, we obtain the equation $y^2 - y - 2 = 0$, wherefrom $y_1 = 2$ and $y_2 = -1$. Since $2^x = y$ cannot be a negative number, the only suitable root is $y_1 = 2$ and, hence, $2^x = 2$, i.e. $x = 1$.

Answer: $x = 1$.

386. Since the binomial coefficients of the terms equidistant from the beginning and the end are equal, instead of the coefficients of the last three terms we may take those of the first three terms, i.e. $1 + m + \frac{m(m-1)}{1\cdot 2} = 22$, whence $m = 6$ (see the preceding problem). Hence, the binomial is $(2^{\frac{x}{2}} + 2^{\frac{1-x}{2}})^6$. By hypothesis,

$$T_3 + T_5 = 135$$

or

$$C_6^2 (2^{\frac{1-x}{2}})^2 (2^{\frac{x}{2}})^4 + C_6^4 (2^{\frac{1-x}{2}})^4 (2^{\frac{x}{2}})^2 = 135$$

After simplifications we obtain

$$2^{x+1} + 2^{2-x} = 9 \text{ or } 2\cdot 2^x + \frac{2^2}{2^x} = 9$$

As in the preceding problem, we find: (1) $2^x = 4$ and (2) $2^x = \frac{1}{2}$.

Answer: $x_1 = 2$; $x_2 = -1$.

387. The numbers a_1, a_3, a_5, which are respectively the first, third and fifth terms of an arithmetic progression, form an arithmetic progression themselves, so that $2a_3 = a_1 + a_5$. Since, by hypothesis, $a_1 = C_m^1$; $a_3 = C_m^2$; $a_5 = C_m^3$, then

$$\frac{2m(m-1)}{1\cdot 2} = m + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}$$

Reducing by m $(m \neq 0)$, we find the equation $m^2 - 9m + 14 = 0$, whose roots are $m_1 = 7$, $m_2 = 2$. Since, by hypothesis, the expansion of the binomial contains at least six terms, then $m \geqslant 5$, hence, only $m_1 = 7$ is suitable. The binomial is

$$[2^{\frac{1}{2}\log(10-3^x)} + 2^{\frac{x-2}{5}\log 3}]^7$$

By hypothesis

$$T_6 = 21$$

or

$$C_7^5 2^{(x-2)\log 3} 2^{\log(10-3^x)} = 21$$

Hence we have

$$2^{(x-2)\log 3 + \log(10-3^x)} = 1 = 2^0$$

consequently,

$$(x-2)\log 3 + \log(10-3^x) = 0$$

Taking antilogarithms, we get

$$3^{x-2}(10-3^x) = 1$$

or

$$\frac{3^x}{3^2}(10-3^x)=1$$

Then proceed in the same way as in Problem 385.
Answer: $x_1 = 2$; $x_2 = 0$.

388. By hypothesis, the numbers $\frac{14}{9}C_m^2$, C_m^3 and C_m^4 constitute a geometric progression; consequently,

$$\frac{14}{9}C_m^2C_m^4=(C_m^3)^2$$

Both members of the equality can be divided by $m^2(m-1)^2(m-2)$ since none of the factors (m, $m-1$, $m-2$) is equal to zero (it follows from the given condition that $m \geqslant 3$); we get $m = 9$. By hypothesis, $T_4 = 16.8$, or

$$C_9^3 5^{3\left[\frac{1}{6}\log(x-1)-\frac{1}{3}\log 5\right]} 5^{6\left[-\frac{1}{6}\log(6-\sqrt{8x})\right]}=16.8$$

Hence we get the equation

$$\frac{1}{2}\log(x-1)-\log 5-\log(6-\sqrt{8x})=-1$$

Taking antilogarithms, we have

$$10\sqrt{x-1}=5(6-\sqrt{8x})$$

Hence, $x_1 = 50$ and $x_2 = 2$. The first root is not suitable, since at $x = 50$ the number $6-\sqrt{8x}$ is negative and, consequently, has no logarithm
Answer: $x = 2$.

389. By hypothesis,

$$\log(3\cdot C_m^3)-\log C_m^1=1$$

or

$$\log\frac{3C_m^3}{C_m^1}=\log 10$$

hence $\frac{3C_m^3}{C_m^1}=10$. After simplifications we find the equation $m^2-3m-18=0$, whose roots are $m_1=6$ and $m_2=-3$. Consequently $m=6$. From the condition $9T_3-T_5=240$ we get the equation

$$9C_6^2 2^{2\left(\frac{2}{3}+\frac{x}{2}\right)} 2^{4\left(\frac{x-1}{2}-\frac{1}{3}\right)}-C_6^4 2^{4\left(\frac{2}{3}+\frac{x}{2}\right)} 2^{2\left(\frac{x-1}{2}-\frac{1}{3}\right)}=240$$

whence

$$9\cdot 2^{3x-2}-2^{3x+1}=16$$

or

$$\frac{9\cdot 2^{3x}}{2^2}-2^{3x}\cdot 2=16$$

Hence,

$$2^{3x}=2^6 \text{ and } x=2$$

Answer: $x=2$.

CHAPTER VII

ALGEBRAIC AND ARITHMETIC PROBLEMS

390. The weight of the round is made up of the weights of the projectile, charge and shell. The weights of the projectile and shell taken together make $\frac{2}{3}+\frac{1}{4}=\frac{11}{12}$ of the total weight of the round. Thus, the weight of the charge makes $1-\frac{11}{12}=\frac{1}{12}$ of the weight of the round, which amounts to 0.8 kg. Hence, the weight of the round is equal to 0.8 kg: $\frac{1}{12}=9.6$ kg.

Answer: 9.6 kg.

391. Out of the total number of workers, men make 100% — 35% = 65%. The number of men exceeds that of women by 65% — 35% = 30%, which amounts to 252. Consequently, the total number of workers is equal to $\frac{252\cdot 100}{30}=$ $=840$.

Answer: 840 workers.

392. The profit percentage is calculated in relation to the prime cost (taken for 100%). Hence, the selling-price (1386 roubles) makes 100% + 10% = 110% of the prime cost. Consequently, the prime cost is equal to

$$\frac{1386\cdot 100}{110}=1260 \text{ (roubles)}$$

Answer: 1260 roubles.

393. The loss is calculated in relation to the prime cost (taken for 100%). Hence, 3348 roubles make 100% — 4% = 96% of the prime cost. Consequently, the prime cost of the goods was equal to

$$\frac{3348\cdot 100}{96}=3487.5 \text{ (roubles)}$$

Answer: 3487 roub. 50 kop.

394. The copper content of the ore is $\frac{34.2\cdot 100}{225}$ %.

Answer: 15.2%.

395. The price was reduced by 29 kop. — 26 kop. = 3 kop., which amounts to $\frac{3\cdot 100}{29}$ % of the old price. The number $\frac{3\cdot 100}{29}=10\frac{10}{29}$ is approximately replaced by a decimal fraction.

Answer: 10.34%.

396. Solved in the same way as the previous problem.

Answer: 10.94%.

397. By hypothesis, 2 kg of raisins make 32% of the total weight of the grapes. The weight of the grapes is equal to $\frac{2\cdot 100}{32}=6.25$.

Answer: 6.25 kg.

398. Let us denote the number of tourists by x. In the first case the collected money amounts to $75x$ kop.; hence, a sum of $(75x+440)$ kop. is needed for organizing the excursion. In the second case the collected money amounts to $80x$ kop.; hence, $(80x - 440)$ kop. is needed. Consequently, $75x + 440 = 80x - 440$.

Answer: 176 persons.

399. Let us denote the number of persons by x; then each was to pay $\frac{72}{x}$. By hypothesis,

$$(x-3)\left(\frac{72}{x}+4\right)=72$$

Answer: 9 persons.

400. Let the price of one copy of the first volume amount to x roub., and that of the second volume, to y roub. The first condition yields the equation $60x + 75y = 405$. With a 15% discount the price of one copy of the first volume amounts to $0.85x$ roub.; with a 10% discount the price of one copy of the second volume comes to $0.9y$ roub. From the second condition we obtain the equation

$$60\cdot 0.85x+75\cdot 0.9y=355\frac{1}{2}$$

Solving the system of the two equations, we find $x = 3$, $y = 3$.

Answer: the price of the first volume is 3 roubles; the price of the second volume is also 3 roubles.

401. Let the first item be bought for x roub. Then the second item was bought for $(225 - x)$ roub. The first item yielded a profit of 25%. Hence, it was sold for $1.25x$ roub. The second item which yielded a 50% profit was sold for $1.5\,(225 - x)$ roub. By hypothesis, the shop sold the two items for 225 roubles and made a total profit of 40%. Hence, the two items were sold for $1.40\cdot 225 = 315$ roub. We get the equation

$$1\frac{1}{4}x+1\frac{1}{2}(225-x)=315$$

Answer: the first item was bought for 90 roubles, the second, for 135 roubles.

402. 40 kg of sea water contain $40\cdot 0.05 = 2$ kg of salt. For 2 kg to amount to 2% of the total weight, the latter must be equal to $2 : 0.02 = 100$ kg.

Answer: 60 kg of fresh water should be added.

403. Let us denote the lengths of the legs (in metres) by x and y. By hypothesis, $x^2 + y^2 = (3\sqrt{5})^2$. After the first leg is increased by $133\frac{1}{3}\%$, i.e. by $133\frac{1}{3} : 100 = 1\frac{1}{3}$ of its length, it is equal to $2\frac{1}{3}x$. Increased by $16\frac{2}{3}\%$ the second leg is equal to $1\frac{1}{6}y$. Thus, we get the equation $2\frac{1}{3}x + 1\frac{1}{6}y = 14$.

Answer: 3 m and 6 m.

404. If we take 12.5% of the flour contained in the first sack, then 87.5% of the flour is left in it, which amounts to 140 kg : 2 = 70 kg. Consequently, the first sack contains $\frac{70\cdot 100}{87.5}$.

Answer: the first sack contains 80 kg, the second, 60 kg.

405. Both factories together could fulfill $\frac{1}{12}$ of the order per day. By hypothesis, B has an efficiency of $66\frac{2}{3}\%$, i.e. $\frac{2}{3}$ of that of A; consequently, the efficiency of both factories is $1\frac{2}{3}$ of that of A. Hence, A can daily fulfill $\frac{1}{12}:1\frac{2}{3}=\frac{1}{20}$ of the order, while B, $\frac{1}{20}\cdot\frac{2}{3}=\frac{1}{30}$ of the order. Before A was closed down $\frac{2}{12}=\frac{1}{6}$ of the order was fulfilled. To fulfill the remaining $\frac{5}{6}$ of the order B needs another $\frac{5}{6}:\frac{1}{30}=25$ days.

Answer: the order will be completed in $27=(25+2)$ days.

406. The 14 students, who obtained correct solutions, make $100\%-(12\%+32\%)=56\%$ of the total number of the students of the class. The total number of the students is $\frac{14\cdot 100}{56}=25$.

Answer: 25 students.

407. The weight of the cut-off piece makes 72% of the total weight of the rail, hence, the weight of the remaining part (45.2 kg) amounts to $100\%-72\%=28\%$ of the total weight of the rail; 1% of this weight is $\frac{45.2}{28}$, and 72% amount to $\frac{45.2}{28}\cdot 72=116\frac{8}{35}$ kg ≈ 116.23 kg.

Instead of determining one percent of the weight of the rail we may set up the proportion $x:45.2=72:28$.

Answer: the weight of the cut-off piece is (approximately) equal to 116.2 kg.

408. The weight of the piece of the alloy (2 kg) makes $100\%+14\frac{2}{7}\%=114\frac{2}{7}\%$ of the weight of the copper. Hence, 1% of the weight of the copper amounts to $\frac{2}{114\frac{2}{7}}$ kg. Consequently, the weight of the silver, which comes to $14\frac{2}{7}\%$ of the weight of the copper, is equal to

$$\frac{2}{114\frac{2}{7}}\cdot 14\frac{2}{7}=\frac{1}{4}\text{ kg}$$

Instead of determining one percent of the weight of the copper, we may set up the proportion

$$x:2=14\frac{2}{7}:114\frac{2}{7}$$

Answer: the weight of the silver is $\frac{1}{4}$ kg.

409. The money received by the second worker amounts to $1\frac{3}{4}:7\frac{1}{2}=\frac{7}{30}$ of that of the first, or, in percent, $\frac{7}{30}\cdot 100\%=23\frac{1}{3}\%$. The total received by the three workers (4080 roubles) makes

$$100\%+23\frac{1}{3}\%+43\frac{1}{3}\%=166\frac{2}{3}\%$$

of what was paid to the first worker. One percent of the money received by the first worker comes to $\frac{4080}{166\frac{2}{3}}$ roub., hence the first worker was paid

$$\frac{4080}{166\frac{2}{3}}\cdot 100=2448 \text{ roub.}$$

The second worker received $23\frac{1}{3}\%$ of this sum, i.e.

$$\frac{2448\cdot 23\frac{1}{3}}{100}=571.2 \text{ roub.}$$

The third worker earned

$$\frac{2448\cdot 43\frac{1}{3}}{100}=1060.8 \text{ roub.}$$

Answer: 2448 roub.; 571 roub. 20 kop.; 1060 roub. 80 kop.

410. If the first box contains x kg of sugar, then the second contains $\frac{4}{5}x$ kg, and the third, $\frac{4}{5}x\cdot\frac{42\frac{1}{2}}{100}=\frac{17}{50}x$ kg. By hypothesis, $x+\frac{4}{5}x+\frac{17}{50}x=$ $=64.2$, whence $x=30$ (kg). Of this number we take first $\frac{4}{5}$, and then $\frac{17}{50}$.

Answer: 30 kg; 24 kg; 10.2 kg.

411. Let us take x tons of the first grade; it contains $0.05x$ tons of nickel, and then it is necessary to take $(140-x)$ tons of the second grade, containing $0.40\,(140-x)$ tons of nickel. By hypothesis, 140 tons of steel contains $0.30\cdot 140$ tons of nickel. Consequently, $0.05x+0.40\,(140-x)=0.30\cdot 140$. Hence, $x=40$.

Answer: 40 tons of the first grade and 100 tons of the second grade.

412. The piece of the alloy contains 12 kg$\cdot 0.45=5.4$ kg of copper. Since in the piece of the new alloy this amount of copper makes 40% of its weight, the piece weighs $5.4:0.40=13.5$ kg. Hence, it is required to add 13.5 kg − 12kg = 1.5 kg of pure tin.

Answer: 1.5 kg.

413. Solved in the same way as the preceding problem: (1) 735 g$\cdot 0.16=$ $=117.6$ g; (2) 117.6 g : 0.10 = 1176 g; (3) 1176 g − 735 g = 441 g.

Answer: 441 g.

414. Let x denote the weight of copper (in kg). Then $24 - x$ is the weight of zinc. The loss of weight is $\frac{1}{9}x$ (for copper) and $\frac{1}{7}(24 - x)$ (for zinc). Consequently, $\frac{1}{9}x + \frac{1}{7}(24 - x) = 2\frac{8}{9}$. Hence, $x = 17$.

Answer: 17 kg of copper, 7 kg of zinc.

415. Let us denote the number of 25-metre lengths by x, and that of 12.5-metre lengths, by y. For a 20-kilometre (20 000-metre) track 40 000 m of rails are needed (two lines). By hypothesis,

$$25x + 0.50 \cdot 12.5y = 40\,000 \text{ and } 12.5y + \frac{66\frac{2}{3}}{100} \cdot 25x = 40\,000$$

Answer: 1200 pieces of 25-metre rails and 1600 pieces of 12.5-metre rails.

416. Let the number of students be x. During the exchange each student received $x - 1$ photographs and all the students received $x(x - 1)$ photographs; by hypothesis, we have the equation $x(x - 1) = 870$.

Answer: 30 students.

417. Let us denote the smaller number by x and the larger number, by y $(x < y)$. The first condition yields $\sqrt{xy} = x + 12$, and the second condition, $\frac{x+y}{2} = y - 24$, i.e. $y - x = 48$. Solving the system, we find $x = 6$, $y = 54$. Since $6 < 54$, this solution is suitable.

Answer: 6 and 54.

418. Let the smallest number be x, the next one, y, and the largest, z. We have three equations

$$y - x = z - y; \quad xy = 85; \quad yz = 115$$

From the first equation we find $z = 2y - x$; substituting it into the third equation, we get $2y^2 - xy = 115$ or, by virtue of the second equation, $2y^2 = 200$. Out of the two solutions ($x_1 = 8.5$, $y_1 = 10$, $z_1 = 11.5$; $x_2 = -8.5$, $y_2 = -10$, $z_2 = -11.5$) the first one is suitable (since $x_1 < y_1 < z_1$), and the second is not (since $x_2 > y_2 > z_2$).

Answer: 8.5; 10; 11.5.

419. Given

$$\frac{x+y+z}{3} = a \quad \text{and} \quad \frac{x^2+y^2+z^2}{3} = b$$

It is required to find $\frac{xy+yz+zx}{3}$. From the first equation we have $x^2 + y^2 + z^2 + 2(xy + yz + zx) = 9a^2$. By virtue of the second equation we have $x^2 + y^2 + z^2 = 3b$. Hence, $3b + 2(xy + yz + zx) = 9a^2$.

Answer: $\frac{xy+yz+zx}{3} = \frac{3a^2-b}{2}$.

420. If the length of the sheet is x cm, and width, y cm, then the box has the following dimensions: length—$(x - 8)$ cm, width—$(y - 8)$ cm and height—4 cm. By hypothesis, $4(x - 8)(y - 8) = 768$ and $2x + 2y = 96$.

Answer: 32 cm × 16 cm.

421. Let the tens digit be x, and units digit, y (x and y are positive integers less than 10). We have the following system of equations:

$$\frac{10x+y}{xy} = 2\frac{2}{3}; \quad (10x+y)-(10y+x)=18$$

Out of the two solutions $\left(x = 6,\ y = 4 \text{ and } x = \frac{1}{8},\ y = -\frac{15}{8}\right)$ only the first one is suitable.

Answer: 64.

422. If the number of tens is x, then the number of units is equal to $x + 2$. We get the equation

$$[10x + (x + 2)]\,[x + (x + 2)] = 144$$

whence $x = 2$ and $x = -3\frac{2}{11}$; by hypothesis, the second solution is not suitable.

Answer: the required number is 24.

423. Let the required number be x. If the figure 5 is adjoined on the right of it, then we get the number $10x + 5$. By hypothesis, we have

$$10x + 5 = (x + 3)(x - 13)$$

Answer: 22.

424. Let the larger number be x, and the smaller, y. If three digits (zero and the two digits of the smaller number) are adjoined to the larger number, then the digits of the latter express the number of thousands, and thus, finally we get $1000x + y$. And from smaller number we get the number $1000y + 10x$. By hypothesis,

$$1000x + y = 2\,(1000y + 10x) + 590; \quad 2x + 3y = 72$$

Solving the system, we find $x = 21$, $y = 10$. Being two-digit numbers, they satisfy the condition of the problem.

Answer: 21 and 10.

425. If the units digit of the factor is x (x an integer, less than 10), then the tens digit is $3x$. The factor is equal to $3 \cdot 10x + x = 31x$. The incorrectly written factor is $10x + 3x = 13x$. The true product is equal to $78 \cdot 31x$, the product obtained by mistake is $78 \cdot 13x$. By hypothesis, $78 \cdot 31x - 78 \cdot 13x = 2808$. Hence, $x = 2$.

Answer: the true product is equal to 4836.

426. The speed of the first train is x km/h, that of the second, $(x - 12)$ km/h. We have the equation

$$\frac{96}{x-12} - \frac{96}{x} = \frac{2}{3}$$

Answer: the speed of the first train is equal to 48 km/h, that of the second, to 36 km/h.

427. Let the rate of the first person be v km/h, then the rate of the second is equal to $(v - 2)$ km/h. The first spends $\frac{24}{v}$ h, the second, $\frac{24}{v-2}$ h. We obtain the equation

$$\frac{24}{v-2} - \frac{24}{v} = 1$$

Answer: 8 km/h; 6 km/h.

428. Let the speed of the train be x km/h; then the speed of the boat is $(x - 30)$ km/h. The train spends $\frac{66}{x}$ h, and the boat, $\frac{80.5}{x-30}$ h. We get the equation

$$\frac{80.5}{x-30} - \frac{66}{x} = 4 + \frac{15}{60}$$

Answer: the speed of the train is 44 km/h, that of the boat is 14 km/h.

429. Let the first tailor shop produce x suits a day; then the second shop makes $x+4$ suits per day. The first shop has completed its order in $\frac{810}{x}$ days; hence, the time given for the fulfilment of the order has been $\left(\frac{810}{x}+3\right)$ days. The time given to the second shop has been the same. Consequently,

$$\frac{810}{x}+3=\frac{900}{x+4}+6$$

Answer: the first shop produces 20, and the second, 24 suits per day.

430. Let the speed of the ship going off to the south be x km/h, and that of the other ship, $(x+6)$ km/h. Since the directions of their travel are mutually perpendicular, by the Pythagorean theorem we have

$$(2x)^2+[2(x+6)]^2=60^2$$

Answer: the speed of the first ship is equal to 18 km/h, that of the second, to 24 km/h.

431. Two dog's leaps cover 4 metres; three fox's leaps, 3 metres. Consequently, each time the dog runs 4 metres the distance between the dog and the fox is reduced by 4 m − 3 m = 1 m. The initial distance between them is 30 times greater. Hence, the dog will catch up the fox, when it covers $4\text{ m}\cdot 30=120$ m.

Answer: at a distance of 120 m.

432. In one minute the minute hand turns through 6°, while the hour hand, only through $\frac{1}{2}^{\circ}$. At four o'clock the angle between the hands is equal to 120°. During x minutes the hands turn through $6x$ and $\frac{1}{2}x$ degrees, respectively. By hypothesis, $6x-\frac{1}{2}x=120$.

Answer: $21\frac{9}{11}$ minutes.

433. Let us denote the time spent by the train to cover the A-C section by t (hours) and the required speed, by v (km/h). By hypothesis, the distance AB was covered by the train in $\frac{t}{2}$ h at a speed of v km/h, and BC, in $\frac{t}{2}$ h at a speed of $0.75\cdot v$ km/h. Hence, $AB=v\frac{t}{2}$ km and $BC=0.75\cdot v\frac{t}{2}$ km. By hypothesis, on the return trip the C-B section was covered at a speed of v, and the B-A section, at a speed of $0.75v$. Hence, the time spent on the C-B section was $\frac{0.75\,vt}{2}:v$, i.e. $\frac{0.75\,t}{2}$ h, and the time spent on the B-A section was $\frac{vt}{2}:0.75\,v$, i.e. $\frac{t}{2\cdot 0.75}$ h. By hypothesis,

$$\frac{t}{2\cdot 0.75}+\frac{0.75t}{2}=\frac{5}{12}+t$$

Answer: 10 hours.

434. Suppose the cyclist travelled at a speed of v km/h; then the required speed was $(v-1)$ km/h. The cyclist actually travelled $\frac{30}{v}$ h, while the scheduled time was $\frac{30}{v-1}$ h. By hypothesis,

$$\frac{30}{v-1}-\frac{30}{v}=\frac{3}{60}$$

The negative solution $v=-24$ is not suitable.

Answer: 25 km/h.

435. Let the scheduled speed be x km/h. The actual speed was $(x+10)$ km/h. The scheduled time is $\frac{80}{x}$ h, but actually it was $\frac{80}{x+10}$ h. By hypothesis,

$$\frac{80}{x}-\frac{80}{x+10}=\frac{16}{60}$$

Answer: 50 km/h.

436. The first half of the distance was covered by the train in x hours. Then, to arrive in time the train had to cover the remaining portion of the route in $x-\frac{1}{2}$ hours. The speed of the train in the first half of the route was $\frac{420}{x}$ km/h, in the second, $\frac{420}{x-\frac{1}{2}}$ km/h. By hypothesis,

$$\frac{420}{x-\frac{1}{2}}-\frac{420}{x}=2$$

The equation has only one positive root.

Answer: 21 hours.

437. Let the speed of the first train be x km/h, that of the second, y km/h. In the first case the first train covers $10x$ kilometres before they meet, the second, $10y$ kilometres. Consequently,

$$10x+10y=650$$

In the second case the first train covers $8x$ kilometres, and the second (which spent en route 8 h + 4 h 20 min $=12\frac{1}{3}$ h), $12\frac{1}{3}y$. Consequently,

$$8x+12\frac{1}{3}y=650$$

Answer: the mean speed of the first train is 35 km/h, that of the second is 30 km/h.

438. Let the speed of the first train be x km/h, and that of the second, y km/h. The distance of 600 km is covered by the first train in $\frac{600}{x}$ hours, and by the second, in $\frac{600}{y}$ hours. By hypothesis,

$$\frac{600}{x}+3=\frac{600}{y}, \quad \frac{250}{x}=\frac{200}{y}$$

Answer: the speed of the first train is 50 km/h, that of the second is 40 km/h.

439. If the distance is x km, then at a rate of 3.5 km/h the commuter would cover this distance in $\frac{x}{3.5}$ hours. And since he would be one hour late, the moment he started out was separated from the train leaving time by $\left(\frac{x}{3.5}-1\right)$ hours. In an hour, during which he had walked 3.5 km, there remained $\left(\frac{x}{3.5}-2\right)$ hours till the train departure, and he had to cover the remainder of the distance of $(x-3.5)$ km. At a rate of 5 km/h the commuter covered this distance in $\frac{x-3.5}{5}$ hours. Since he arrived half an hour before the train leaving time, we have

$$\frac{x}{3.5}-2-\frac{x-3.5}{5}=\frac{1}{2}$$

Answer: 21 km.

440. Let the speed of the cyclist be x km/min, and that of the car, y km/min. The car had travelled 10 minutes, and the cyclist, $10+15=25$ minutes when he was caught up by the car. By this moment they had covered one and the same distance. Consequently, $25x=10y$. By the time the car on its return trip encountered the cyclist the car had covered $50y$ km, and the cyclist, $65x$ km. The sum of these distances is equal to twice the distance between A and B. Therefore $65x+50y=38$. Solving the system of equations, we find $x=0.2$; $y=0.5$.

Answer: the speed of the cyclist is equal to 0.2 km/min = 12 km/h, and that of the car, to 0.5 km/min = 30 km/h.

441. Let the trains pass each other in x hours after the fast train departure. Then, by the time of the encounter the mail train had travelled $(x+3)$ hours. Each train had covered $1080:2=540$ (km) before they met. Hence, the speed of the first train is equal to $\frac{540}{x}$ km/h and that of the second, to $\frac{540}{x+3}$ km/h. By hypothesis, $\frac{540}{x}-\frac{540}{x+3}=15$. Only one root is suitable: $x=9$.

Answer: in 9 hours after the fast train departure.

442. Let the first cyclist travel x hours. Reasoning in the same way as in the preceding problem, we set up the equation

$$\frac{36}{x-1}-\frac{42}{x}=4$$

Answer: the speed of the first cyclist is equal to 14 km/h and that of the second, to 18 km/h; the first is 3 hours en route prior to the encounter, the second, 2 hours.

443. Let the distance AB between the starting points be x km, and let the first hiker cover it in y hours. By hypothesis, the second hiker covers the distance BA in $(y-5)$ hours. Hence, the first covers $\frac{x}{y}$ kilometres per hour, and the second, $\frac{x}{y-5}$ kilometres per hour. During an hour the distance between the hikers is reduced by $\left(\frac{x}{y}+\frac{x}{y-5}\right)$ km, during $3\frac{1}{3}$ hours, by $3\frac{1}{3}\left(\frac{x}{y}+\frac{x}{y-5}\right)$. Since they meet in $3\frac{1}{3}$ hours, we have $3\frac{1}{3}\left(\frac{x}{y}+\frac{x}{y-5}\right)=x$. Since $x \neq 0$, we can divide both members by x. We get:

$$3\frac{1}{3}\left(\frac{1}{y}+\frac{1}{y-5}\right)=1$$

Hence, we find y. The value of x remains undetermined.

Answer: the first hiker covers the whole distance in 10 hours and the second, in 5 hours.

444. Let us denote the point of encounter by C. Let $AC = x$ km; then, by hypothesis, $CB = (x + 12)$ km. Furthermore, by hypothesis, the first hiker covers the distance CB in 8 hours. Hence, his rate is equal to $\frac{x+12}{8}$ km/h. In the same way we find that the rate of the second hiker is $\frac{x}{9}$ km/h. Consequently, the distance AC is covered by the first one in $x : \frac{x+12}{8} = \frac{8x}{x+12}$ hours, while the second covers the distance BC in $\frac{9(x+12)}{x}$ hours. And since the second travels 6 hours more than the first one, we have

$$\frac{9(x+12)}{x}-\frac{8x}{x+12}=6$$

When solving this equation we may introduce an auxiliary unknown $\frac{x+12}{x} = z$. We get $9z - \frac{8}{z} = 6$. Out of the two roots $\left(z_1 = \frac{4}{3} \text{ and } z_2 = -\frac{2}{3}\right)$ the second one is not suitable, since both quantities $x = AC$ and $x + 12 = CB$ must be positive. From the equation $\frac{x+12}{x} = \frac{4}{3}$ we find $x = 36$. Hence, $AC = 36$ km, $CB = 48$ km.

Answer: $AB = 84$ km. The rate of the first hiker is 6 km/h and that of the second one is 4 km/h.

445. The problem is similar to the preceding one. Let the dirigible fly to the passing point x km; then the airplane has made $(x+100)$ km. The speed of the dirigible is equal to $\frac{x+100}{3}$ km/h and that of the airplane, to $\frac{x}{1\frac{1}{3}}$ km/h. From its terminal to the passing point the dirigible flies $x : \frac{x+100}{3} = \frac{3x}{x+100}$ hours; whereas the airplane covers the distance between the airport and the passing point in $\frac{1\frac{1}{3}(x+100)}{x}$ hours. We obtain the

equation

$$\frac{3x}{x+100}=\frac{\frac{4}{3}(x+100)}{x}, \text{ i.e. } \left(\frac{x}{x+100}\right)^2=\frac{4}{9}$$

Consequently, $\frac{x}{x+100} = \pm\frac{2}{3}$, whence $x = 200$; the second root is not suitable.

Answer: the distance between the airports is equal to 500 km; the speed of the dirigible is 100 km/h and that of the airplane is 150 km/h.

446. *First method*. The problem may be solved in the same way as the preceding one. We get the equation

$$\left(\frac{x}{x-a}\right)^2=\frac{n}{m}, \text{ i.e. } \frac{x}{x-a}=\frac{\sqrt{n}}{\sqrt{m}}$$

Hence,

$$x=\frac{a\sqrt{n}}{\sqrt{n}-\sqrt{m}}$$

Then we find the speeds of the hikers:

$$v_1=\frac{x-a}{m} \text{ and } v_2=\frac{x}{n}$$

Second method. Let us denote the point of encounter by C. Since the first hiker covers the distance CB in m hours, we have $CB=v_1m$ km. Similarly, $CA=v_2n$ km. By hypothesis, $CA-CB=a$. We get the equation $nv_2-mv_1=a$. The section AC is covered by the first hiker in $\frac{AC}{v_1}$ hours; hence, the distance between the starting and the encounter points he covers in $\frac{v_2n}{v_1}$ hours. Similarly, for the second hiker: $\frac{v_1m}{v_2}$ hours. Since they start out at the same time, we have $n\frac{v_2}{v_1}=m\frac{v_1}{v_2}$, whence $v_1:v_2=\sqrt{n}:\sqrt{m}$. Let us solve this equation together with the first one. For the sake of symmetry it is useful to make the following substitution: $t=\frac{v_1}{\sqrt{n}}=\frac{v_2}{\sqrt{m}}$. Substituting the expressions $v_1=\sqrt{n}t$ and $v_2=\sqrt{m}t$ into the first equation, we get $(n\sqrt{m}-m\sqrt{n})t=$ $=a$, whence $t=\frac{a}{n\sqrt{m}-m\sqrt{n}}$; now we find

$$v_1=\frac{a\sqrt{n}}{n\sqrt{m}-m\sqrt{n}}, \quad v_2=\frac{a\sqrt{m}}{n\sqrt{m}-m\sqrt{n}}$$

Note. The problem is solvable only if $n\sqrt{m}>m\sqrt{n}$; dividing both members of this inequality by the positive number $\sqrt{m}\sqrt{n}$, we get $\sqrt{n}>\sqrt{m}$, i.e. $n>m$. This condition can be obtained immediately from the given one: since the first hiker covers a greater distance before they meet, his rate is higher than that of the second hiker. On the other hand, the first hiker has to cover a shorter distance to arrive at B than the second to arrive at A. Consequently, the first will reach B faster than the second will reach A.

Answer: the speed of the first hiker is $\frac{a\sqrt{n}}{n\sqrt{m}-m\sqrt{n}}$ km/h and that of the second, $\frac{a\sqrt{m}}{n\sqrt{m}-m\sqrt{n}}$ km/h.

447. Let the first body make x degrees in one second, and the second, y degrees. From the first condition we find $\frac{360}{y}-\frac{360}{x}=5$. Each second the distance between the bodies (as measured along the arc) is increased by $(x-y)$ degrees. Every 100 seconds the distance must be increased by 360°. Therefore, $100(x-y)=360$. The obtained system has two solutions ($x_1=18$, $y_1=14.4$; $x_2=-14.4$, $y_2=-18$). Both of them are suitable, since they have one and the same physical meaning, only the numbers of the bodies and the direction of motion being changed.

Answer: 18°; 14°24′.

448. Let us denote the speed of one body (in m/min) by x and that of the other, by y, assuming that $x>y$. Let the bodies move in the same direction and come together at some point A. Let the next nearest encounter take place at a point B (these points may coincide: for instance, in case the speed of the first body is twice that of the second; in this event by the moment they come together once again the first body will complete two revolutions, whereas the second, only one).

Moving from A to B (this path may overlap itself for one or both bodies), the second body lags behind the first one so that by the moment of the nearest encounter the delay will be equal to the full circumference. Since the bodies come together every 56 minutes, during which the first body covers a distance of $56x$ metres, and the second one, $56y$ metres, the circumference is equal to $56x-56y$.

Let now the bodies move in opposite directions. Then the sum of the distances covered by them during the interval between the two nearest encounters, i.e. during 8 minutes, will make the whole circumference. Consequently, the circumference is equal to $8x+8y$. Thus we have the equation $56x-56y=8x+8y$.

By hypothesis, in 24 seconds the distance (along the circumference) between the approaching bodies decreases by $40-26=14$ metres. During this time the bodies do not come together; therefore the decrease in the distance between the bodies equals the sum of the distances covered by them in 24 seconds $=\frac{2}{5}$ minute. And so we get the second equation

$$\frac{2}{5}x+\frac{2}{5}y=14$$

Answer: 20 m/min; 15 m/min; 280 m.

449. Let x and y be positive numbers expressing the speed of the points in corresponding units (if c is the circumference in metres, then the unit of speed is 1 m/sec, and so on; it is not clear from the given conditions in what units the length is measured). Assuming that $x>y$, we have the system of equations

$$tx-ty=c;\quad \frac{c}{y}-\frac{c}{x}=n$$

(for the setting up of the first equation see the preceding problem). Making a substitution, we find the equation

$$nty^2+ncy-c^2=0$$

Its positive root is $y=\frac{c\left(\sqrt{n^2+4nt}-n\right)}{2nt}$ (the second root is negative)

Answer: the higher speed is numerically equal to $\frac{c\left(\sqrt{n^2+4nt}+n\right)}{2nt}$, the lower, to $\frac{c\left(\sqrt{n^2+4nt}-n\right)}{2nt}$.

450. Let the speed of the ship in still water be x km/h. Then we have the equation $\frac{80}{x+4}+\frac{80}{x-4}=8\frac{1}{3}$.

Answer: 20 km/h.

451. *Answer*: 9 km/h.

452. Let the rate of the current of water be x km/h, and the speed of the boat in still water, y km/h. The first condition yields the equation $\frac{20}{y+x}+\frac{20}{y-x}=10$; the second condition, the equation $\frac{2}{y-x}=\frac{3}{y+x}$. In solving this system it is convenient to put

$$\frac{1}{y+x}=u;\quad \frac{1}{y-x}=v$$

Solving the system,

$$20u+20v=10;\ 2v=3u,$$

we find

$$u=\frac{1}{5};\quad v=\frac{3}{10},\text{ i.e. } y+x=5;\quad y-x=\frac{10}{3}$$

whence $x=\frac{5}{6}$.

Answer: $\frac{5}{6}$ km/h.

453. Let the raft float down the river over the distance (a km) between A and B in x days. Then its rate, equal to the rate of the current of water is $\frac{a}{x}$ km/day. By hypothesis, the speed of the ship going downstream is equal to $\frac{a}{2}$ km/day. Consequently, the speed of the ship in still water is $\left(\frac{a}{2}-\frac{a}{x}\right)$ km/day. And since the speed of the ship going upstream is equal to $\frac{a}{3}$ km/day, its speed in still water is $\left(\frac{a}{3}+\frac{a}{x}\right)$ km/day. We have the equation

$$\frac{a}{2}-\frac{a}{x}=\frac{a}{3}+\frac{a}{x}$$

Answer: 12 days.

454. Let the speed of the body M_1 be x m/s, and that of M_2, y m/s. By the moment of the first encounter M_1 has been in motion during 21 seconds, and M_2, during $21-15=6$ seconds. Thus, we get the equation

$$21x+6y=60$$

By the moment of the second encounter M_1 has been in motion during 45 seconds, and M_2, during $45-15=30$ seconds. Let C be the point of the second encounter; then M_1 by the moment of the second encounter has covered the distance $AB+BC$, and M_2, the distance $BA+AC$. The sum of these distances is $3AB$, i.e. 180 m. And so we obtain the second equation

$$45x+30y=180$$

Answer: the speed of the body M_1 is equal to 2 m/s, and that of the body M_2, to 3 m/s.

455. Let the speed of the messenger when going uphill be equal to x km/h, over the level ground, to y km/h, and downhill, to z km/h. Before returning the messenger has covered half the distance, i.e. $14:2=7$ km; he has gone 3 km uphill, 4 km over the level ground, then (on his way back) another 4 km over the level ground and, finally, 3 km downhill. By hypothesis,

$$\frac{3}{x}+\frac{4}{y}+\frac{4}{y}+\frac{3}{z}=3\frac{3}{5}, \text{ i.e. } \frac{3}{x}+\frac{8}{y}+\frac{3}{z}=3\frac{3}{5}$$

The other two conditions yield:

$$\frac{3}{x}+\frac{5}{y}+\frac{6}{z}=3\frac{9}{20}, \quad \frac{6}{x}+\frac{5}{y}+\frac{3}{z}=3\frac{17}{20}$$

We find $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ and then x, y, z.

Answer: uphill—3 km/h, over the level ground—4 km/h, downhill—5 km/h.

456. Let the quota be x pages a day and the time limit, y days. Then, by hypothesis,

$$(x+2)(y-3)=xy \quad \text{and} \quad (x+4)(y-5)=xy$$

Answer: 120 pages, 15 days.

457. Let the operator make x parts in y days. Then he produced $\frac{x}{y}$ parts per day. By hypothesis, if he had turned out $\frac{x}{y}+10$ parts, he would have completed the job in $y-4\frac{1}{2}$ days. Hence, $\left(\frac{x}{y}+10\right)\left(y-4\frac{1}{2}\right)=x$. The other condition yields the equation $\left(\frac{x}{y}-5\right)(y+3)=x$. We get the following system of equations

$$\begin{cases} 10y-4\frac{1}{2}\frac{x}{y}=45 \\ -5y+3\frac{x}{y}=15 \end{cases}$$

Multiplying the second equation by 2 and adding the product to the first one, we get $\frac{x}{y}=50$. Substituting this value into the second equation, we find $y=27$. Consequently, $x=50y=1350$.

Note. This problem may be solved in the same way as the preceding one, if instead of the unknown x we introduce the quantity z denoting the number of parts produced daily. We would obtain the same system of equations, where the quantity $\frac{x}{y}$ would be replaced by z.

Answer: the worker made 1350 parts in 27 days.

458. Let the daily quota of the typist be x pages, and the time limit, y days; then the job involves the typing of xy pages. By hypothesis, typing $x + 2$ pages per day, the typist would spend $y - 2$ days. Hence, the job involves $(x + 2)(y - 2)$ pages. Consequently,

$$(x + 2)(y - 2) = xy$$

Reasoning in the same way, we get another equation:

$$(x + 0.60x)(y - 4) = xy + 8.$$

Answer: the quota was 10 pages per day, and the time limit, 12 days.

459. Let the first worker complete the task in x hours. Then we have the equation $\frac{1}{x} + \frac{1}{x+12} = \frac{1}{8}$.

Answer: the first worker can do the job individually in 12 hours, the second, in 24 hours.

460. If the first pipe fills the swimming pool in x hours, then the second fills it in $(x + 5)$ hours. The given condition yields the equation $\frac{1}{x} + \frac{1}{x+5} = \frac{1}{6}$.

Answer: the first pipe fills the pool in 10 hours, the second, in 15 hours.

461. Let the first worker, working individually, be able to complete the task in x hours, and the second, in y hours. Then in one hour the first fulfills $\frac{1}{x}$ of the whole assignment, and the second, $\frac{1}{y}$. By hypothesis, $7\frac{1}{x} + 4\frac{1}{y} = \frac{5}{9}$. Since then they worked together for another 4 hours, they did $\left(\frac{4}{x} + \frac{4}{y}\right)$ of the job, which was equal to $1 - \left(\frac{5}{9} + \frac{1}{18}\right) = \frac{7}{18}$. Thus, we have the second equation $\frac{4}{x} + \frac{4}{y} = \frac{7}{18}$. Subtracting it from the first equation, we get $\frac{3}{x} = \frac{3}{18}$; whence $x = 18$. Then we find $\frac{1}{y} = \frac{1}{24}$ and $y = 24$.

Answer: it would take the first worker 18 hours and the second, 24 hours to do the whole assignment.

462. Let us denote the required numbers by x and y. Four high-power cranes operated $2 + 3 = 5$ hours; two low-power cranes worked three hours. Therefore (see solution of the preceding problem)

$$4 \cdot 5 \cdot \frac{1}{x} + 2 \cdot 3 \cdot \frac{1}{y} = 1$$

The second condition yields

$$4 \cdot 4.5 \cdot \frac{1}{x} + 2 \cdot 4.5 \cdot \frac{1}{y} = 1$$

Answer: 24 hours, 36 hours.

463. Let one three-ton truck be able to deliver the load during x hours, and one five-ton truck, during y hours. By hypothesis (see solutions of Problems 461 and 462),

$$30\cdot 8\cdot\frac{1}{x}+9\cdot 6\cdot\frac{1}{y}=1 \quad\text{and}\quad 9\cdot 8\cdot\frac{1}{y}+30\cdot 6\cdot\frac{1}{x}=\frac{13}{15}$$

Answer: $x = 300$; $y = 270$; 30 five-ton trucks will deliver all the material in $270 : 30 = 9$ hours.

464. Let it take the first typist x hours and the second, y hours to do the whole job. When the first was busy typing for three hours, the second was busy only for 2 hours. Both of them did $1-\frac{9}{20}=\frac{11}{20}$ of the whole work. We get the equation

$$\frac{3}{x}+\frac{2}{y}=\frac{11}{20}$$

When the assignment was completed, it turned out that each typist had done half the work. Hence, the first spent $\frac{x}{2}$ hours, and the second, $\frac{y}{2}$ hours. And since the first had begun one hour before the second, we have

$$\frac{x}{2}-\frac{y}{2}=1$$

The system has two solutions, but one of them is not suitable, since it yields a negative value for y.

Answer: 10 hours (the first typist), 8 hours (the second typist).

465. The problem is similar to the preceding one. We get

$$\frac{2}{x}+\frac{1.5}{y}=\frac{11}{30};\quad \frac{x}{2}-\frac{y}{2}=\frac{1}{2}$$

where x and y are the times (in hours) for each train to travel the distance between A and B. Out of the two solutions yielded by the system only one is suitable.

Answer: 10 hours; 9 hours.

466. Let x litres of water per minute flow in through the first pipe, and y litres per minute flow out through the second pipe. By hypothesis, a full bath containing $2\times 9\times 2.5 = 45$ litres can be emptied in one hour, if both pipes are open. Hence, the amount of water is reduced by $\frac{45}{60}=\frac{3}{4}$ litre per minute. Consequently, $y-x=\frac{3}{4}$. On the other hand, when only the first pipe is used, the bath can be filled in $\frac{45}{x}$ minutes; when only the second pipe is used, the bath can be emptied in $\frac{45}{y}$ minutes. By hypothesis,

$$\frac{45}{x}-\frac{45}{y}=5$$

The system of equations

$$\begin{cases} y-x=\frac{3}{4} \\ \frac{45}{x}-\frac{45}{y}=5 \end{cases}$$

has two solutions $\left(x_1=2\frac{1}{4};\ y_1=3 \text{ and } x_2=-3;\ y_2=-2\frac{1}{4}\right)$. The second solution is not suitable (x and y must be positive numbers).

Answer: $2\frac{1}{4}$ l/min; 3 l/min.

467. Let the time limit be x days; then the daily plan is $\frac{8000}{x}$ cubic metres. The team of navvies completed the job in $x-8$ days; hence, the daily output was $\frac{8000}{x-8}$ cubic metres. By hypothesis, $\frac{8000}{x-8}-\frac{8000}{x}=50$. Out of the two roots yielded by this equation ($x_1=40$ and $x_2=-32$) only the positive one is suitable. Hence, the daily plan amounted to $\frac{8000}{x}=200$ cubic metres. The overfulfilment by 50 m^3 made

$$\frac{50\cdot 100}{200}=25\%$$

Answer: the original time limit was 40 days, the daily overfulfilment of the plan being 25 per cent.

468. The first team repaired x km per day; then the second repaired $(4.5-x)$ km per day. The first worked $\frac{10}{x}$ days; the second, $\frac{10}{4.5-x}$ days. By hypothesis, $\frac{10}{x}-\frac{10}{4.5-x}=1$. This equation yields two roots: $x_1=2$ and $x_2=22.5$. The second root is not suitable, since the number $4.5-x$ must be positive.

Answer: the first team repaired 2 km, and the second, 2.5 km daily.

469. Let the first worker be able to do the whole job in x hours, and the second, in y hours. Hence, half the assignment was done by the first in $\frac{x}{2}$ hours; the remaining half will be done by the second in $\frac{y}{2}$ hours. By hypothesis, $\frac{x}{2}+\frac{y}{2}=25$. The other condition (see solution of Problem 461) yields

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{12}$$

Answer: one of the workers (either the first, or the second) can do the whole job in 20 hours, the other, in 30 hours.

470. Let one tractor be able to plough the field in x days, and the second, in y days. We have (see the preceding problem) the system of equations:

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{t}; \quad \frac{x}{2}+\frac{y}{2}=k$$

It can be replaced by the system $x+y=2k$; $xy=2kt$.

Answer: it would take one of the tractors $(k+\sqrt{k^2-2kt})$ days and the other, $(k-\sqrt{k^2-2kt})$ days. The problem is solvable for $\frac{k}{2}>t$.

471. Let all the three dredgers working together be able to complete the job in x days. Then the first one working alone can do the job in $(x+10)$ days, the second alone, in $(x+20)$ days, and the third alone, in $6x$ days. In one day the first dredger alone fulfills $\frac{1}{x+10}$ of the job, the second alone, $\frac{1}{x+20}$ and the third, $\frac{1}{6x}$, and all of them together, $\frac{1}{x}$ of the job. Thus we have the equation

$$\frac{1}{x+10}+\frac{1}{x+20}+\frac{1}{6x}=\frac{1}{x}$$

Answer: the job can be done by the first dredger alone in 20 days, by the second, in 30 days, and by the third, in 60 days.

472. The second worker can complete the assignment in x days, the first can do it in $(x+3)$ days. In 7 days the first worker will fulfill $\frac{7}{x+3}$ of the job, in $7-1\frac{1}{2}=5\frac{1}{2}$ days the second worker will do $\frac{5\frac{1}{2}}{x}$ of the whole job. Thus, we obtain the equation

$$\frac{7}{x+3}+\frac{5\frac{1}{2}}{x}=1$$

Answer: it would take the first worker 14 days and the second one, 11 days to do the job individually.

473. Let it be possible for the first tractor to plough the whole field in x days, for the second one, in y days. The first condition yields

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{8}$$

The first tractor can plough half the field in $\frac{x}{2}$ days, the remaining half will be ploughed by both tractors in 4 days (the whole field was ploughed by them in 8 days). And so we have the second equation $\frac{x}{2}+4=10$, hence $x=12$ (days). From the first equation we find $y=24$ (days).

Answer: it would take the first tractor 12 days, and the second, 24 days to plough the field.

474. Since the workers began working one after another, the intervals between the starting times being the same, and the first to begin worked five times as long as the last to begin, the number of workers was equal to 5. If the last to begin worked x hours, then the total of the working hours amounted to $x + 2x + 3x + 4x + 5x = 15x$. By hypothesis, the men could have finished the job in 6 hours, if they had begun at the same time. Consequently, $15x = 5\cdot 6$, whence $x = 2$. The job lasted as long as the first worker digged, i.e. for $5x$ hours.

Answer: they worked 10 hours.

475. Let the first worker be able to complete the task in x hours; we get the equation

$$\frac{1}{x}+\frac{1}{x+1}+\frac{1}{2x}=\frac{1}{t}$$

Answer: $x=\frac{1}{4}\left(5t-2+\sqrt{25t^2+4t+4}\right)$.

476. Let the first tap fill the tank in x hours, and the second, in y hours. In one hour the first tap fills $\frac{1}{x}$ of the tank, and, by hypothesis, it was open $\left(\frac{1}{3}y\right)$ hours; hence, the water from the first tap filled $\frac{\frac{1}{3}y}{x}$ of the tank. Similarly, we find that the water from the second tap filled $\frac{\frac{1}{3}x}{y}$ of the tank. Since, when this was done, the tank was $\frac{13}{18}$ full, we have

$$\frac{1}{3}\cdot\frac{y}{x}+\frac{1}{3}\cdot\frac{x}{y}=\frac{13}{18}$$

The second condition yields the equation

$$\frac{1}{x}+\frac{1}{y}=\frac{5}{18}$$

The system may be solved in the following way. If we put $\frac{y}{x}=z$, then the first equation takes the form

$$\frac{1}{3}z+\frac{1}{3}\cdot\frac{1}{z}=\frac{13}{18}$$

whence $z_1=\frac{3}{2}$; $z_2=\frac{2}{3}$. Transform the second equation to

$$\frac{y}{x}+1=\frac{5}{18}y$$

Substituting $\frac{y}{x}=\frac{3}{2}$, we find $y=9$; hence, $x=\frac{2}{3}y=6$. Substituting $\frac{y}{x}=\frac{2}{3}$, we find $y=6$ and $x=9$.

Answer: one of the taps fills the tank in 6 hours, the other, in 9 hours.

477. If the daily quota of bricklaying was x thousand bricks, and the actual number of bricks laid daily was y thousand, then we have the system of equations

$$\begin{cases} \dfrac{120}{x}-\dfrac{120}{y}=4 \\ 3y-4x=5 \end{cases}$$

Answer: the daily quota of bricklaying was 10 000 bricks and the actual number of bricks laid amounted to 15 000.

478. The consecutive amounts of water (in litres) in the three vessels (I, II, III) are tabulated below:

I	x	$\frac{2}{3}x$	$\frac{1}{10}\left[\frac{1}{4}\left(\frac{1}{3}x+y\right)+z\right]+\frac{2}{3}x$
II	y	$\frac{3}{4}\left(\frac{1}{3}x+y\right)$	$\frac{3}{4}\left(\frac{1}{3}x+y\right)$
III	z	$\frac{1}{4}\left(\frac{1}{3}x+y\right)+z$	$\frac{9}{10}\left[\frac{1}{4}\left(\frac{1}{3}x+y\right)+z\right]$

By hypothesis, each of the expressions in the last column is equal to 9.

Alternate solution. First find the amount u of water contained in the second vessel *after the first pouring*. By hypothesis, the second pouring reduced this amount by $\frac{1}{4}u$, leaving there 9 litres of water. Consequently, $\frac{3}{4}u = 9$, i.e. $u = 12$. Now find the *original amount* z of water in the third vessel. The first pouring left it unchanged; the second one increased it by $\frac{1}{4}u=3$ litres, so that the third vessel turned out to contain $(z + 3)$ litres. The third pouring reduced this amount by $\frac{1}{10}(z + 3)$. Consequently, $\frac{9}{10}(z + 3) = 9$, i.e. $z = 7$. Then find the original amount of water x in the first vessel. The first pouring left in it $\frac{2}{3}x$ litres; the second pouring left this amount unchanged; the third pouring increased it by $\frac{1}{10}(z + 3) = 1$. Consequently, $\frac{2}{3}x + 1 = 9$, i.e. $x = 12$. Finally, find the original amount y of water in the second vessel. After the first pouring it was increased by $\frac{1}{3}x=4$ and became equal to 12 litres (as it has been found). Consequently, $y = 12 - 4 = 8$.

Answer: 12 litres; 8 litres; 7 litres.

479. If for the first time x litres of alcohol was poured out, then $(64 - x)$ litres of alcohol was left; for the second time $\frac{64-x}{64}x$ litres of pure alcohol was poured out, leaving $64-x-\frac{64-x}{64}x=\frac{1}{64}(64-x)^2$ litres of pure alcohol.

We get the following equation

$$\frac{1}{64}(64-x)^2=49$$

Answer: for the first time 8 litres of alcohol was poured out and for the second, 7 litres.

480. Having poured x litres of alcohol into the second vessel and made it full by adding water, we have in the second vessel $\frac{x}{20}$ litres of alcohol per litre of the mixture. Then x litres of the mixture, containing $\frac{x}{20}x=\frac{x^2}{20}$ litres of alcohol, is poured back. As a result, the first vessel now contains $\left(20-x+\frac{x^2}{20}\right)$ litres of alcohol. Then $6\frac{2}{3}$ litres of the mixture is poured out from the first vessel $\left(\frac{6\frac{2}{3}}{20}\text{ make }\frac{1}{3}\text{ of the total amount of the mixture}\right)$. Thus, the amount of alcohol is reduced by $\frac{1}{3}$, i.e. now the first vessel contains $\frac{2}{3}\left(20-x+\frac{x^2}{20}\right)$ litres of alcohol. Since the amount of alcohol contained in both vessels is constant and is equal to 20 litres, and by hypothesis, both vessels now contain the same amount of alcohol (i.e. 10 litres each), we have

$$\frac{2}{3}\left(20-x+\frac{x^2}{20}\right)=10$$

Answer: 10 litres.

481. Let x litres of air be let out of the vessel, and the same amount of nitrogen put in. The remaining amount of air of $(8-x)$ litres contains $(8-x)\,0.16$ litres of oxygen. Thus, 8 litres of the mixture contains this amount of oxygen, i.e. 1 litre of the mixture contains $\frac{(8-x)\,0.16}{8}$ l of oxygen. Consequently, when for the second time x litres of the mixture is replaced by x litres of nitrogen, the remaining amount of the mixture of $(8-x)$ litres contains $\frac{(8-x)\,0.16}{8}\cdot(8-x)=(8-x)^2\,0.02$ litres of oxygen. Hence, in relation to the total amount of the mixture (8 litres) the oxygen content is $\frac{(8-x)^2\,0.02}{8}\times 100=\frac{(8-x)^2}{4}$. By hypothesis, $\frac{(8-x)^2}{4}=9$. Out of the two roots ($x_1=2$, $x_2=14$) only the first one is suitable, since it is impossible to let out more than 8 litres.

Answer: 2 litres.

482. Let the first woman have x eggs, and the second one, y eggs. If the first had sold y eggs, then, by hypothesis, she would have received 72 roubles. Consequently, she sold her eggs at $\frac{72}{y}$ roub. per piece and received $\frac{72}{y}x$ roub. Reasoning in the same way, we find that the second woman received $\frac{32}{x}y$ roub. Thus we have two equations

$$\frac{32}{x}y=\frac{72}{y}x;\ x+y=100$$

From the first we find $\left(\frac{y}{x}\right)^2=\frac{72}{32}$, whence $\frac{y}{x}=\frac{3}{2}$ $\Big($the negative value $\frac{y}{x}=-\frac{3}{2}$ is not suitable$\Big)$.

Answer: the first had 40 eggs; the second, 60 eggs.

483. With the notation of the preceding problem we get the following system

$$m\frac{x}{y}=n\frac{y}{x};\ x+y=a$$

From the first equation we find $x:y=\sqrt{n}:\sqrt{m}$. Divide then a into parts proportional to $\sqrt{n}$ and $\sqrt{m}$.

Answer: the first had $\frac{a\sqrt{n}}{\sqrt{m}+\sqrt{n}}$ litres; the second, $\frac{a\sqrt{m}}{\sqrt{m}+\sqrt{n}}$ litres.

484. Let the first engine consume x grams of petrol per hour, and the second, y grams; then 600 grams of petrol was consumed by the first engine in $\frac{600}{x}$ hours, and 384 grams, by the second in $\frac{384}{y}$ hours. By hypothesis, $\frac{600}{x}-\frac{384}{y}=2$. If the first engine had consumed y grams of petrol per hour, then during $\frac{600}{x}$ hours of operation it would have consumed $\frac{600}{x}\cdot y$ grams of petrol, and if the second had consumed x grams per hour, then during $\frac{384}{y}$ hours it would have consumed $\frac{384}{y}x$ grams of petrol; by hypothesis, $\frac{600y}{x}=\frac{384x}{y}$.

Answer: the first engine consumes 60 g/h; the second, 48 g/h.

485. Suppose we need x kg of the first alloy. Then x kg will contain $\frac{2}{5}x$ kg of gold, and $(8-x)$ kg of the second alloy will contain $\frac{3}{10}(8-x)$ kg of gold. By hypothesis, 8 kg of the new alloy must contain $\frac{5}{16}\cdot 8$ kg $=2.5$ kg of gold. Consequently, $\frac{2}{5}x+\frac{3}{10}(8-x)=2.5$. Hence, $x=1$ (kg) and $8-x=7$ (kg).

Answer: 1 kg of the first alloy and 7 kg of the second.

486. See solution of the preceding problem.

Answer: 9 pails from the first barrel and 3 pails from the second.

487. Let the third alloy contain x parts of the first and y parts of the second alloy, i.e. x kg of the first and y kg of the second alloy. Then $(x+y)$ kg of the third alloy will contain $\left(\frac{1}{3}x+\frac{2}{5}y\right)$ kg of the first metal and $\left(\frac{2}{3}x+\frac{3}{5}y\right)$ kg of the second. By hypothesis,

$$\left(\frac{1}{3}x+\frac{2}{5}y\right):\left(\frac{2}{3}x+\frac{3}{5}y\right)=17:27$$

Reducing the dividend and divisor to a common denominator (15) and dividing them by y, we get

$$\left(5\frac{x}{y}+6\right):\left(10\frac{x}{y}+9\right)=17:27$$

whence $\frac{x}{y}=\frac{9}{35}$.

Answer: 9 parts of the first alloy, and 35 parts of the second.

488. Let the larger wheel make x revolutions per minute, and the smaller one, y r.p.m., $y>x$. We have two equations:

$$y-x=400; \quad \frac{5}{x}-\frac{5}{y}=\frac{1}{60}$$

The second equation may be transformed to $xy=300(y-x)$, i.e. $xy=120{,}000$.

Answer: The larger wheel makes 200 r.p.m., the smaller one, 600 r.p.m.

489. Let the circumference of the front wheel be equal to x dm, and that of the rear wheel, to y dm. We have two equations:

$$\frac{180}{x}-\frac{180}{y}=10 \quad \text{and} \quad \frac{180}{x+6}-\frac{180}{y-6}=4$$

The first one is transformed to $18(y-x)=xy$; the second, to $39(y-x)=$ $=xy+504$. From them we find $y-x=24$; $xy=432$.

Answer: the circumference of the front wheel is 12 dm; that of the rear wheel, 36 dm.

490. $600\cdot\frac{2}{3}=400$ tons was unloaded during the first and the third days; 600 tons $-$ 400 tons $=$ 200 tons was unloaded during the second day. Let x tons be unloaded during the first day; then $(400-x)$ tons was unloaded during the third day. The reduction of the amount of goods unloaded on the second day (as compared with the first day) came to $(x-200)$ tons, which made $\frac{(x-200)100}{x}$ % of the amount unloaded on the first day. The reduction of the amount of goods unloaded on the third day relative to that on the second day was $200-(400-x)=(x-200)$ tons, which made $\frac{(x-200)100}{200}$ % or $\frac{x-200}{2}$ % of the amount unloaded on the second day. By hypothesis,

$$\frac{x-200}{2}-\frac{(x-200)\,100}{x}=5$$

We find two roots: $x_1=250$; $x_2=160$. The second one is not suitable, since, by hypothesis, the amount of the unloaded goods was reduced from day to day, whereas at $x=160$ the amount of the unloaded goods would be 160 tons on the first day, 200 tons on the second day, and 240 tons on the third.

Answer: 250 tons was unloaded during the first day, 200 tons during the second, and 150 tons during the third.

491. Let the first solution weigh x kg, then the second weighs $(10-x)$ kg. The percentage of anhydrous sulphuric acid in the first solution is $\frac{0.8\cdot 100}{x}=\frac{80}{x}$ and that in the second, $\frac{0.6\cdot 100}{10-x}=\frac{60}{10-x}$. By hypothesis,

$$\frac{80}{x}-\frac{60}{10-x}=10$$

The equation has two positive roots $x_1=20$ and $x_2=4$. Since, by hypothesis, $x<10$, the first solution is not suitable.

Answer: 4 kg and 6 kg.

492. The first alloy contained $x\%$ of copper, the second, $(x+40)\%$. The first alloy weighed $\frac{6\cdot 100}{x}$ kg, and the second, $\frac{12\cdot 100}{x+40}$ kg. We get the following equation: $\frac{600}{x}+\frac{1200}{x+40}=50$.

Answer: 20% and 60%.

493. Let the speed of the freight train be x m/s, and that of the passenger train, y m/s. In 28 seconds the freight train covered $28x$ (m), and the passenger train, $28y$ (m); we obtain the first equation

$$28x+28y=700$$

The freight train passes the signal lights during $\frac{490}{x}$ seconds, and the passenger train, during $\frac{210}{y}$ seconds. Thus, we get the second equation

$$\frac{490}{x}-\frac{210}{y}=35$$

Answer: the speed of the freight train is equal to 10 m/s, i.e. 36 km/h and that of the passenger train, to 15 m/s, i.e. 54 km/h.

494. If the number of the eight-wheel tank-cars is equal to x, then that of the four-wheel cars, to $(x+5)$. If one four-wheel car weighs y tons, then one eight-wheel car weighs $3y$ tons. The net weight of the oil contained in a four-wheel car is equal to $(40\cdot 0.3)$ tons, i.e. to 12 tons. An eight-wheel car filled with oil weighs $(3y+40)$ tons, and a four-wheel car, $(y+12)$ tons. We have the first equation

$$x(3y+40)+(x+5)(y+12)=940$$

The weight of oil contained in all the eight-wheel cars is $40x$ tons, and that of all the loaded four-wheel cars is $(x+5)(y+12)$ tons. Thus, we have the second equation

$$40x-(x+5)(y+12)=100$$

Answer: there are 10 eight-wheel] tank-cars, each weighing 24 tons, and 15 four-wheel cars, each weighing 8 tons.

495. Let the first machine drive x m per day, and the second, y m. In the first case the first machine would have done 30% of the work, i.e. it would have driven $\frac{60x\cdot 30}{100}=18x$ (m), and the second, $\frac{60\cdot y\cdot 80}{300}=16y$ (m). We have the first equation

$$18x + 16y = 60$$

In the second case the first machine would have driven $\frac{2}{3}\cdot 60y$ (m) in $\frac{2}{3}\cdot 60\cdot\frac{y}{x}$ days. The second machine would have done the work in $\frac{3}{10}\cdot 60\cdot\frac{x}{y}$ days. And so we have the second equation

$$\frac{40y}{x}-\frac{18x}{y}=6$$

The obtained system is easily solved, if we put $\frac{y}{x}=z$. Only the positive value $z=\frac{3}{4}$ is suitable.

Answer: the first machine drives 2 metres of the tunnel per day, the second, $1\frac{1}{2}$ metres per day.

496. Let the first crew be able to repair the whole section of the track in x days, and the second, in y days. By hypothesis, we have the following system of equations

$$\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{1}{6} \\ \frac{40x}{100}-\frac{40y}{300}=2 \end{cases}$$

Answer: the first crew can complete the whole repair job in 10 days, the second, in 15 days.

497. Let the first portion of the goods $\left(\text{amounting to } \frac{25}{46}\cdot 690 = 375 \text{ tons}\right)$ be transported in x hours, and each three-ton truck accomplish y trips per hour. Then each $1\frac{1}{2}$-ton truck made $(y+1)$ trips per hour. By hypothesis, the remaining portion of the goods (i.e. $690 - 375 = 315$ tons) was transported in $(x-2)$ hours, the three-ton trucks making $(y+1)$ trips per hour, and $1\frac{1}{2}$-ton trucks, $(y+1)+1=(y+2)$ trips per hour. We get the system of equations

$$\begin{cases} 5\cdot 3xy+10\cdot 1\frac{1}{2}x(y+1)=375 \\ 5\cdot 3(x-2)(y+1)+10\cdot 1\frac{1}{2}(x-2)(y+2)=315 \end{cases}$$

After simplifications these equations take the form

$$\begin{cases} 2xy+x=25 \\ 2xy+3x-4y=27 \end{cases}$$

Subtracting the first equation from the second one, we get $2x-4y=2$. Hence, $2y=x-1$. Substituting it into the first equation, we get $x^2=25$, i.e. $x=5$.

The first portion of the goods was transported in 5 hours, the second, in $5-2=$ $=3$ hours.

Answer: all the goods were transported in 8 hours; at the beginning the three-ton trucks made 2 trips per hour and $1\frac{1}{2}$-ton trucks, 3 trips per hour.

498. If x is the width of the track, then the area of the sports ground together with the track, is equal to $(a+2x)(b+2x)$ m^2. And so we have the equation $(a+2x)(b+2x)=2ab$.

Answer: $\frac{1}{4}[\sqrt{(a+b)^2+4ab}-(a+b)]$.

499. Let x denote the number of chairs in each row; then the number of rows is $\frac{a}{x}$. We get the equation

$$(x+b)\left(\frac{a}{x}-c\right)=1.1a$$

After simplifications we have

$$10cx^2+(a+10bc)x-10ab=0$$

Hence,

$$x=\frac{-(a+10bc)\pm\sqrt{(a+10bc)^2+400abc}}{20c}$$

If the radical is taken with the minus sign, then $x<0$; if it is taken with the plus sign, then $x>0$.

Answer: the number of chairs in each row is equal to

$$\frac{\sqrt{(a+10bc)^2+400abc}-(a+10bc)}{20c}$$

500. Let us denote the speeds of the bodies (in m/s) by v_1 and v_2; let v_1 be higher than v_2. The first condition yields the equation $av_1+av_2=d$; the second, $bv_1-bv_2=d$.

Answer: $v_1=\frac{d}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$; $v_2=\frac{d}{2}\left(\frac{1}{a}-\frac{1}{b}\right)$. The problem is solvable only if $a<b$.

501. Let us denote the speed of the motorcyclist (in km/h) by x, and that of the cyclist, by y. We get the following system of equations

$$2x+2y=d;\quad \frac{d}{y}-\frac{d}{x}=t$$

Answer: the speed of the motorcyclist is equal to $d\frac{t-4+\sqrt{16+t^2}}{4t}$ km/h, and that of the cyclist, to $d\frac{t+4-\sqrt{16+t^2}}{4t}$ km/h.

502. If it takes the cyclist x hours to cover the whole distance, then the hiker requires $(x+c)$ hours. Let us denote the distance AB (say, in kilometres) by y. The hiker covered $\frac{y(a+b)}{x+c}$ km before he met the cyclist, while

the latter covered $\frac{by}{x}$ km. We have the equation $\frac{(a+b)\,y}{x+c}+\frac{by}{x}=y$. Since $y\neq 0$, we have $\frac{a+b}{x+c}+\frac{b}{x}=1$, or $x^2-(a+2b-c)\,x-bc=0$. This equation has one positive and one negative root (since the product of the roots is equal to the negative number $-bc$). Here only the positive solution is suitable:

$$x=\frac{a+2b-c+\sqrt{(a+2b-c)^2+4bc}}{2}.$$

The distance y remains undetermined. The quantity $x+c$ may be found either from the above expression, or from the equation $\frac{a+b}{x+c}+\frac{b}{x}=1$, putting $x+c=z$. We get the equation $\frac{a+b}{z}+\frac{b}{z-c}=1$. We take only the positive solution.

Answer: it takes the cyclist

$$\frac{a+2b-c+\sqrt{(a+2b-c)^2+4bc}}{2}\ \text{hours}$$

and the hiker

$$\frac{a+2b+c+\sqrt{(a+2b-c)^2+4bc}}{2}=\frac{(a+2b+c)+\sqrt{(a+2b+c)^2-4\,(a+b)\,c}}{2}\ \text{hours}$$

to cover the whole distance AB.

503. Let us denote the distance (in kilometres) by x. By hypothesis, according to the schedule train A must catch up with train B in $\frac{x}{v}$ hours after departure. Actually, it caught up with train B after having covered $(x-a)$ km, i.e. in $\frac{x-a}{v}$ hours. Consequently, both trains travelled $\frac{a}{v}$ hours less than required by the schedule before they met. Train B had to travel $\frac{x}{v_1}$ hours to meet train A, but actually it covered a distance of $\frac{2}{3}x$ at a speed of v_1 and a distance of $\frac{1}{3}x-a$ at a speed of $\frac{1}{2}v_1$, covering the whole distance in $\left(\frac{\frac{2}{3}x}{v_1}+\frac{\frac{1}{3}x-a}{\frac{1}{2}v_1}\right)$ hours. Consequently,

$$\frac{x}{v_1}-\left(\frac{\frac{2}{3}x}{v_1}+\frac{\frac{1}{3}x-a}{\frac{1}{2}v_1}\right)=\frac{a}{v}$$

Answer: the distance to the terminal station is equal to $\frac{3a\,(2v-v_1)}{v}$ km. The problem is solvable only if $v_1<2v$.

504. Let the interest be $x\%$. Then the originally deposited sum was $\frac{1500}{x}$ roub. At the beginning of the second year the total sum was $\frac{1500}{x}+15+85$, i.e. $\left(\frac{1500}{x}+100\right)$ roub. At the end of the second year this sum turned into $\left(\frac{1500}{x}+100\right)\left(1+\frac{x}{100}\right)$ roub. Hence, we get the equation

$$\left(\frac{1500}{x}+100\right)\left(1+\frac{x}{100}\right)=420$$

Answer: 300 roub., 5%.

505. Let us denote the output of machines A, B, C by x, y, z, respectively. By hypothesis,

$$x=\frac{m}{100}(y+z), \quad y=\frac{n}{100}(x+z)$$

We find x and y in terms of z from these equations; adding them, we get

$$x+y=\frac{100\,(m+n)+2mn}{10\,000-mn}\,z$$

The required percentage is equal to $\frac{z}{x+y}\cdot 100$.

Answer: $100\cdot\frac{10\,000-mn}{100\,(m+n)+2mn}$.

506. Let us take for the unit of measurement the output for the preceding year. Then the output for the first year is $1+\frac{p}{100}$. And compared with it, the output for the second year is increased by $q\%$, i.e. by $\left(1+\frac{p}{100}\right)\frac{q}{100}$, to be equal to

$$\left(1+\frac{p}{100}\right)+\left(1+\frac{p}{100}\right)\frac{q}{100}=\left(1+\frac{p}{100}\right)\left(1+\frac{q}{100}\right)$$

If the output for the third year is increased by $x\%$, the increase amounts to $\left(1+\frac{p}{100}\right)\left(1+\frac{q}{100}\right)\frac{x}{100}$. By hypothesis,

$$\frac{1}{3}\left[\frac{p}{100}+\left(1+\frac{p}{100}\right)\frac{q}{100}+\left(1+\frac{p}{100}\right)\left(1+\frac{q}{100}\right)\frac{x}{100}\right]=\frac{r}{100}$$

Answer: $$\frac{3r-p-q-\frac{pq}{100}}{\left(1+\frac{p}{100}\right)\left(1+\frac{q}{100}\right)}\,\%.$$

507. Let the prime cost of the total quantity of goods amount to m roubles. Then the prime cost of the first batch sold makes $a\%$ of m, i.e. $\frac{ma}{100}$ roubles. By hypothesis, the profit made by selling this batch is $p\%$ of this

sum, i.e. $\frac{ma}{100}\cdot\frac{p}{100}$ roub. The prime cost of the rest of the goods is equal to $m-\frac{ma}{100}=m\left(1-\frac{a}{100}\right)$ roubles. The prime cost of the second batch sold amounts to $b\%$ of this sum, i.e. to $m\left(1-\frac{a}{100}\right)\frac{b}{100}$ roubles. The profit made on selling the second batch is $q\%$; consequently, this profit amounts to $m\left(1-\frac{a}{100}\right)\frac{b}{100}\cdot\frac{q}{100}$ roubles. The prime cost of the remaining goods is equal to

$$m-\frac{ma}{100}-m\left(1-\frac{a}{100}\right)\frac{b}{100}=m\left(1-\frac{a}{100}\right)\left(1-\frac{b}{100}\right)\text{ roub.}$$

Let the remaining goods be sold at a profit of $x\%$. Then the profit made on their selling amounts to $m\left(1-\frac{a}{100}\right)\left(1-\frac{b}{100}\right)\frac{x}{100}$ roub. The total profit is

$$m\left[\frac{a}{100}\cdot\frac{p}{100}+\left(1-\frac{a}{100}\right)\frac{b}{100}\cdot\frac{q}{100}+\left(1-\frac{a}{100}\right)\left(1-\frac{b}{100}\right)\frac{x}{100}\right]$$

By hypothesis, the total profit must be $r\%$ of m roub, i.e. $\frac{mr}{100}$ roub. Consequently,

$$m\left[\frac{a}{100}\cdot\frac{p}{100}+\left(1-\frac{a}{100}\right)\frac{b}{100}\cdot\frac{q}{100}+\left(1-\frac{a}{100}\right)\left(1-\frac{b}{100}\right)\frac{x}{100}\right]=\frac{mr}{100}$$

The quantity m is reduced.

Answer: $$\frac{r-\frac{ap}{100}-\frac{bq}{100}\left(1-\frac{a}{100}\right)}{\left(1-\frac{a}{100}\right)\left(1-\frac{b}{100}\right)}\%.$$

508. *First method.* Let us assume that each of the cut-off pieces weighs x kg. For the sake of brevity, let us call the first alloy (weighing m kg) "alloy A", and the second, "alloy B". Out of the two newly produced ingots the first one contains $(m-x)$ kg of alloy A and x kg of alloy B, and the second, x kg of alloy A and $(n-x)$ kg of alloy B. By hypothesis, the copper content in both alloys is the same, which is possible only if the amounts of alloy A and alloy B, contained in the new alloys, are proportional. We get the equation

$$\frac{m-x}{x}=\frac{x}{n-x},\quad\text{whence}\quad x=\frac{mn}{m+n}$$

Second method. Let u kg be the weight of copper in 1 kg of alloy A, and v — the weight of copper in 1 kg of alloy B. Then the first ingot contains $(m-x)u+{}$ $+xv$ kg of copper, i.e. 1 kg of the first ingot contains $\frac{(m-x)u+xv}{m}$ kg of copper. The weight of copper contained in 1 kg of the second ingot is expressed in a similar way. Equating the two expressions thus found, we get the equation in three unknowns (x, u, v):

$$n[(m-x)u+xv]=m[(n-x)v+xu]$$

which may be transformed to

$$(u-v)(mx+nx-mn)=0$$

By hypothesis, alloys A and B are of different copper content, i.e. the quantity $u-v$ cannot be equal to zero. Consequently,

$$mx+nx-mn=0$$

Answer: each of the cut-off pieces weighs $\frac{mn}{m+n}$ kg.

509. Let there be originally x_1 roubles in the first pile, x_2 roubles in the second and so on, and x_n roubles in the nth pile. As is obvious, the first pile is treated in a special way, since at first an nth part of the money is taken from it and only by the last shifting operation an nth part of the nth pile is put into it, whereas each of the rest of the piles first is enlarged on the account of the preceding pile and then an nth part of it is taken away. Therefore, let us consider any pile, except for the first one. Let k designate its number. Originally, it had x_k roubles, then some amount of y roubles of $(k-1)$th pile was put into it, and, finally, an nth part of the total sum $y+x_k$ was taken from it. After this operation pile k had $(y+x_k)\frac{n-1}{n}$ roubles. By hypothesis, we have the equation

$$(y+x_k)\frac{n-1}{n}=A \tag{1}$$

A roubles must remain in the preceding $(k-1)$th pile, if it is not the first (i.e. if $k\neq 2$) (the money in the first pile amounts to A roubles only after it is replenished from the nth pile). Hence, prior to the shifting operation it had $A+y$ roubles. By hypothesis, the money taken from it makes an nth part of $A+y$, i.e.

$$y=\frac{1}{n}(A+y) \tag{2}$$

Hence, $y=\frac{1}{n-1}A$. Substituting it into (1), we get $x_k=A$.

Thus, each of the piles, except, perhaps, for the second and first (previously excluded from consideration) originally had A roubles:

$$x_3=x_4=\ldots=x_n=A \tag{3}$$

The unknown x_1 may be found in the following way. By hypothesis, at first an nth part is taken out of the amount of x_1 roubles. There remains $x_1\frac{n-1}{n}$ roubles. At the end of the shifting process a certain amount of money (y roubles) from the last pile is put into the first pile. We obtain the equation

$$y+x_1\frac{n-1}{n}=A \tag{4}$$

Reasoning (conformably to the nth pile) in the same manner we find as before $y=\frac{1}{n-1}A$. Substituting it into (4), we get

$$x_1=\frac{(n-2)\,n}{(n-1)^2}A, \tag{5}$$

to find x_2 we have the equation

$$\left(\frac{1}{n}x_1+x_2\right)\frac{n-1}{n}=A \tag{6}$$

where x_1 is determined by the formula (5). Solving the equation, we find

$$x_2=\frac{n\,(n-1)-(n-2)}{(n-1)^2}A.$$

Answer:

$$x_1=\frac{n^2-2n}{(n-1)^2}A;\quad x_2=\frac{n^2-2n+2}{(n-1)^2}A;\quad x_3=x_4=\ldots=x_n=A$$

PART TWO
GEOMETRY AND TRIGONOMETRY

CHAPTER VIII
PLANE GEOMETRY

510. Let a and b be the legs of the right-angled triangle and c, its hypotenuse. By hypothesis, $a+b+c=132$ and $a^2+b^2+c^2=6050$. Since $a^2+b^2=c^2$, then $2c^2=6050$, whence $c=\sqrt{3025}=55$. Therefore $a+b=77$. Squaring this equality and taking into account the relation $a^2+b^2=3025$, we get $ab=1452$. Consequently, a and b are the roots of the equation

$$x^2-77x+1452=0$$

Answer: the legs of the triangle are equal to 44 and 33 respectively, the hypotenuse, to 55.

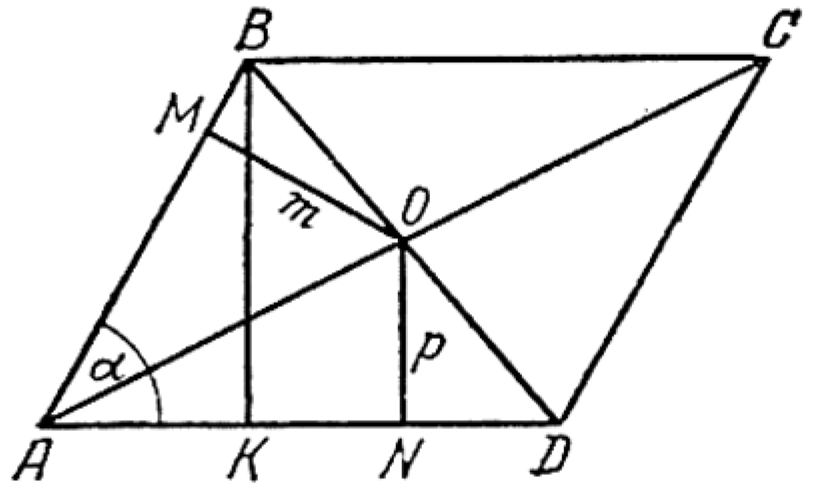


Fig. 1

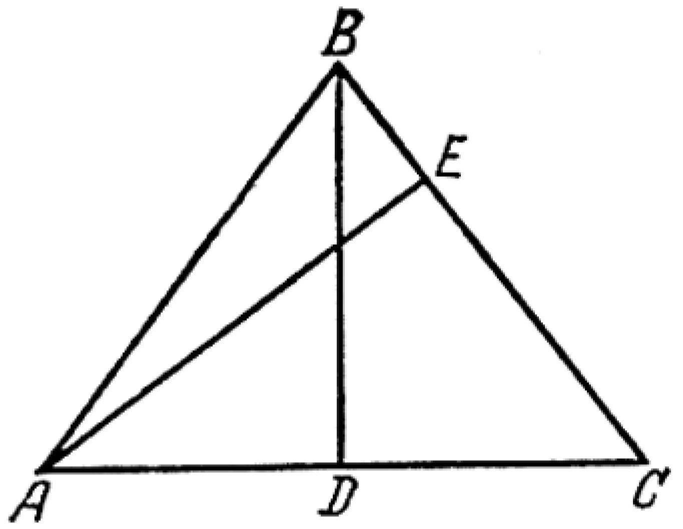


Fig. 2

511. The altitude BK (Fig. 1) of the parallelogram $ABCD$ is equal to $2ON=2p$. Since $\angle BAK=\alpha$, $AB=\frac{2p}{\sin\alpha}$. Similarly, $AD=\frac{2m}{\sin\alpha}$. We find:

$$S=AD\cdot BK=\frac{4mp}{\sin\alpha}$$

The diagonals are found by the law of cosines.

Answer: $S=\frac{4mp}{\sin\alpha}$

$$BD=\frac{2\sqrt{p^2+m^2-2mp\cos\alpha}}{\sin\alpha}$$

$$AC=\frac{2\sqrt{p^2+m^2+2mp\cos\alpha}}{\sin\alpha}$$

512. By hypothesis, $AC=30$ cm and $BD=20$ cm (Fig. 2). The altitude AE may be found proceeding from the similarity of the right-angled triangles BDC

and AEC (having the common angle C), or, which is easier, by comparing two expressions of the area S of the triangle ABC. Namely,

$$S=\frac{1}{2}AC\cdot BD \quad \text{and} \quad S=\frac{1}{2}BC\cdot AE$$

Hence,

$$AE=\frac{AC\cdot BD}{BC}=\frac{30\cdot 20}{\sqrt{20^2+\left(\frac{30}{2}\right)^2}}=24 \text{ cm}$$

Answer: 24 cm.

513. From the triangle BDE, where $BD=12$ cm and $BE=13$ cm, we find $DE=\sqrt{13^2-12^2}=5$ (cm) (Fig. 3). Consequently, $AD=AE-DE=\frac{1}{2}AC-DE=\frac{1}{2}\cdot 60-5=25$ (cm) and $DC=EC+DE=35$ (cm). The sides are found from the triangles ADB and DCB.

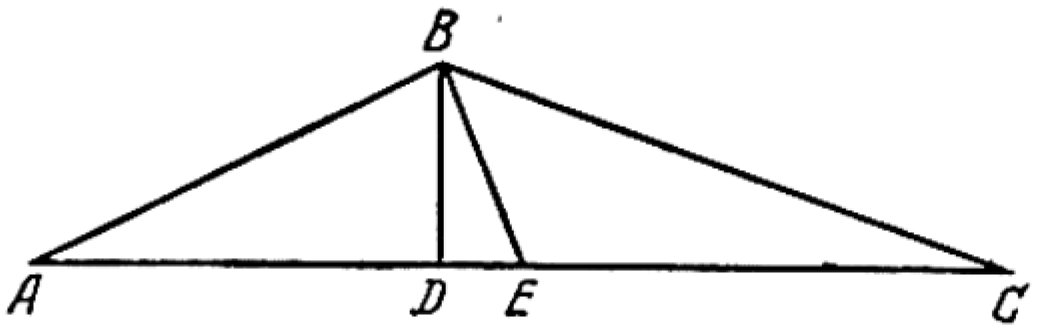


Fig. 3

Answer: $AB=\sqrt{769}\approx 27.7$ cm, $BC=\sqrt{1369}=37$ cm.

514. Let ABC be the given triangle ($AC=CB=b$). It is required to determine the area S of the triangle $O_1O_2O_3$ (Fig. 4).

We have $S=\frac{1}{2}O_2O_3\cdot O_1C$, where $O_2O_3=AB$ and $O_1C=AB$. Hence, $S=\frac{1}{2}AB^2=b^2$.

Alternate solution. The triangle O_1O_2C is equal to the triangle O_1BC, since they have the common base O_1C and equal altitudes. The triangle O_1O_3C is equal to the triangle O_1AC (for the same reason). Hence, the triangle $O_1O_2O_3$ is equal to the square O_1BCA.

Answer: $S=b^2$.

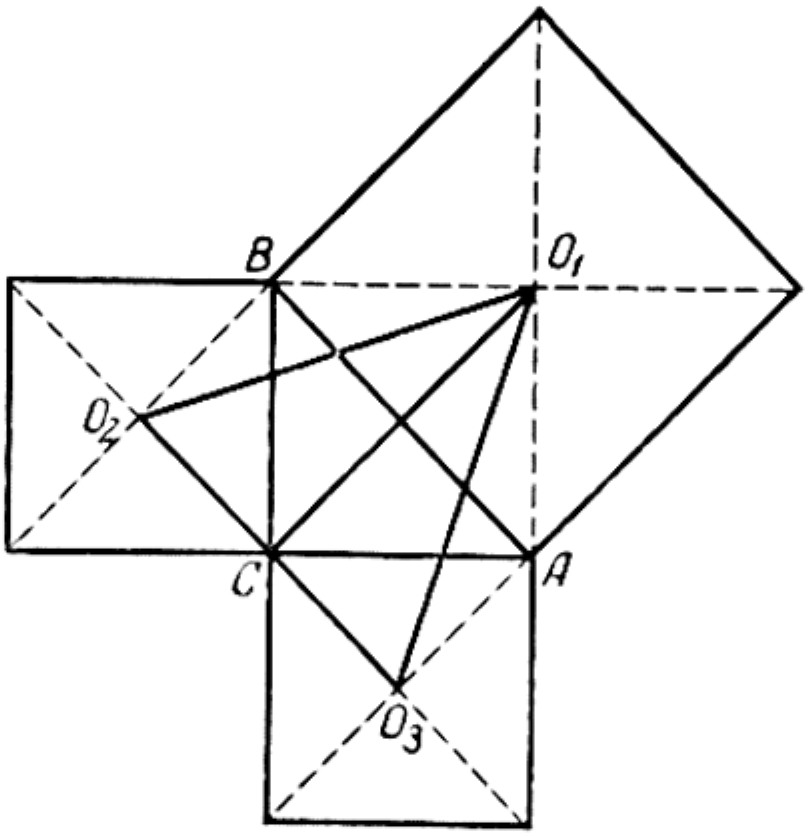


Fig. 4

515. By hypothesis, the line-segment $AB=a$ is divided by the point M in the ratio $m:n$ (Fig. 5). Therefore $AM=\frac{ma}{m+n}$ and $MB=\frac{na}{m+n}$. In the same way

$$BN=CK=DL=\frac{ma}{m+n} \quad \text{and} \quad NC=KD=LA=\frac{na}{m+n}$$

Consequently,

$$LM = MN = NK = KL = \sqrt{\frac{m^2a^2}{(m+n)^2} + \frac{n^2a^2}{(m+n)^2}} = \frac{a}{m+n}\sqrt{m^2+n^2}$$

Furthermore, all the angles of the quadrilateral $LMNK$ are the right ones (since the triangles ALM and BMN are congruent, we have $\angle LMA = \angle MNB = 90° - \angle NMB$; hence, $\angle LMA + \angle NMB = 90°$; therefore $\angle LMN = 90°$). Consequently, the quadrilateral $LMNK$ is a square.

Answer: $S = \frac{a^2(m^2+n^2)}{(m+n)^2}$.

Alternate solution. Subtract the total area of the four triangles from the area of the square $ABCD$.

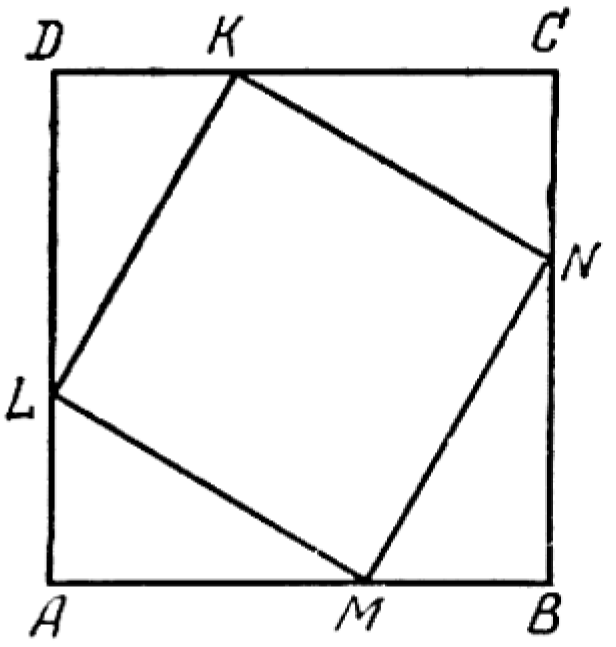


Fig. 5

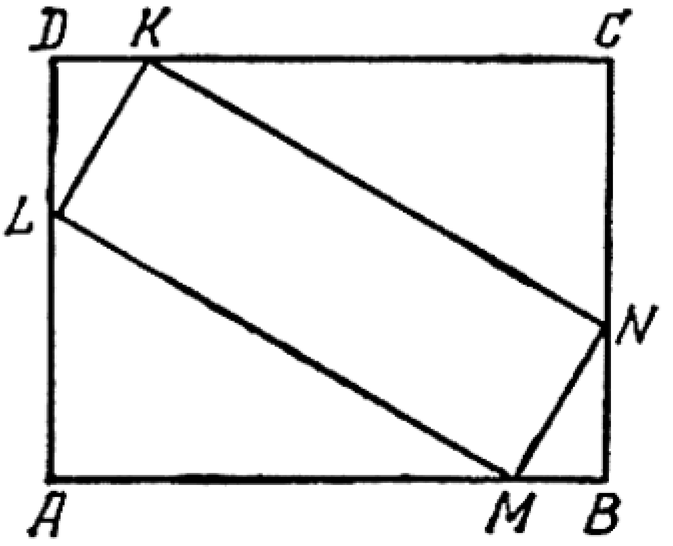


Fig. 6

516. By hypothesis, $\angle LMA = 30°$ (Fig. 5). Consequently,

$$AL = \frac{1}{2}ML \quad \text{and} \quad AM = \frac{\sqrt{3}}{2}ML$$

Hence,

$$AB = AM + MB = AM + AL = \frac{1}{2}(1+\sqrt{3})\,ML$$

Consequently,

$$\text{area } ABCD : \text{area } LMNK = AB^2 : ML^2 = (1+\sqrt{3})^2 : 4,$$

i.e.

$$\text{area } LMNK = \frac{4}{(1+\sqrt{3})^2} \text{ area } ABCD$$

Answer: the ratio is $\frac{4}{(1+\sqrt{3})^2} = 2(2-\sqrt{3}) \approx 0.54$.

517. Let us denote AM (Fig. 5) by x. Then $AL = MB = a - x$. Consequently,

$$\text{area } KLMN = LM^2 = AL^2 + AM^2 = (a-x)^2 + x^2$$

By hypothesis, $(a-x)^2 + x^2 = \frac{25}{49}a^2$. Solve this equation.

Answer: the required segments are equal to $\frac{3a}{7}$ and $\frac{4a}{7}$.

518. *A preliminary.* It will become clear from the solution how to find the position of the vertices of the inscribed rectangle $KLMN$ (Fig. 6). At the moment

it is necessary to carry out the drawing schematically, beginning with the construction of the rectangle $KLMN$.

Solution. Find the line-segments $MB = x$ and $BN = y$. Since $AB = 4$, $AM = 4 - x$. The triangles DLK and BNM are congruent (prove it!); consequently, $DL = BN = y$ and $LA = 3 - y$. The triangles LAM and MNB are similar, since their acute angles ALM and NMB are equal (as angles with mutually perpendicular sides). And since, by hypothesis, ML is three times greater than MN, we have $LA = 3MB$, and, also, $AM = 3BN$, i.e. $3 - y = 3x$ and $4 - x = 3y$. Hence, $x=\frac{5}{8}$, $y=\frac{9}{8}$. Now we have

$$MN=\sqrt{\left(\frac{5}{8}\right)^2+\left(\frac{9}{8}\right)^2}=\frac{\sqrt{106}}{8}$$

and

$$ML=\frac{3\sqrt{106}}{8}$$

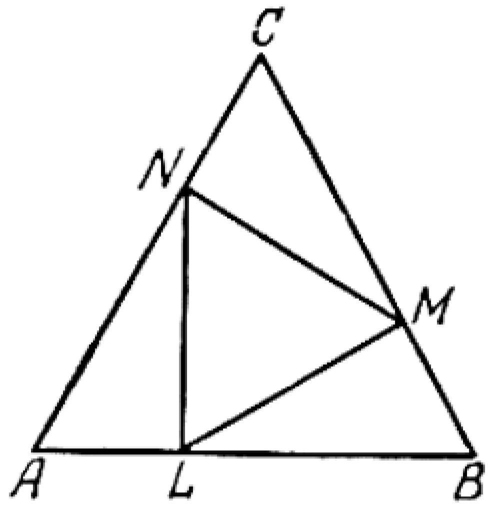


Fig. 7

Answer: the sides of the rectangle are equal to $\frac{\sqrt{106}}{8}\approx 1.29$ m and $\frac{3\sqrt{106}}{8}\approx 3{,}87$ m.

519. The area of the equilateral triangle ABC (Fig. 7) is equal to $\frac{1}{2}a\cdot\frac{\sqrt{3}}{2}a=\frac{\sqrt{3}}{4}a^2$. The triangle ANL, in which, by hypothesis, $AL=\frac{1}{3}a$ and $AN=\frac{2}{3}a$ has its angle A in common with the triangle ABC. Hence, their areas are in the same ratio as the products of the sides: $\frac{S_{ANL}}{S_{ABC}}=\frac{\frac{1}{3}a\cdot\frac{2}{3}a}{a\cdot a}$. Therefore

$$S_{ANL}=\frac{1}{3}\cdot\frac{2}{3}S_{ABC}=\frac{2}{9}S_{ABC}$$

Hence,

$$S_{NLM}=S_{ABC}-3S_{ANL}=\frac{1}{3}S_{ABC}=\frac{a^2\sqrt{3}}{12}$$

Note. The triangle LMN, as well as the triangle ABC, is an equilateral one (prove it). The same method may be used for determining the area of the triangle LMN in a general case, when the triangle ABC is an arbitrary one and its sides are divided in arbitrary ratios.

Answer: $\frac{a^2\sqrt{3}}{12}$.

520. We have (see Fig. 8) $a+b+c=2p$; hence $a+b=2p-c$ and $a^2+2ab+b^2=(2p-c)^2$. But $a^2+b^2=c^2$ and $ab=ch$ (see solution of Problem 512). Therefore $c^2+2ch=4p^2-4pc+c^2$, whence

$$c=\frac{2p^2}{h+2p}$$

Now we have $a+b=\frac{2p(h+p)}{h+2p}$ and $ab=\frac{2p^2h}{h+2p}$. Hence, a and b are the roots of the equation

$$x^2-\frac{2p(h+p)}{h+2p}x+\frac{2p^2h}{h+2p}=0$$

Answer: $c=\frac{2p^2}{h+2p}$

$$a=\frac{p}{h+2p}\left[h+p+\sqrt{(p-h)^2-2h^2}\right]$$

$$b=\frac{p}{h+2p}\left[h+p-\sqrt{(p-h)^2-2h^2}\right]$$

The problem is solvable only if $p \geqslant h(\sqrt{2}+1$.

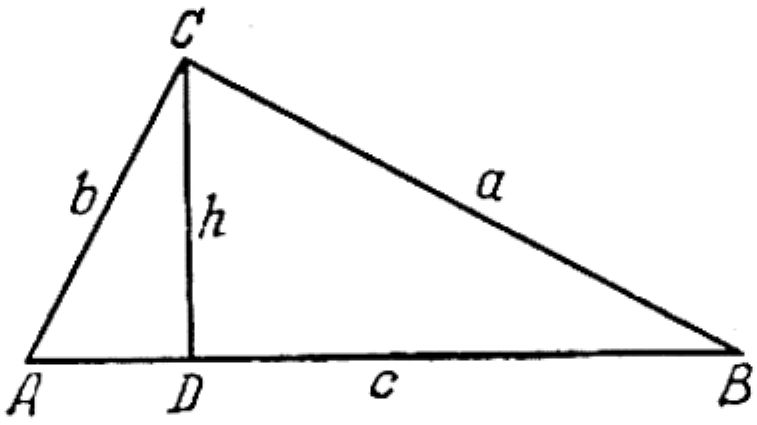


Fig. 8

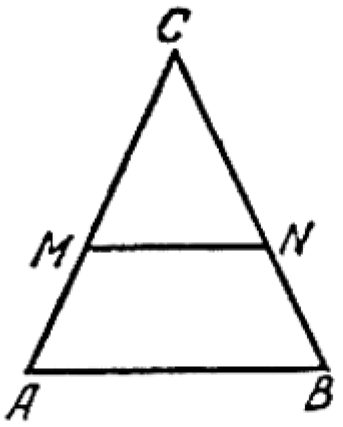


Fig. 9

521. Either of the sides AC and BC (Fig. 9) of the triangle ABC is equal to $\frac{2P-2a}{2}=P-a$. Let x be the length of the line-segment CM ($x=CM=CN$). The perimeter $2p$ of the trapezoid $AMNB$ is obtained from the perimeter $2P$ of the triangle ABC by subtracting $CM+CN=2x$ from $2P$ and then adding MN to the difference thus obtained. From the similarity of the triangles ABC and MNC we find

$$MN=\frac{AB\cdot MC}{AC}=\frac{2ax}{P-a}$$

Hence,

$$2P-2x+\frac{2ax}{P-a}=2p$$

whence

$$x=\frac{(P-a)(P-p)}{P-2a}$$

Answer: $CM=CN=\frac{(P-a)(P-p)}{P-2a}$.

522. It is required to determine the distance $NP = x$ between the point N (Fig. 10) and the base $AD = a$*, and the distance $NM = y$ between the point N and the side $AB = c$. From the similarity of the triangles AMN and ABC

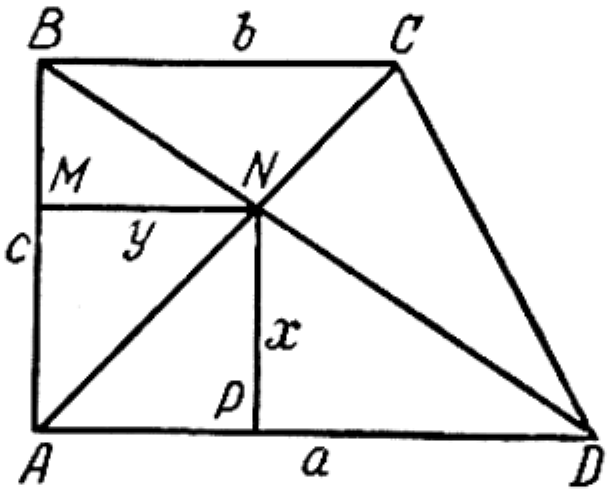


Fig. 10

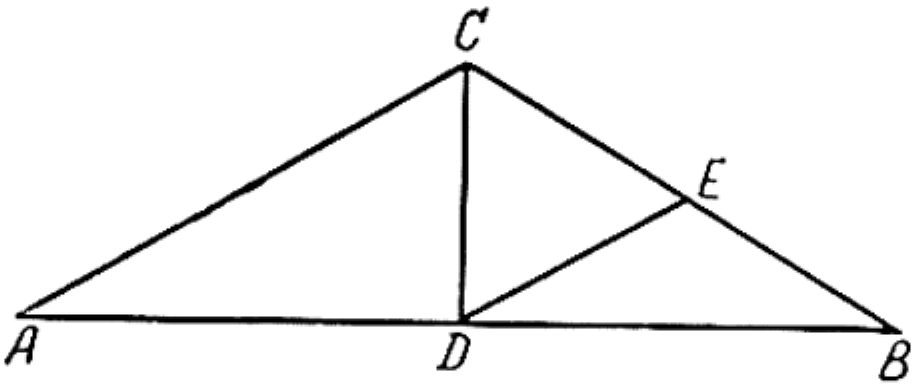


Fig. 11

(where $BC = b$) we find $\frac{MN}{BC} = \frac{AM}{AB}$, i.e. $\frac{y}{b} = \frac{x}{c}$, and from the similarity of the triangles NPD and BAD we have $\frac{NP}{BA} = \frac{PD}{AD}$, i.e. $\frac{x}{c} = \frac{a-y}{a}$. Then we solve these two equations.

Answer: $x = \frac{ac}{a+b}$; $y = \frac{ab}{a+b}$.

523. Let ABC (Fig. 11) is the given triangle. Since DE is a midline of the triangle and $DE = CD$, we have $CD = \frac{1}{2} AC$. Consequently, $\angle CAD = 30°$. Therefore $CD = \frac{AD\sqrt{3}}{3} = 2\sqrt{3}$ cm.

Answer: $S = 12\sqrt{3}$ cm².

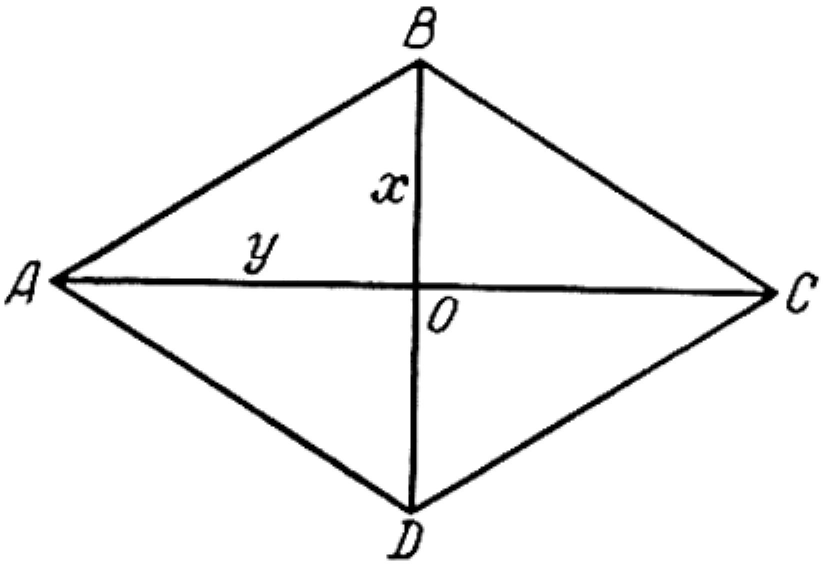


Fig. 12

524. Put $x = BO$, $y = AO$ (Fig. 12). Then the area S of the rhombus $ABCD$ is equal to $2xy$. By hypothesis, $x + y = \frac{m}{2}$; besides, from the right-angled triangle AOB, where $AB = \frac{1}{4} 2p = \frac{p}{2}$, we find $x^2 + y^2 = \left(\frac{p}{2}\right)^2$. Squaring both members of the first equation and subtracting the second one, we find $2xy = \frac{m^2 - p^2}{4}$.

Answer: $S = \frac{m^2 - p^2}{4}$ cm².

---

* The solution is independent of whether *a* is the larger or the smaller base.

525. Let x denote the altitude BE (Fig. 13). Then $AE=x$ and $FD=x\sqrt{3}$. Since $AD=AE+EF+FD$, $a=x+b+x\sqrt{3}$. Hence, $x=\frac{a-b}{\sqrt{3}+1}=$ $=\frac{(a-b)(\sqrt{3}-1)}{2}$.

Answer: $S=\frac{(a^2-b^2)(\sqrt{3}-1)}{4}$.

526. By hypothesis, $AD=44$ cm and $BC=16$ cm (Fig. 14). Hence, $AE+FD=28$ cm. Denoting the length AE (in centimetres) by x, we have $FD=28-x$. By hypothesis, $AB=17$ cm and $CD=25$ cm. Consequently, $BE^2=17^2-x^2$ and $CF^2=25^2-(28-x)^2$. And so, we get the equation

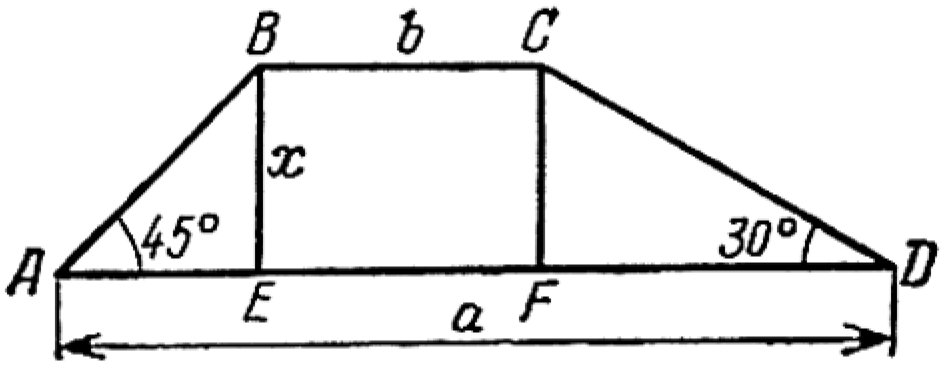


Fig. 13

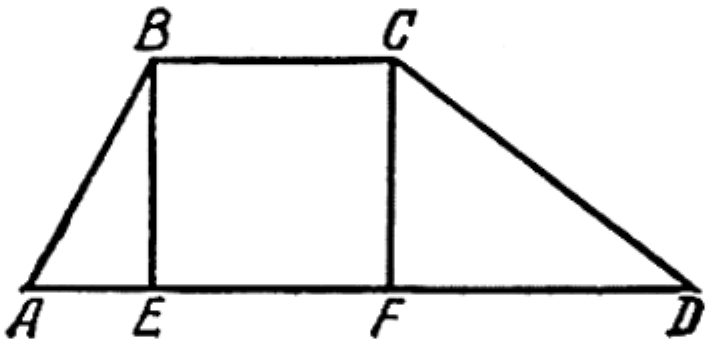


Fig. 14

$$17^2-x^2=25^2-(28-x)^2$$

whence $x=8$ (cm). Hence, we find the altitude h:

$$h=BE=\sqrt{17^2-x^2}=15 \text{ (cm)}$$

Now we find $S=\frac{(a+b)\,h}{2}$.

Answer: $S=450$ cm^2.

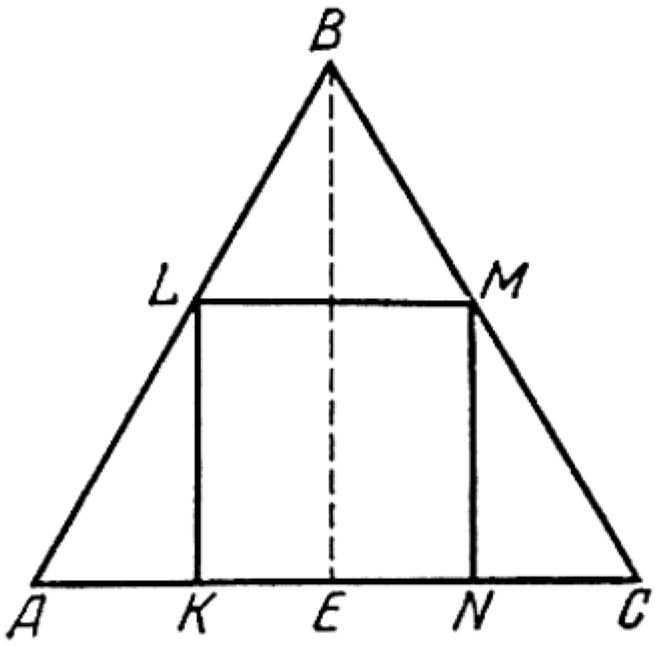


Fig. 15

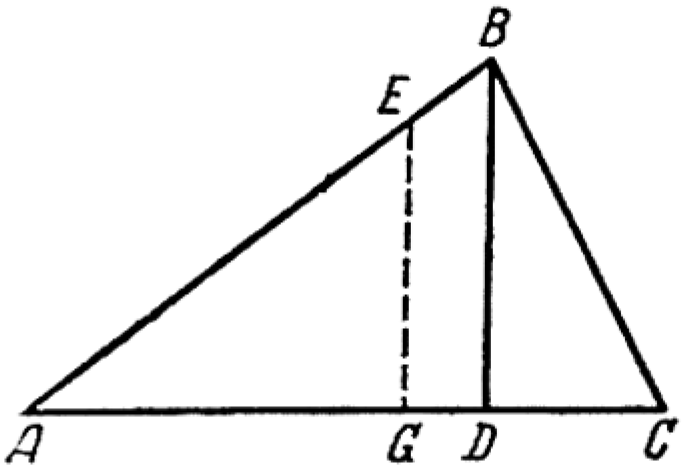


Fig. 16

527. Denote the side of the inscribed square (Fig. 15) by x. From the similarity of the triangles AKL $\left(\text{wherein } AK=\frac{AC-LM}{2}=\frac{a-x}{2}, \text{ and } LK=X\right)$

and $AEB\left(\text{wherein } AE=\frac{a}{2} \text{ and } BE=\frac{a\sqrt{3}}{2}\right)$ we get the equation $\frac{a-x}{2}:\frac{a}{2}=$ $=x:\frac{a\sqrt{3}}{2}$ wherefrom we find $x=\frac{a\sqrt{3}}{2+\sqrt{3}}=a\sqrt{3}(2-\sqrt{3})$.

Answer: $S=3a^2(2-\sqrt{3})^2=3(7-4\sqrt{3})a^2$.

528. By hypothesis, $AD=36$ cm and $DC=14$ cm (Fig. 16). The areas S_1 and S_2 of the triangles ADB and CBD with a common altitude are in the same ratio as the bases, i.e.

$$S_1 : S_2 = 36 : 14 = \frac{18}{7}$$

Consequently, $S_1=\frac{18}{25}S$, where $S=S_1+S_2$ is the area of the triangle ABC. By hypothesis, the straight line EG divides the area S into two equal parts, which means that this line intersects the base AC between the points A and D (but not between D and C). We get the triangle AGE, whose area S_3 is equal to $\frac{1}{2}S$. Since the areas of the similar triangles AGE and ADB are in the same ratio as the squares of the sides AG and AD, then

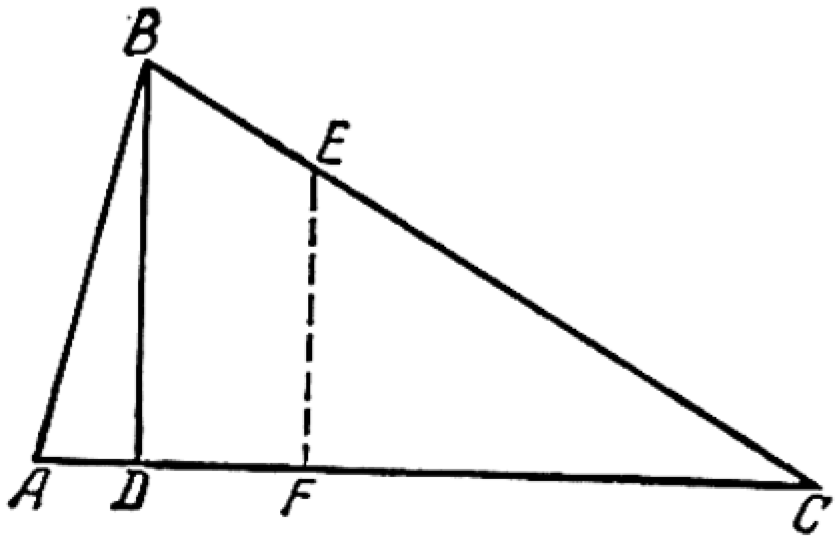


Fig. 17

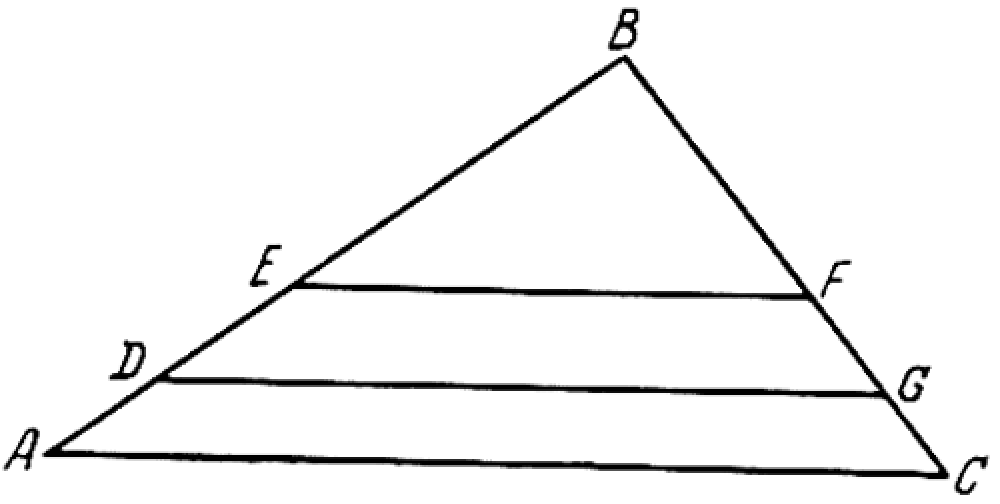


Fig. 18

$$\frac{18}{25}S : \frac{1}{2}S = 36^2 : AG^2$$

whence we find

$$AG = 30 \text{ (cm)}.$$

Hence,

$$GC = AC - AG = (36+14) - 30 = 20 \text{ cm}$$

Answer: 30 cm and 20 cm.

529. See the solution of the preceding problem. From the condition $AD : DC = 1 : 8$ we find that the area of the triangle BDC (Fig. 17) is $\frac{8}{9}$ of the area S of the triangle ABC. Since, by hypothesis, $BD=4$, we have

$$EF^2 : 16 = \frac{1}{2}S : \frac{8}{9}S$$

Answer: $EF=3$.

530. Since $S_{EBF}=S_{DEFG}=S_{ADGC}$ (Fig. 18), the area of the triangle EBF is half the area of the triangle DBG and three times as small as the area of

the triangle ABC. Since these triangles are similar, $EB^2 : DB^2 : AB^2 = 1 : 2 : 3$. By hypothesis, $AB = a$; hence, $EB = \frac{a}{\sqrt{3}}$ and $DB = \frac{a\sqrt{2}}{\sqrt{3}}$.

Answer: the side AB is divided into the following parts: $\frac{a}{\sqrt{3}}$, $\frac{a}{\sqrt{3}}(\sqrt{2}-1)$ and $\frac{a}{\sqrt{3}}(\sqrt{3}-\sqrt{2})$.

531. By hypothesis, the area of the triangle ABC (Fig. 19) is equal to S, and that of the triangle KBM, to q. Three vertices of the quadrilateral coincide with the points K, B, and M; and the fourth vertex L may be arbitrarily taken on the side AC. Indeed, the area S_1 of the quadrilateral $LKBM$ is the sum of the area q of the triangle KBM and the area of the triangle KLM, and the latter remains unchanged as the vertex L moves along the straight line AC parallel to the base KM. Let the altitude BE of the triangle ABC pass through the point E of the base AC. Placing the point L at the point E, we get the quadrilateral $KBME$, whose diagonals are mutually perpendicular; consequently, $S_1 = \frac{1}{2} KM \cdot BE$. And since $q = \frac{1}{2} KM \cdot BD$, $S_1 : q = BE : BD$. But since the triangle ABC is similar to the triangle KBM, we have $S : q = BE^2 : BD^2$. Consequently,

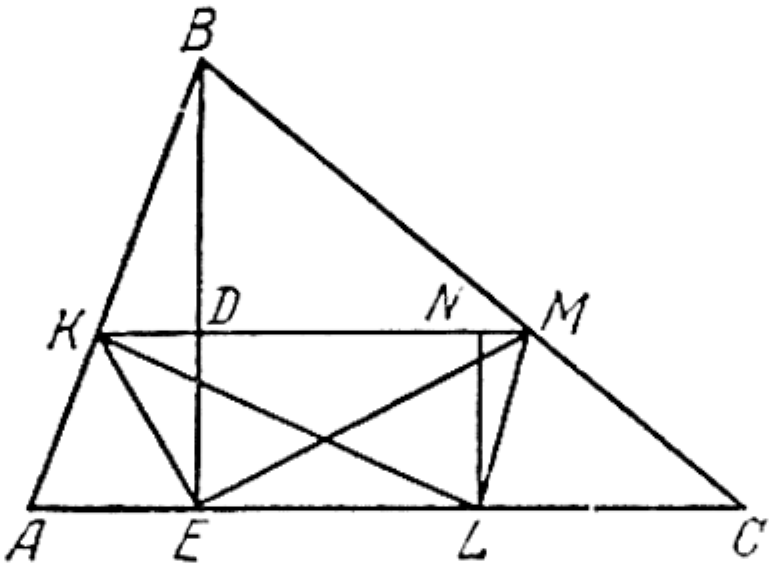


Fig. 19

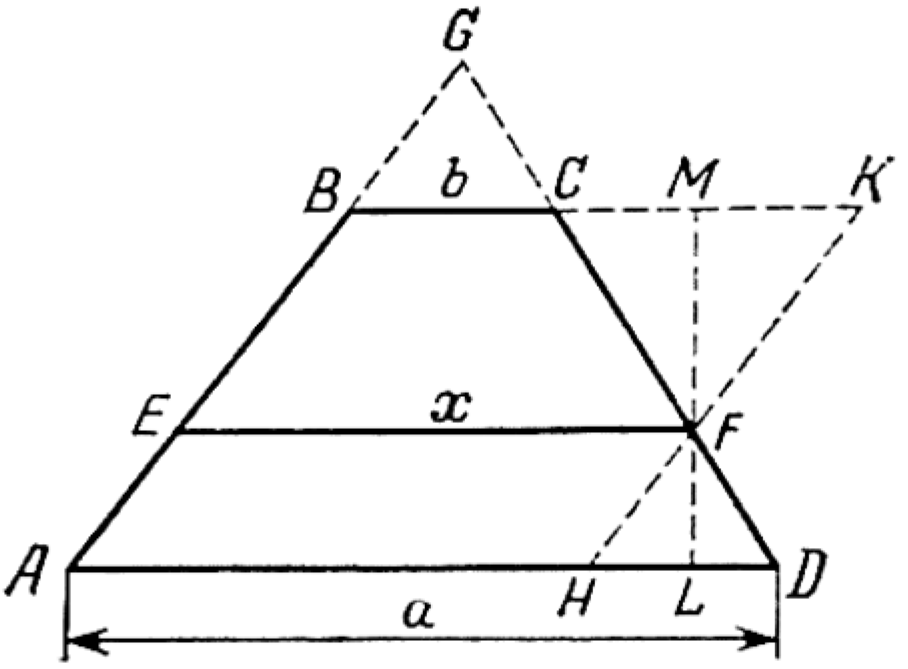


Fig. 20

$$S_1 = q \cdot \frac{BE}{BD} = q\sqrt{\frac{S}{q}} = \sqrt{Sq}$$

Note. If the point L does not coincide with the point E, the solution is modified: find

$$S_1 = \frac{1}{2} KM \cdot BD + \frac{1}{2} KM \cdot NL = \frac{1}{2} KM\,(BD + NL) = \frac{1}{2} KM \cdot BE$$

and then proceed in the same way as above.

Answer: $\sqrt{Sq}$.

532. Let the line-segment $EF = x$ (Fig. 20) divide the area of the trapezoid $ABCD$ ($AD = a$, $BC = b$) into two equal parts. Then

$$\frac{(a+x)\,FL}{2} = \frac{(x+b)\,FM}{2}, \quad \text{i.e.} \quad (a+x)\,FL = (x+b)\,FM$$

The altitudes FL and FM cannot be found separately (the length of one of them may be taken arbitrarily), but the ratio $FL : FM$ is of a definite value. Namely, from the similarity of the triangles HFD and CFK (wherein $HD = a - x$ and $CK = x - b$) we find

$$\frac{a-x}{FL}=\frac{x-b}{FM}$$

Multiplying this equality by the preceding one, we get

$$a^2-x^2=x^2-b^2$$

whence

$$x=\sqrt{\frac{a^2+b^2}{2}}$$

Alternate method. Extending the nonparallel sides, we get the similar triangles BGC, EGF and AGD. Their areas S_1, S_2, S_3 are proportional to the squares of

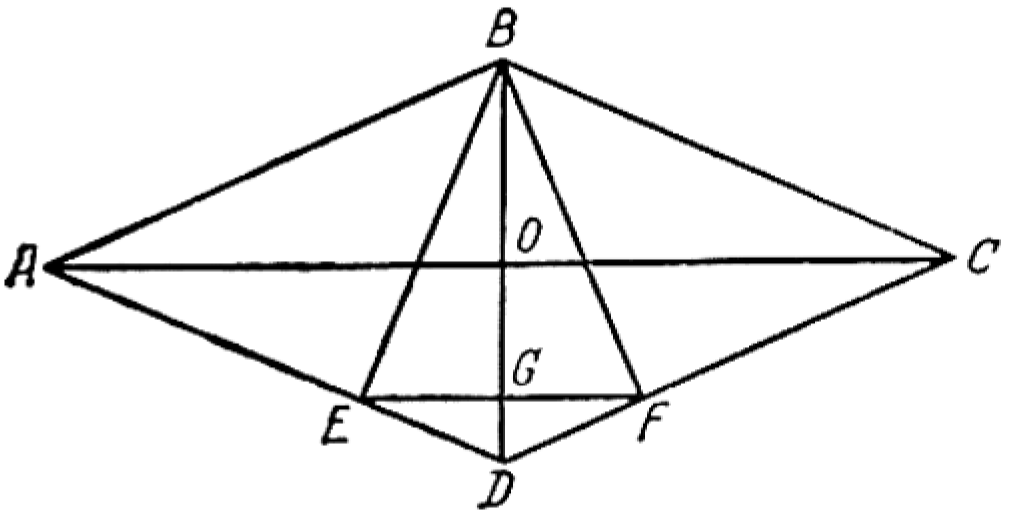


Fig. 21

corresponding sides b, x, a so that $S_1 = qb^2$, $S_2 = qx^2$, $S_3 = qa^2$, where q is a certain coefficient of proportionality, whose magnitude depends on the altitude of the trapezoid. By hypothesis, $S_2 - S_1 = S_3 - S_2$, i.e.

$$q(x^2-b^2)=q(a^2-x^2)$$

and since $q \neq 0$,

$$x^2-b^2=a^2-x^2$$

Answer: $x=\sqrt{\frac{a^2+b^2}{2}}$.

533. By hypothesis, $BE=BF=a$ (Fig. 21) and $EF=b$. Hence, $EG=\frac{b}{2}$ and $BG=\sqrt{a^2-\left(\frac{b}{2}\right)^2}$.

By the theorem on proportional lines in the right-angled triangle (BDE) we find $BD=\frac{BE^2}{BG}=\frac{a^2}{\sqrt{a^2-\left(\frac{b}{2}\right)^2}}$. Now we find the side of the rhombus (AD). The isosceles triangles ABD and BEF are similar, since their angles (all of them are acute) are respectively equal (as the angles with mutually perpendicular sides). Consequently,

$$AD:BD=BE:EF,$$

i.e.

$$AD : \frac{a^2}{\sqrt{a^2-\left(\frac{b}{2}\right)^2}} = a : b$$

wherefrom we find AD and then the area of the rhombus $S = AD \cdot a$.

Answer: $\frac{2a^4}{b\sqrt{4a^2-b^2}}$.

534. Let $AB = 27$ cm and $AC = 29$ cm (Fig. 22); then the median $AD = 26$ cm. Extend AD as long as $DE = AD$. The quadrilateral $ABEC$ is a parallelogram (prove it!) with sides of 27 and 29 cm.

The area of the triangle ABC constitutes half the area of the parallelogram obtained, on the other hand, the area of the triangle ABE is also equal to half the area of the parallelogram $ABEC$. Consequently, the area of the triangle ABC is equal to the area of the triangle ABE, whose sides are known ($AB = 27$ cm; $BE = 29$ cm; $AE = 52$ cm). Now the area of the triangle may be computed by using Heron's formula:

$$S = \sqrt{p(p-a)(p-b)(p-c)}$$

Answer: 270 cm².

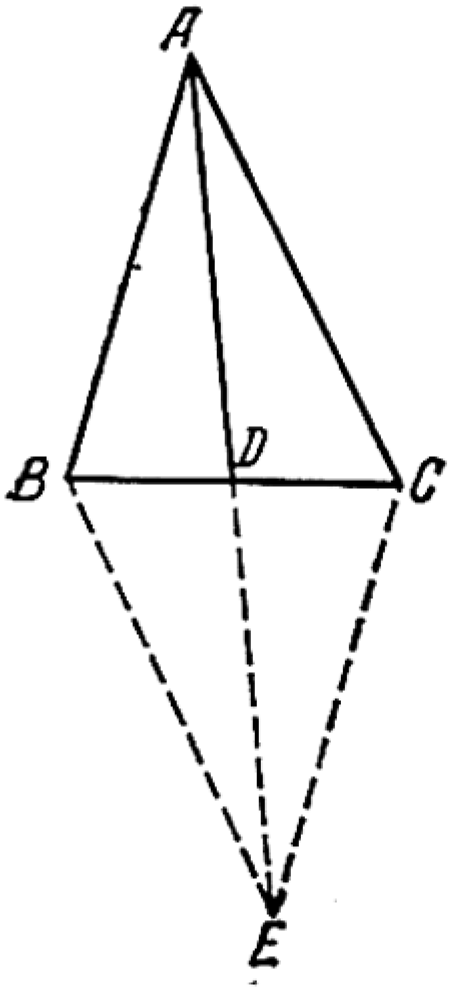


Fig. 22

535. By the law of cosines $a^2 = b^2 + c^2 - 2bc\cos A$, and since $S = \frac{1}{2}bc\sin A$, i.e. $\sin A = \frac{2S}{bc} = \frac{4}{5}$, we have

$$\cos A = \pm\sqrt{1-\sin^2 A} = \pm\frac{3}{5}$$

We get two solutions; both of them are suitable (in one case A is an acute angle, in the other it is an obtuse one).

Answer: $a = \sqrt{b^2+c^2-\frac{6}{5}bc}$ or $a = \sqrt{b^2+c^2+\frac{6}{5}bc}$.

536. From the triangle ABC (see Fig. 23) we have:

$$m^2 = b^2 + c^2 - 2bc\cos B$$

and since $\cos B = \cos(180° - A) = -\cos A$, then

$$m^2 = b^2 + c^2 + 2bc\cos A$$

From the triangle ADC we find

$$m^2 = a^2 + d^2 - 2ad\cos D$$

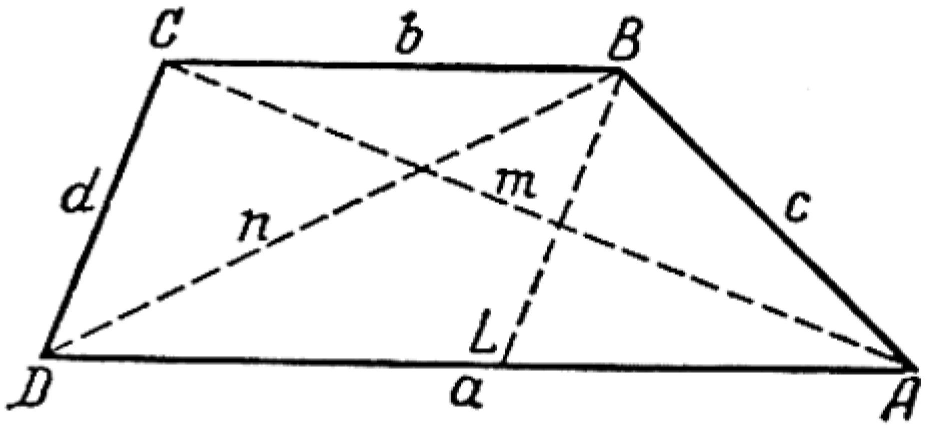


Fig. 23

Equating this expression to the preceding one, we get

$$2bc\cos A + 2ad\cos D = a^2 - b^2 + d^2 - c^2 \quad (1)$$

In the same way, considering the triangles ABD and CBD, we get

$$2ac\cos A + 2bd\cos D = a^2 - b^2 - (d^2 - c^2) \qquad (2)$$

Equations (1) and (2) yield $\cos A$ and $\cos D$, and then we find m^2 and n^2. Proceed as follows: multiply (1) by b, and (2), by a, and then subtract the first equation from the second. We get

$$2(a^2 - b^2)c\cos A = (a^2 - b^2)(a - b) - (d^2 - c^2)(a + b)$$

Dividing both members of the equality by $(a^2 - b^2)$ $[\neq 0]$, we obtain

$$2c\cos A = a - b - \frac{d^2 - c^2}{a - b}$$

Now we find

$$m^2 = b^2 + c^2 + (2c\cos A)\,b = c^2 + ab - \frac{(d^2 - c^2)\,b}{a - b} = \frac{a(c^2 - b^2) + b(a^2 - d^2)}{a - b}$$

Similarly, we find

$$2d\cos D = a - b + \frac{d^2 - c^2}{a - b}$$

and then

$$n^2 = b^2 + d^2 + (2d\cos D)\,b = \frac{a(d^2 - b^2) + b(a^2 - c^2)}{a - b}$$

Note. The line-segment $AD = a$ is smaller than the broken line $ABCD$. Therefore the problem is solvable only if $a < b + c + d$. But this condition alone is not sufficient, which is seen from the following. Let $a > b$ and $c \geqslant d$ (if these inequalities are not fulfilled then we can always change the notation to make the inequalities valid). Draw a straight line BL parallel to the side CD to complete the parallelogram $DCBL$. Now we find: $BL = CD = d$ and $DL = CB = b$. In the triangle ALB the side $LA = DA - DL = a - b$ is larger than the difference of the sides $AB = c$ and $BL = d$. Therefore another condition should be satisfied, namely $a - b > c - d$. If either of the conditions is not fulfilled, then at least one of the expressions obtained for m^2 and n^2 will turn out to be negative.

The two conditions $a < b + c + d$ and $a - b > c - d$ are quite sufficient for the problem to be solvable. Indeed, the first condition may be written in the form $a - b < c + d$. Consequently, we can construct a triangle ABL with the sides $AL = a - b$, $AB = c$ and $BL = d$. Extending the side AL by $LD = b$ and constructing a parallelogram $DLBC$, we get a quadrilateral $ABCD$, which is a trapezoid with the bases $AD = a$, $BC = b$ and nonparallel sides $AB = c$ and $DC = d$.

Answer: $m^2 = \dfrac{a(c^2 - b^2) + b(a^2 - d^2)}{a - b}$

$$n^2 = \frac{a(d^2 - b^2) + b(a^2 - c^2)}{a - b}$$

537. For the notation in Fig. 24, where $\angle A = 60^\circ$, we have

$$BD^2 = AD^2 + AB^2 - 2\cdot BA\cdot AD\cdot\cos 60^\circ = a^2 + b^2 - ab,$$

$$AC^2 = a^2 + b^2 + ab$$

Since AC is longer than BD, the given ratio $\frac{19}{7}$ is equal to $\frac{AC^2}{BD^2}$ $\left(\text{but not to } \frac{BD^2}{AC^2}\right)$. From the equation

$$\frac{a^2+b^2+ab}{a^2+b^2-ab}=\frac{19}{7} \quad \text{or} \quad \frac{\left(\frac{a}{b}\right)^2+1+\frac{a}{b}}{\left(\frac{a}{b}\right)^2+1-\frac{a}{b}}=\frac{19}{7}$$

we find $\frac{a}{b}=\frac{3}{2}$ and $\frac{a}{b}=\frac{2}{3}$. Both of these values give one and the same parallelogram (we may alter the notation in Fig. 24, denoting AB by a and AD, by b).

Answer: the sides are in the ratio 3 : 2.

538. Let O be an arbitrary point within the equilateral triangle ABC (Fig. 25). Join the point O with the vertices. The sum of the areas of the triangles AOB, BOC and COA is equal to the area of the triangle ABC. Denoting the side of this triangle by a, and the altitude, by h, we get

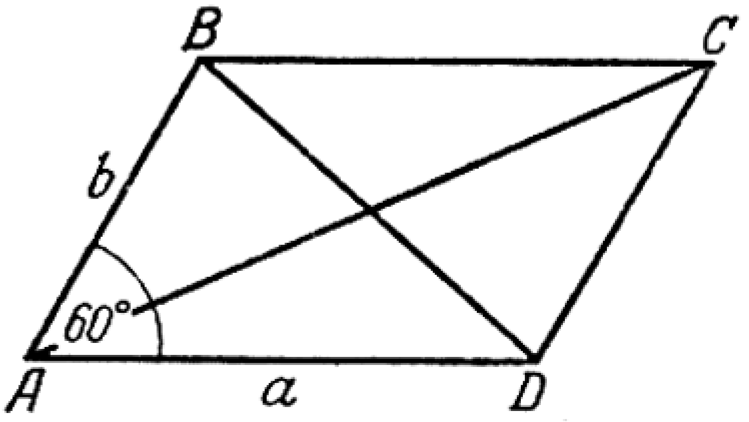


Fig. 24

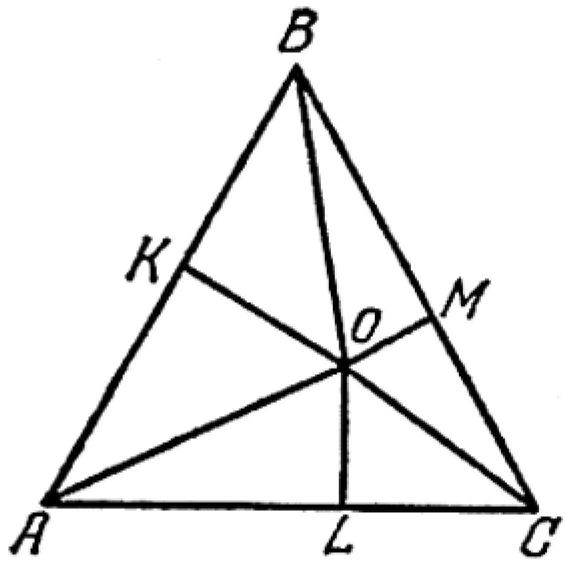


Fig. 25

$$(OK+OL+OM)\frac{a}{2}=\frac{ah}{2}$$

Hence,

$$h = OK + OL + OM$$

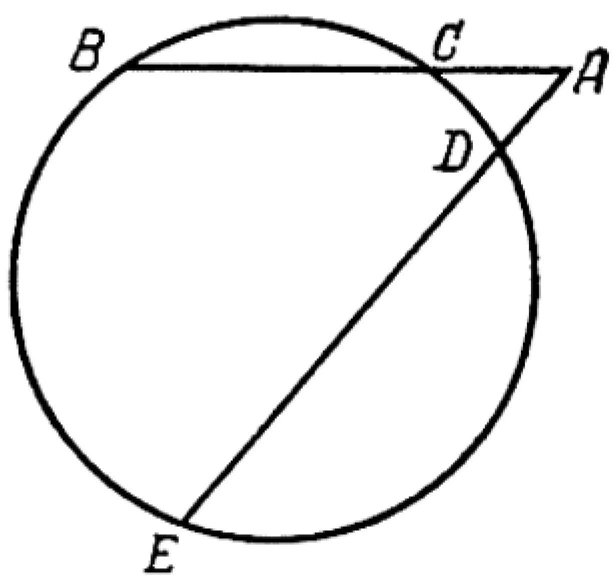


Fig. 26

539. By hypothesis, $BC = 47$ m and $CA = 9$m (Fig. 26; the drawing is made not to scale); hence, $BA = 56$ m. Consequently, $AD \cdot AE = 9 \cdot 56 = 504$. Let $AD = x$; then $DE = x + 72$ and, hence, $AE = 2x + 72$. From the equation $x(2x + 72) = 504$ we find $x = 6$.

Answer: $AE = 84$ m.

540. The problem is reduced to finding one of the legs of the triangle OAB (Fig. 27), given the hypotenuse $OA = m$ and altitude $BD = \frac{a}{2}$. Let us denote the larger leg by x, and the smaller one, by y. The area of the triangle OAB expressed in two

different ways (see solution of Problem 512) gives the equation $xy = a\frac{m}{2}$, i.e. $2xy = am$; besides, $x^2 + y^2 = m^2$. Adding and subtracting these equations by members, we get

$$x + y = \sqrt{m^2 + am}$$

and

$$x - y = \sqrt{m^2 - am}$$

Both x and y can serve as the required radius.

Answer: $\frac{1}{2}(\sqrt{m^2+am} + \sqrt{m^2-am})$, or $\frac{1}{2}(\sqrt{m^2+am} - \sqrt{m^2-am})$.

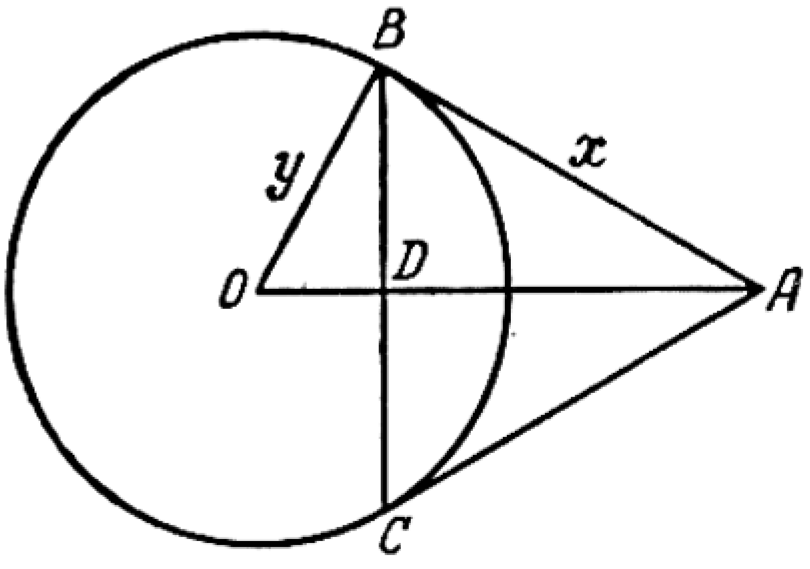


Fig. 27

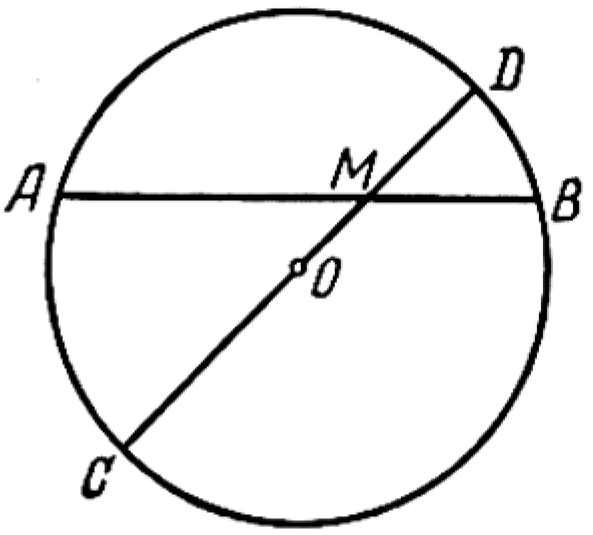


Fig. 28

541. Since the radius of the circle is equal to 13 cm and $MO = 5$ cm, then $MD = 8$ cm, $MC = 18$ cm (Fig. 28). Let us denote MB by x. Then $AM = 25 - x$. Since $AM \cdot MB = MD \cdot MC$, we have

$$(25 - x)x = 18 \cdot 8$$

Hence $x_1 = 16$, $x_2 = 9$.

Answer: the segments are 16 cm and 9 cm long.

542. From the triangle EBO_2 (Fig. 29), wherein $BE = \frac{1}{2}AB$, we find

$$R = O_2B = \frac{AB}{2\cos\frac{\alpha}{2}}$$

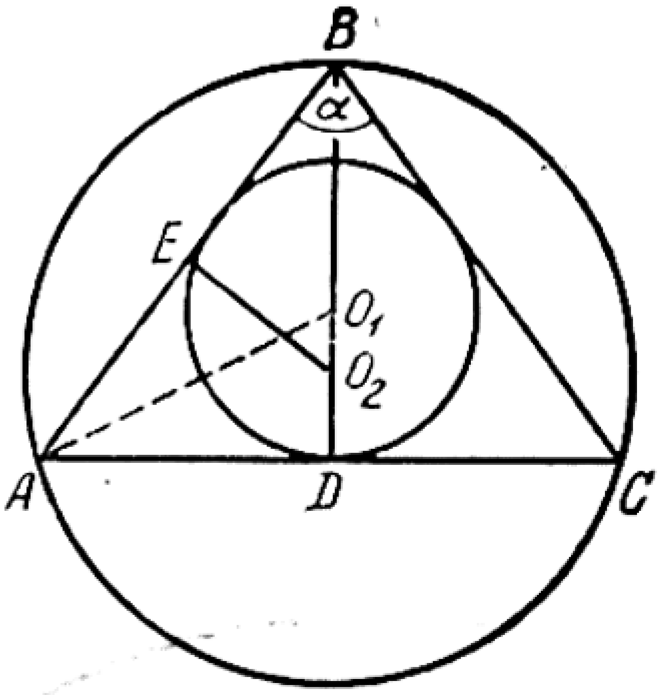


Fig. 29

From the triangle ADO_1, wherein $\angle DAO_1 = \frac{1}{2}\angle DAB = \frac{1}{2}\left(90° - \frac{\alpha}{2}\right)$, we find

$$r = O_1D = AD \cdot \tan\left(45° - \frac{\alpha}{4}\right)$$

8091
SRINAGAR

Since $AD = AB \cdot \sin\frac{\alpha}{2}$ (from the triangle ABD), we have

$$R : r = \frac{\cot\left(45^\circ - \frac{\alpha}{4}\right)}{\sin\alpha}$$

Answer: $\frac{R}{r} = \frac{\cot\left(45^\circ - \frac{\alpha}{4}\right)}{\sin\alpha}$.

543. By hypothesis, $a = BC = 13$ cm, $b = CA = 14$ cm, $c = AB = 15$ cm (Fig. 30). Denote $OE = OF$ by R. The area of the triangle ABC is equal to the sum of the areas of the triangles BOC and AOC. Since the areas of these triangles are equal to $\frac{13R}{2}$ and $\frac{14R}{2}$, respectively, then

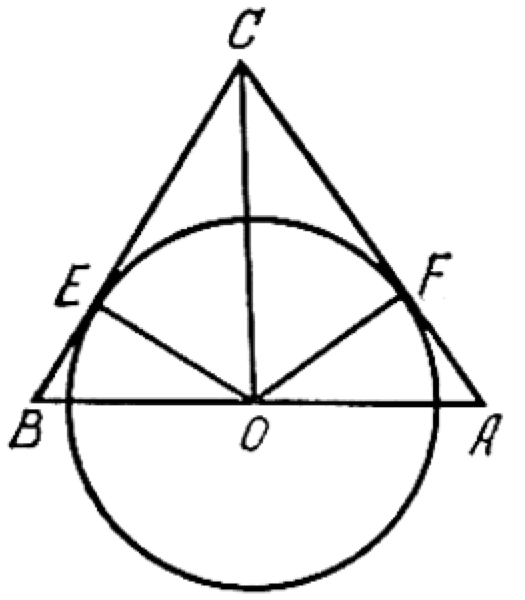


Fig. 30

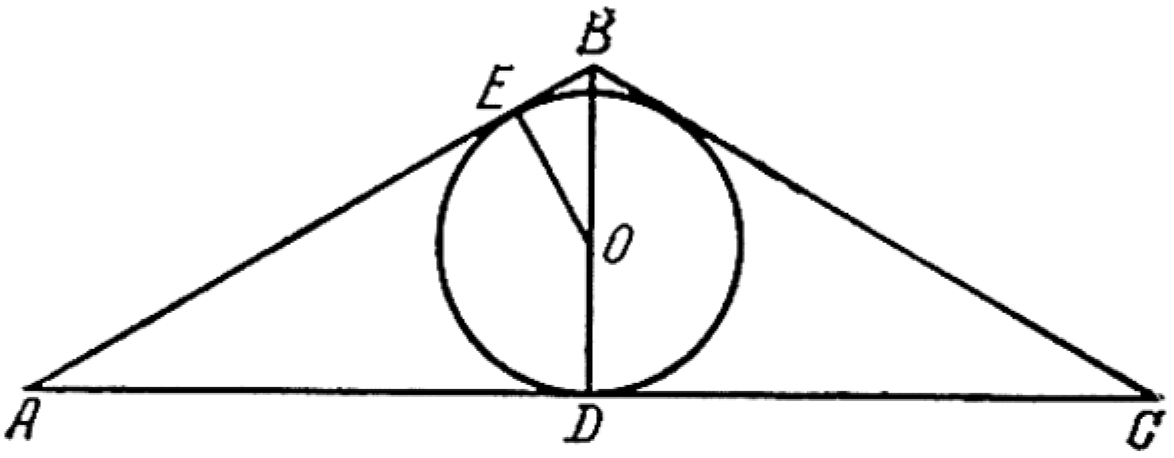


Fig. 31

$$S_{ABC} = \frac{27R}{2}$$

On the other hand, by Heron's formula

$$S_{ABC} = \sqrt{21\,(21-15)\,(21-14)\,(21-13)} = 84 \text{ cm}^2$$

Equate the two expressions for the area.

Answer: $R = 6\frac{2}{9}$ cm.

544. In the right-angled triangle OEB (Fig. 31) the angle EBO is equal to 60°. Therefore

$$BO = EO \cdot \frac{2}{\sqrt{3}} = \frac{2R}{\sqrt{3}}$$

Hence,

$$BD = R\left(1 + \frac{2}{\sqrt{3}}\right) = \frac{R(\sqrt{3}+2)}{\sqrt{3}}$$

From the triangle ABD we find

$$AB = \frac{2R(\sqrt{3}+2)}{\sqrt{3}} \quad \text{and} \quad AD = R(\sqrt{3}+2)$$

hence,

$$AC = 2R(\sqrt{3}+2)$$

Answer: $AB = BC = \dfrac{2R(\sqrt{3}+2)}{\sqrt{3}}$, $AC = 2R(\sqrt{3}+2)$.

545. From the triangle ABD (Fig. 32) we have

$$BD = \sqrt{BA^2 - AD^2} = 18 \text{ cm}$$

Since

$$BC \cdot BD = BA^2,$$

then

$$BC = \frac{BA^2}{BD} = 50 \text{ cm}$$

Consequently,

$$AC = \sqrt{BC^2 - BA^2} = 40 \text{ cm}$$

Answer: the semicircumference is equal to 20π.

546. Since the angles B, D, and E of the quadrilateral $ODBE$ are the right ones and $DO = OE$ (Fig. 33), this quadrilateral is a square. The sought-for arc DE is equal to one fourth of the circumference of the circle. Let us denote its radius by R. From the similarity of the triangles ADO and OEC we have

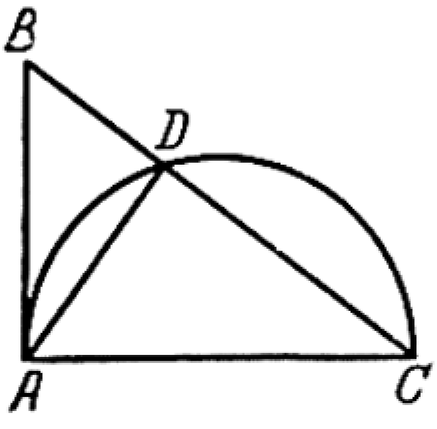


Fig. 32

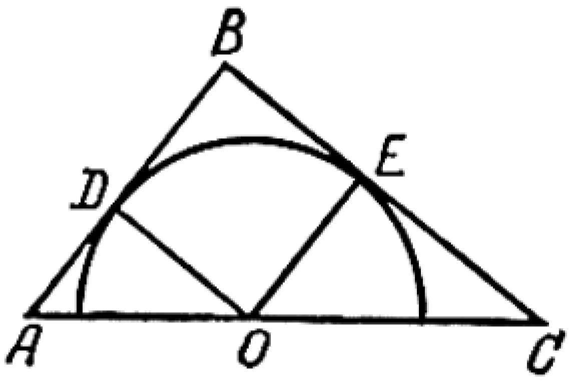


Fig. 33

$$\frac{AD}{AO} = \frac{OE}{OC}$$

Since

$$AD = \sqrt{AO^2 - OD^2} = \sqrt{15^2 - R^2},$$

then

$$\frac{\sqrt{15^2 - R^2}}{15} = \frac{R}{20}$$

Hence, $R = 12$.

Answer: 6π.

547. The area S of the quadrilateral $ADEB$ (Fig. 34) is

$$S = S_{ABC} - S_{DEC}$$

We have

$$S_{ABC} = \frac{AC}{2} BD = 12 \text{ cm}^2$$

To find S_{DEC}, let us notice that the triangles DEC and DBC have the common vertex D and one and the same altitude (not shown in the drawing) and that $S_{DBC} = \frac{1}{2} S_{ABC} = 6$ cm^2. Consequently, $S_{DEC} : 6 = CE : CB$. The unknown segment CE is found proceeding from the property of secants drawn from one point (C). We have $CE \cdot CB = CD \cdot CA$, whence $CE = \frac{CD \cdot CA}{CB}$. Hence,

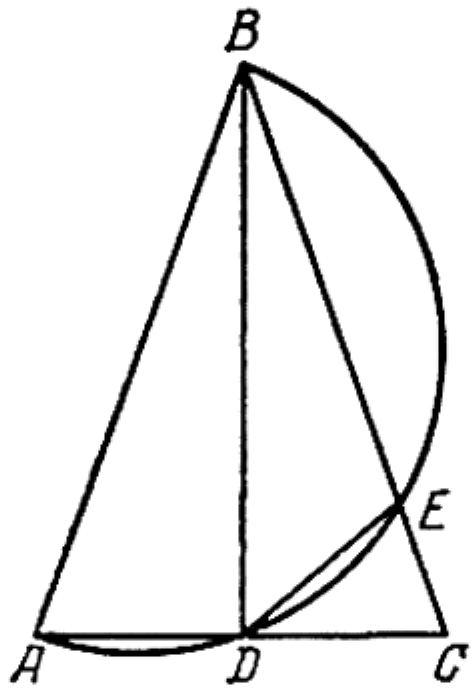


Fig. 34

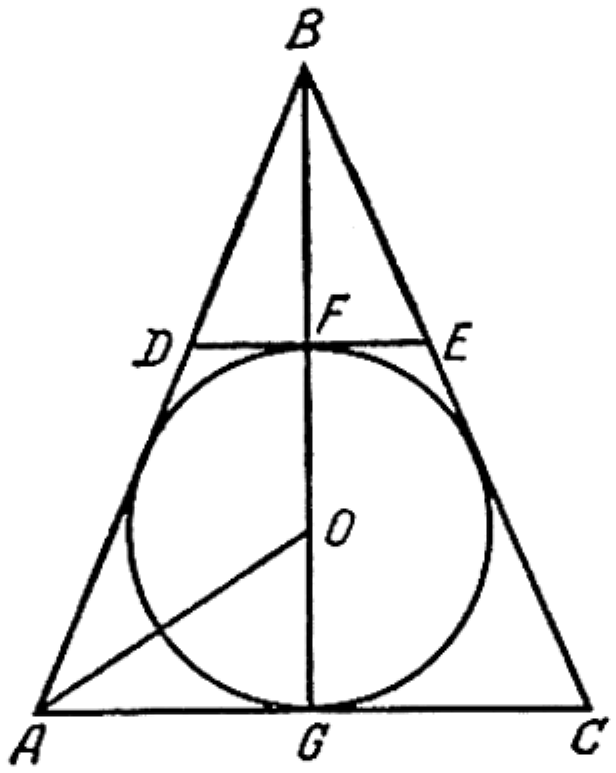


Fig. 35

$$S_{DEC} = 6 \frac{CE}{CB} = 6 \frac{CD \cdot CA}{CB^2} = 6 \frac{2 \cdot 4}{2^2 + 6^2} = 1.2 \text{ cm}^2$$

Answer: $S = 10.8$ cm^2.

548. The area S of the triangle ABC (Fig. 35) is equal to the product of its perimeter $2a + 2\sqrt{a^2 + h^2}$ and $\frac{r}{2}$ (r is the radius of the inscribed circle):

$$S = (a + \sqrt{a^2 + h^2})\, r$$

On the other hand,

$$S = \frac{1}{2} AC \cdot BG = ah$$

Equating the two expressions, we find

$$r = \frac{ah}{a + \sqrt{a^2 + h^2}}$$

The segment DE is found from the proportion

$$DE : AC = BF : BG$$

where

$$AC = 2a, \quad BF = h - 2r \quad \text{and} \quad BG = h$$

Note. We may find r in a different way: the straight line AO is the bisector of the angle A. Hence, the line-segments $GO = r$ and $OB = h - r$ are proportional to the sides AG and AB, i.e.

$$\frac{r}{h-r} = \frac{a}{\sqrt{a^2+h^2}}$$

Answer: $$r = \frac{ha}{\sqrt{a^2+h^2}+a}$$

$$DE = 2a\,\frac{\sqrt{a^2+h^2}-a}{\sqrt{a^2+h^2}+a} = \frac{2a\left(\sqrt{a^2+h^2}-a\right)^2}{h^2}$$

549. Since $OB \cdot OA = OC \cdot OD$ (Fig. 36) and $OB = OC$, then $OA = OD$. The opposite sides AB and CD of the quadrilateral $ABCD$ are equal to each other; hence, the given lengths (6m and 2.4m) belong to the sides AD and BC ($AD = 6$m, $BC = 2.4$m). The lines BC and AD cutting equal segments off the sides of the angle AOD are parallel, which means that the quadrilateral $ABCD$ is a trapezoid (an isosceles one). From the similarity of the triangles BOC and AOD we find

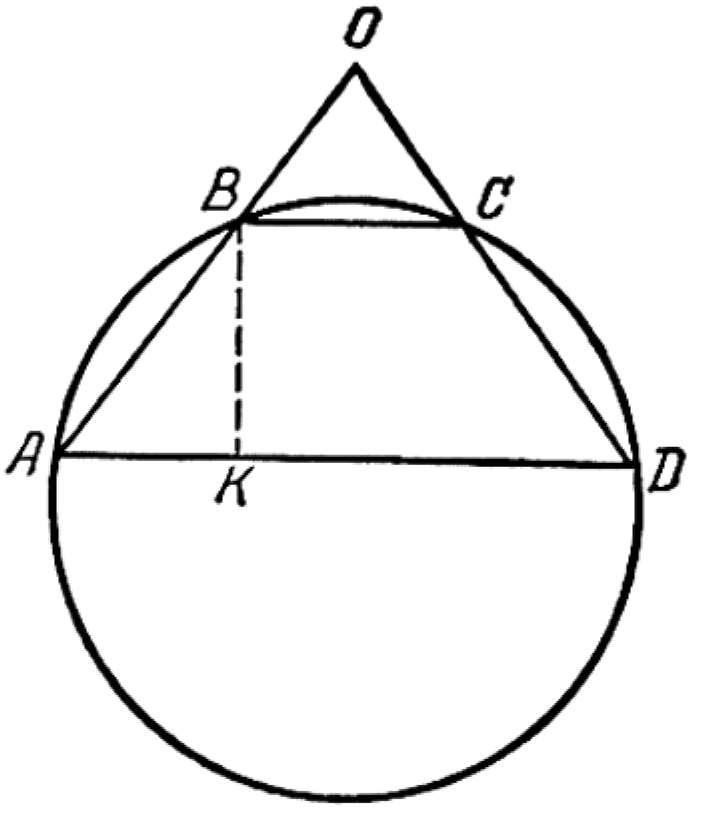


Fig. 36

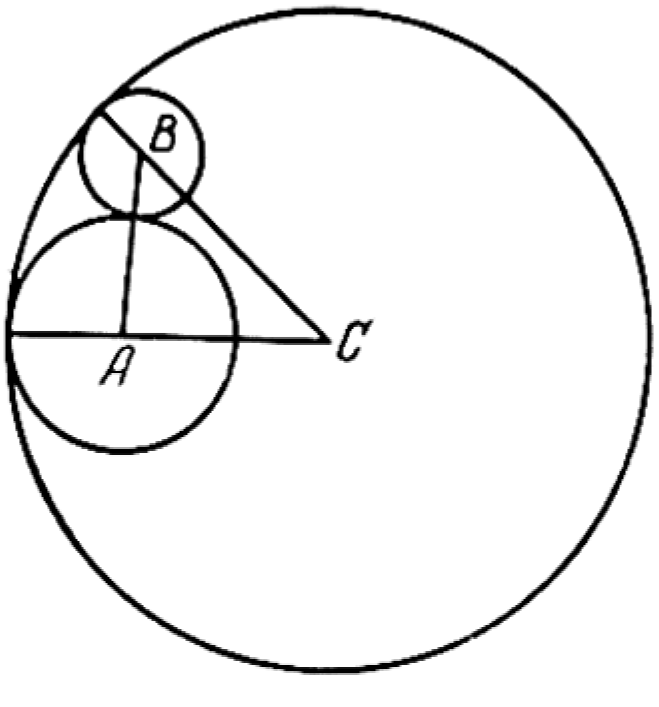


Fig. 37

$$BO : AO = BC : AD$$

whence

$$AO = \frac{BO \cdot AD}{BC} = \frac{2 \cdot 6}{2.4} = 5 \text{ m}$$

hence, $AB = 3$ m. Now find the altitude of the trapezoid

$$h = BK = \sqrt{AB^2 - AK^2} = \sqrt{3^2 - \left(\frac{6-2.4}{2}\right)^2} = 2.4 \text{ m}$$

Answer: $S = 10.08\ \text{m}^2$.

550. By hypothesis, $AB = 6$m, $AC = 7$m, $BC = 9$m (Fig. 37). Let R_A, R_B and R_C be the required radii of the circles with their centres at A, B and C.

Then $R_A + R_B = 6$, $R_C - R_A = 7$, $R_C - R_B = 9$, wherefrom we find the radii R_A, R_B and R_C.

Answer: $R_A = 4\text{m}$, $R_B = 2\text{m}$, $R_C = 11\text{m}$.

551. Draw O_2E parallel to AB and O_2P parallel to DC (Fig. 38). By hypothesis, $AB = \frac{3}{2}CD$. Denote CD by x. Then $O_2P = x$, $O_2E = \frac{3}{2}x$. From the triangles O_1EO_2 and O_1PO_2 we have

$$O_1O_2^2 = O_1E^2 + \frac{9}{4}x^2 \quad \text{and} \quad O_1O_2^2 = O_1P^2 + x^2$$

Equate these two expressions and take into account that

$$O_1E = O_1A - EA = O_1A - O_2B = 5 - 2 = 3 \text{ cm}$$

and, similarly,

$$O_1P = O_1C + O_2D = 7 \text{ cm}$$

Then we get

$$9 + \frac{9}{4}x^2 = 49 + x^2$$

whence $x^2 = 32$. Therefore

$$O_1O_2^2 = 49 + 32 = 81$$

Answer: $O_1O_2 = 9$ cm.

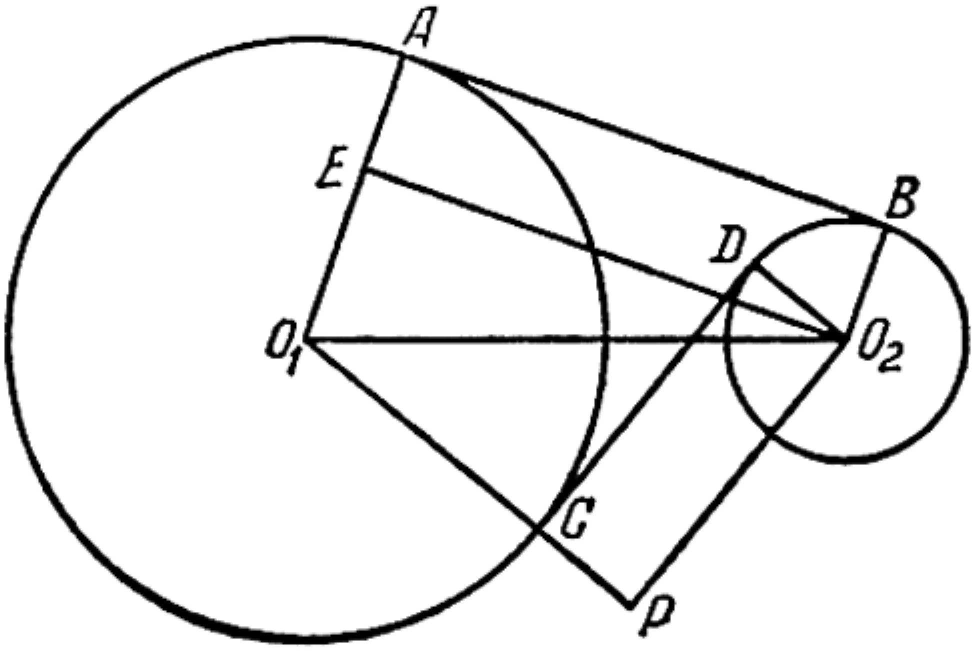


Fig. 38

552. Since the distance between the centres of the circles is less than the sum of their radii, but exceeds the difference between them, the circles intersect each other; hence, they have a common exterior tangent and no common interior tangent. Put $O_1C = x$ and $O_2C = y$ (Fig. 39). We have

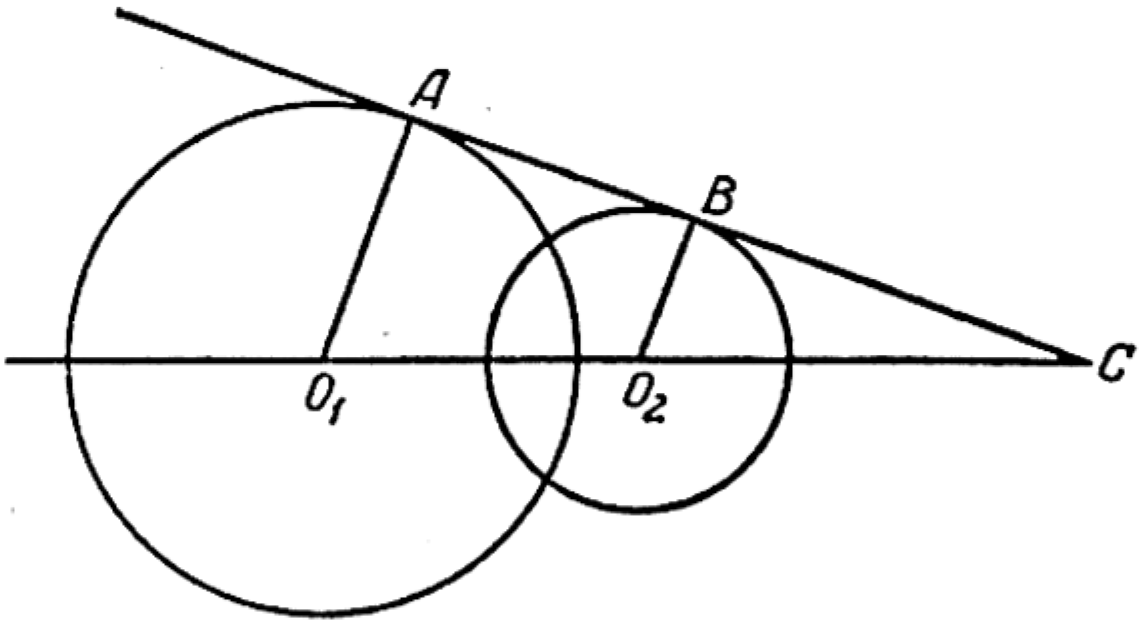


Fig. 39

$$x - y = O_1O_2 = 21 \text{ cm and } x : y = O_1A : O_2B = 17 : 10$$

Answer: $O_1C = 51$ cm, $O_2C = 30$ cm.

553. Two tangents to the circle O_1 (MD and MA) pass through the point M (Fig. 40). Hence, $MD = MA$. In the same way we prove that $MD = MB$.

Consequently,

$$MN = 2MD = AM + MB = AB$$

To find AB draw the straight line O_2C parallel to AB. From the triangle O_1O_2C, wherein $O_2C = AB$, $O_1O_2 = R + r$ and $O_1C = R - r$, we get

$$AB = \sqrt{(R+r)^2 - (R-r)^2}$$

or

$$AB = 2\sqrt{Rr}$$

Answer: $MN = 2\sqrt{Rr}$.

554. Let MN be a common tangent to the two circles (Fig. 41). Since $AM = MP = MB$, MN is the median of the trapezoid $ABCD$. We have $MN = AB = 2\sqrt{Rr}$ (see solution of the preceding problem). Let us now find the altitude BG of the trapezoid. According to the theorem on proportional lines in the right-angled triangle (EAB), we have

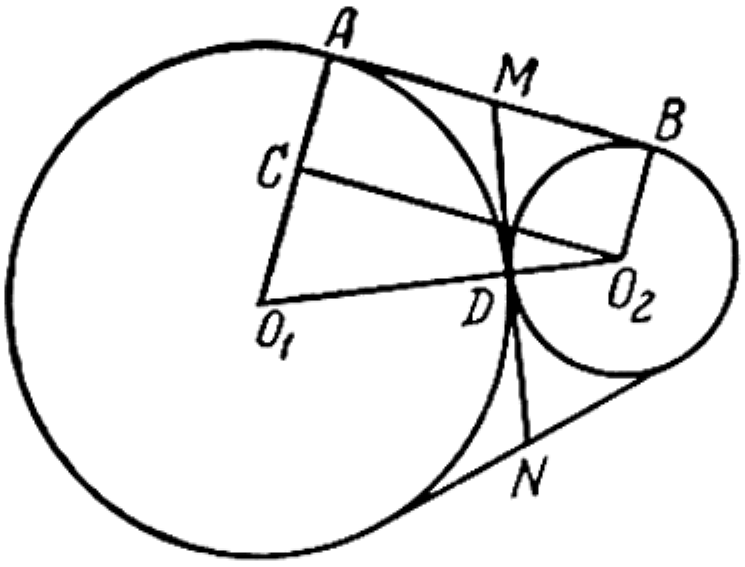


Fig. 40

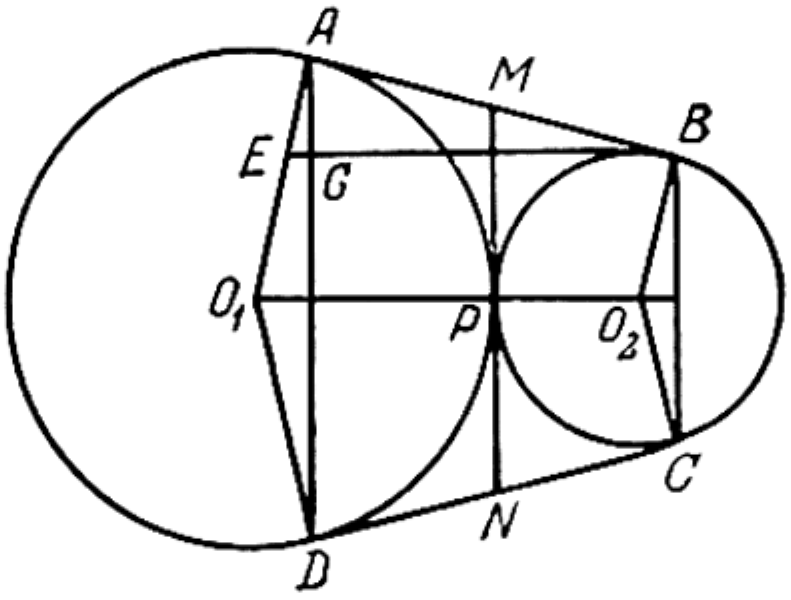


Fig. 41

$$BG = \frac{AB^2}{BE}$$

But

$$|BE = O_1O_2 = R + r$$

Hence,

$$BG = \frac{4Rr}{R+r}$$

Answer: $S = \frac{8(Rr)^{3/2}}{R+r}$.

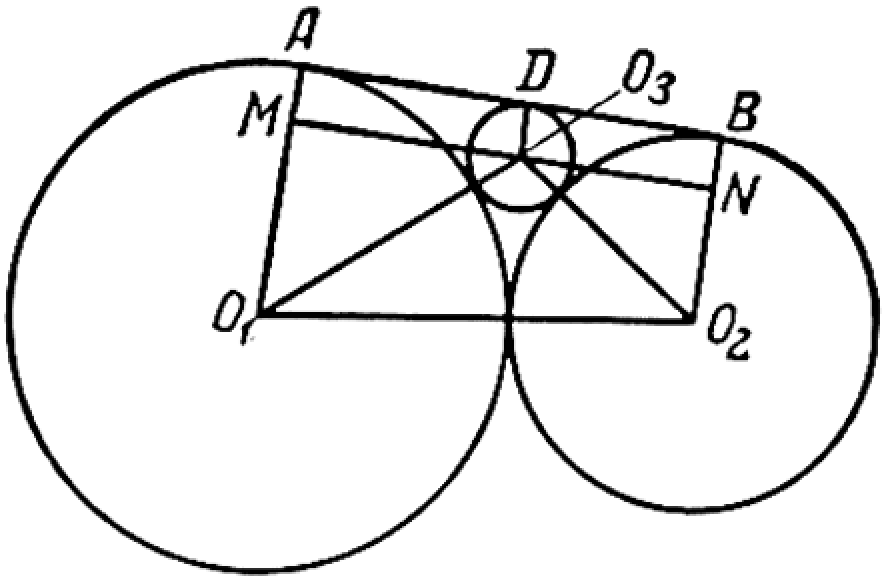


Fig. 42

555. Denote the radius of the required circle by x. Draw the straight line $MN \parallel AB$ through its centre O_3 (Fig. 42). Since AB is perpendicular to the radii O_1A, O_2B and O_3D, then $AM = BN = x$, and, hence, $O_1M = R - x$ and $O_2N = r - x$. Furthermore, we have $O_1O_3 = R + x$ and $O_2O_3 = r + x$. Consequently,

$$MO_3 = \sqrt{(R+x)^2 - (R-x)^2} = 2\sqrt{Rx}$$

similarly,

$$NO_3 = 2\sqrt{rx}$$

And since $MN = 2\sqrt{Rr}$ (see Problem 553), we have

$$2\sqrt{Rx} + 2\sqrt{rx} = 2\sqrt{Rr}$$

whence

$$\sqrt{x} = \frac{\sqrt{Rr}}{\sqrt{R}+\sqrt{r}}$$

Answer: the radius of the circle is $\frac{Rr}{(\sqrt{R}+\sqrt{r})^2}$.

556. Since $S = \frac{1}{2}ab\sin C$, where C is the angle between the chords, the problem has no solution for $S > \frac{1}{2}ab$. If $S < \frac{1}{2}ab$, then we find $\sin C = \frac{2S}{ab}$, and there exist two triangles with the sides a and b and area S: in one triangle C is an acute angle, in the other it is obtuse. In the first case $\cos C = \sqrt{1-\frac{4S^2}{a^2b^2}}$, in the second case $\cos C = -\sqrt{1-\frac{4S^2}{a^2b^2}}$. Hence,

$$c^2 = a^2+b^2-2ab\cos C = a^2+b^2 \mp 2\sqrt{a^2b^2-4S^2}$$

(the minus for an acute C, the plus for an obtuse one). At $S = \frac{1}{2}ab$ we get a right-angled triangle, so that $c^2 = a^2+b^2$. The radius of the circle circumscribed about the triangle is found by the formula $R = \frac{c}{2\sin C}$.

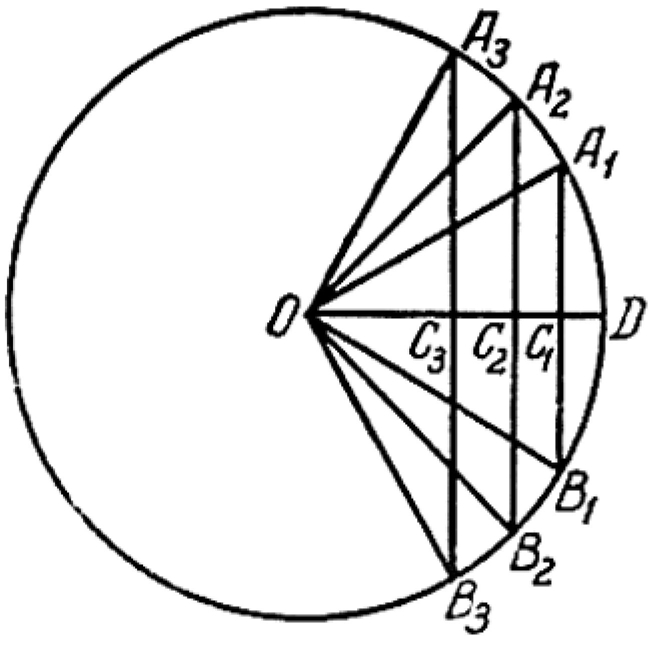


Fig. 43

Answer: $R = \frac{ab\sqrt{a^2+b^2 \mp 2\sqrt{a^2b^2-4S^2}}}{4S}$. For $S > \frac{1}{2}ab$ there is no solution, for $S < \frac{1}{2}ab$ — two solutions (the minus sign if the angle between the chords is acute, the plus one if it is obtuse). At $S = \frac{1}{2}ab$ we have one solution (the chords are mutually perpendicular).

557. By hypothesis (Fig. 43), $A_1B_1 = a_6 = R$, $A_2B_2 = a_4 = R\sqrt{2}$ and $A_3B_3 = a_3 = R\sqrt{3}$. The altitudes of the triangles OA_1B_1, OA_2B_2 and OA_3B_3 are $OC_1 = \frac{R\sqrt{3}}{2}$; $OC_2 = \frac{R\sqrt{2}}{2}$; $OC_3 = \frac{R}{2}$, respectively. Hence, we determine the areas of these triangles. Then we find the area of the sector OA_1DB_1; it is equal to one sixth of the area of the circle; therefore

$$S_{OA_1DB_1} = \frac{1}{6}\pi R^2$$

Similarly, $S_{OA_2DB_2}=\frac{1}{4}\pi R^2$ and $S_{OA_3DB_3}=\frac{1}{3}\pi R^2$. Subtracting the area of each triangle from the area of the respective sector, we find the area of the segments:

$$S_1=R^2\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)$$

$$S_2=R^2\left(\frac{\pi}{4}-\frac{1}{2}\right)$$

$$S_3=R^2\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)$$

The area of the portion of the circle contained between the chords A_1B_1 and A_2B_2 is

$$S_2-S_1=\frac{R^2}{12}(\pi+3\sqrt{3}-6)$$

the area contained between A_2B_2 and A_3B_3 is

$$S_3-S_2=\frac{R^2}{12}(\pi-3\sqrt{3}+6)$$

Answer: the ratio of the areas is equal to $\frac{\pi+3(2-\sqrt{3})}{\pi-3(2-\sqrt{3})}$.

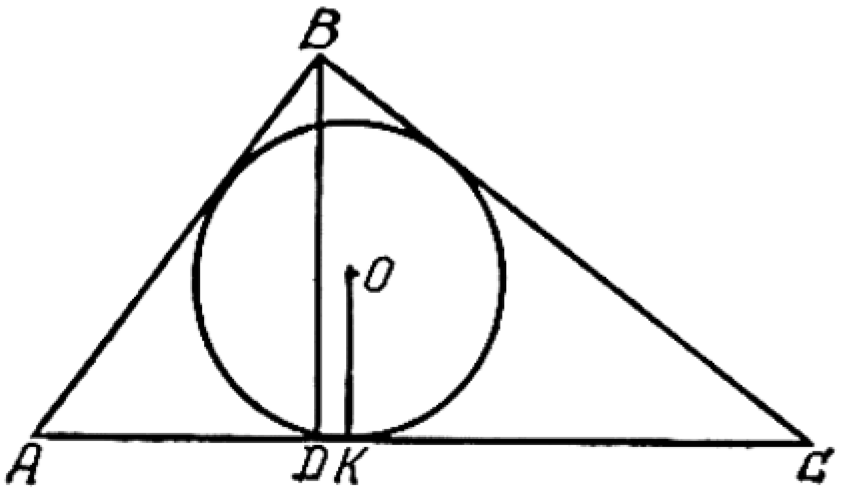


Fig. 44

558. For determining the radius $OK=r$ (Fig. 44) of the inscribed circle let us make use of the formula for the area of the triangle: $S=pr$ (p is the semiperimeter of the triangle). By hypothesis, $AD=14.4$ cm, $DC=25.6$ cm, therefore $AC=40$ cm. Hence, $AB=\sqrt{AD\cdot AC}=24$ (cm), $BC=\sqrt{DC\cdot AC}=32$ (cm). Consequently, $p=48$ cm and $S=384$ cm².

Answer: the area of the circle is equal to 64π cm².

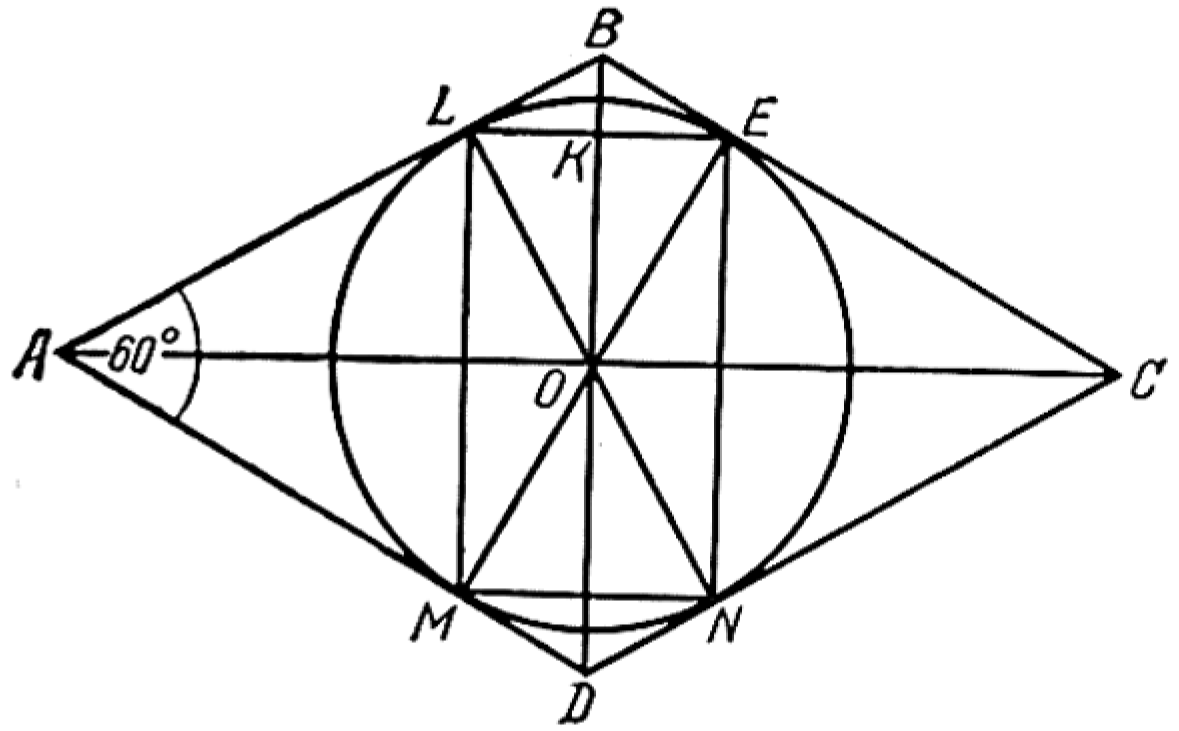


Fig. 45

559. The line LN joining the points of tangency of the two parallel lines AB and CD (Fig. 45) is the diameter of the circle. Therefore the inscribed angles

LEN and *LMN* (and, similarly, the angles *MLE* and *MNE*) are the right ones. Hence, the quadrilateral *LENM* is actually a rectangle. *ABD* is an equilateral triangle (since $AB = AD$ and $\angle A = 60°$); the line-segment *LN* (the altitude of the rhombus) is equal to the altitude of the triangle *ABD*, i.e. $LN = \frac{a\sqrt{3}}{2}$.

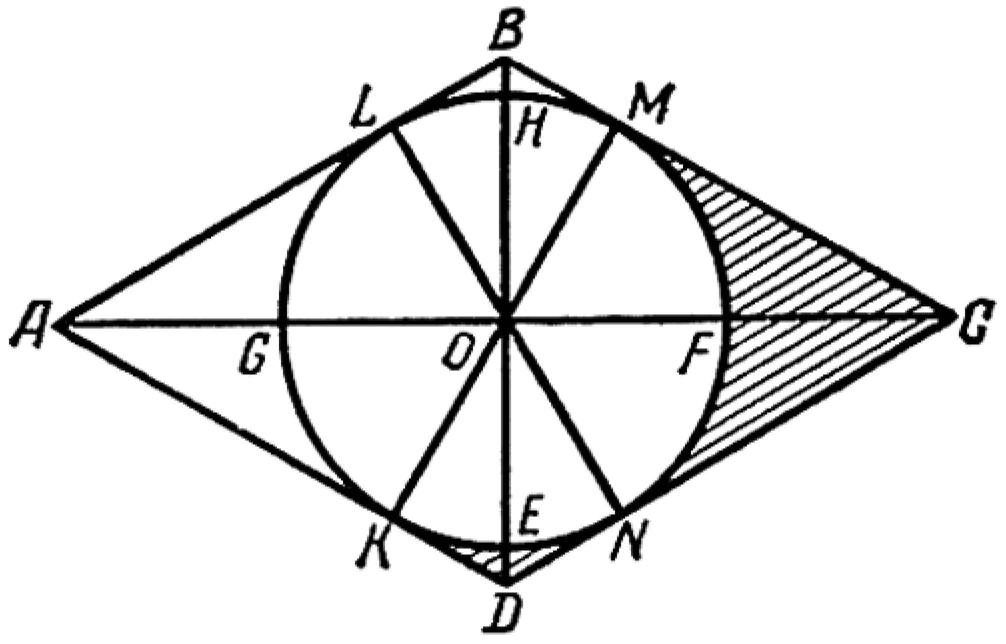


Fig. 46

The area S of the rectangle is equal to

$$\frac{1}{2} LN^2 \cdot \sin \angle LOE =$$

$$= \frac{1}{2} LN^2 \cdot \sin \angle BAD$$

(the sides of the angles *LOE* and *BAD* are mutually perpendicular).

Hence, $S = \frac{1}{2}\left(\frac{a\sqrt{3}}{2}\right)^2 \sin 60°$.

Answer: $S = \frac{3a^2\sqrt{3}}{16}$.

560. It is required to determine the area S_1 of the figure *MCNF* (Fig. 46) and the area S_2 of the figure *KDNE* (the areas of the figures *KALG* and *LBMN* are equal to S_1 and S_2, respectively). Since, by hypothesis, $AC = 4R$, then $OC = 2 \cdot OM$; hence, $\angle OCM = 30°$. Then we have $\angle MON = 180° - 2 \cdot 30° = 120°$ and $\angle KON = 60°$. The area of the quadrilateral *CMON* is equal to $R^2\sqrt{3}$, and the area of the sector *MONF*, to $\frac{1}{3}\pi R^2$. Hence, $S_1 = R^2\sqrt{3} - \frac{\pi R^2}{3}$; similarly, $S_2 = \frac{\sqrt{3}}{3} R^2 - \frac{\pi R^2}{6}$.

Answer: $S_1 = \frac{R^2(3\sqrt{3} - \pi)}{3}$; $S_2 = \frac{R^2(2\sqrt{3} - \pi)}{6}$.

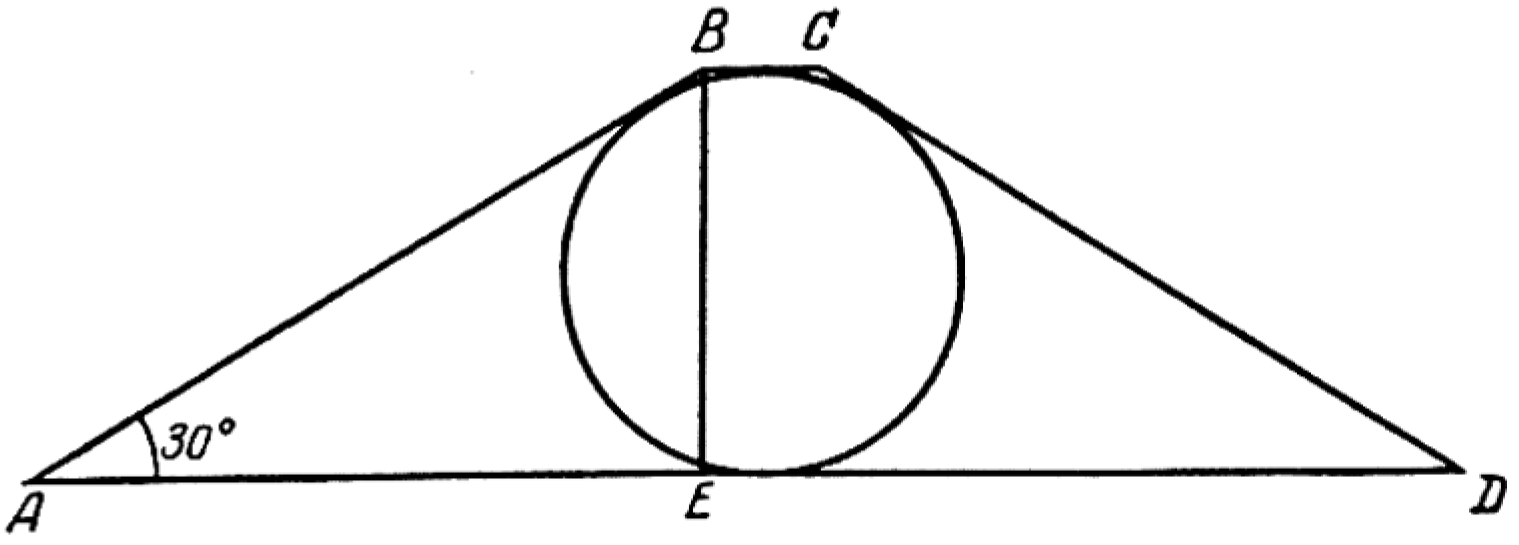


Fig. 47

561. Since $\angle A = 30°$ (Fig. 47), the altitude $BE = h$ of the trapezoid is equal to $\frac{1}{2} AB$. By the property of the circumscribed quadrilateral, $BC + AD =$

$=AB+CD=AB$. Therefore

$$S=\frac{AB+CD}{2}h=\frac{1}{2}AB^2$$

Answer: $AB=\sqrt{2S}$.

562. Given the area $S=20$ cm² and altitude $BE=2r=4$ cm (Fig. 48), we find the half-sum of the bases $\frac{AD+BC}{2}=5$ cm. Consequently, $AB=5$ cm (see the preceding problem). Now we find $AE=\sqrt{AB^2-BE^2}=3$ cm. But AE is the half-difference between the bases of the trapezoid. Knowing the half-sum and the half-difference, we find the bases themselves.

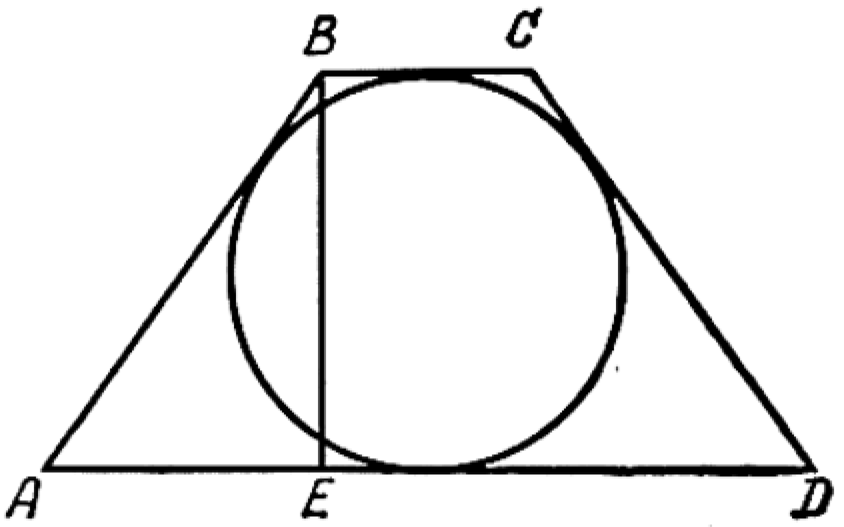


Fig. 48

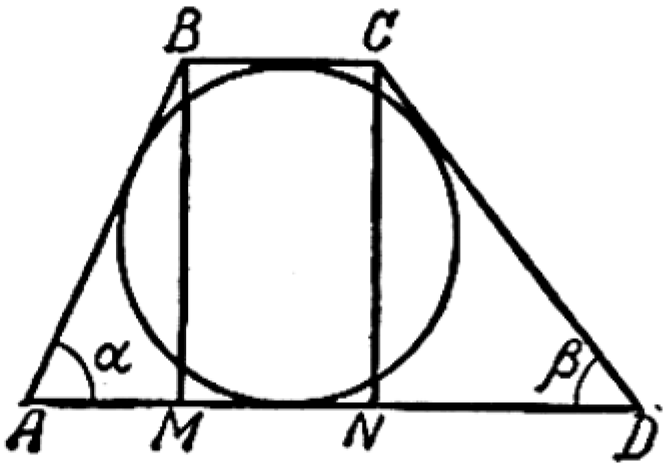


Fig. 49

Answer: $AD=8$ cm
$BC=2$ cm
$AB=CD=5$ cm

563. The area Q of the trapezoid $ABCD$ (Fig.49) is equal to

$$\frac{BC+AD}{2}BM=(BC+AD)R$$

(R is the radius of the inscribed circle). Since this trapezoid is circumscribed about the circle, $BC+AD=AB+CD$. But $AB=\frac{2R}{\sin\alpha}$, and $CD=\frac{2R}{\sin\beta}$. Therefore

$$Q=2R^2\left(\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}\right)=2R^2\frac{\sin\alpha+\sin\beta}{\sin\alpha\sin\beta}=$$

$$=\frac{4R^2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\sin\alpha\sin\beta}$$

Answer: $R=\frac{1}{2}\sqrt{\frac{Q\sin\alpha\sin\beta}{\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}}$.

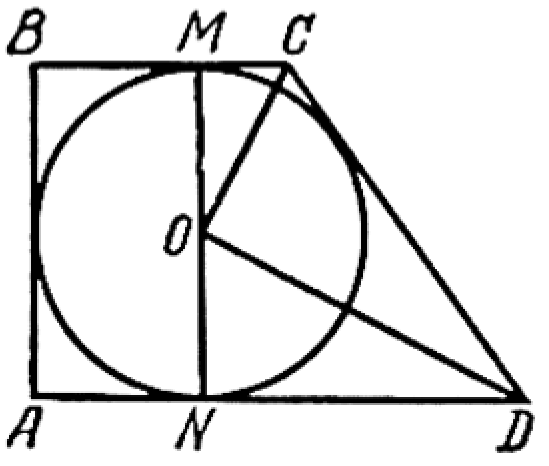


Fig. 50

564. Since the lateral side AB (Fig. 50), perpendicular to the bases, is equal to $2r$, the inclined side CD is greater than $2r$. Consequently, the least side of the trapezoid, equal to $\frac{3}{2}r$, is the (smaller) base BC. To find the larger base AD draw straight lines OC and OD which are respectively the bisectors of the angles MCD and NDC, whose sum amounts to 180°. Hence, $\angle MCO + \angle ODN = 90°$. From the right-angled triangle ODN we find $\angle NOD + \angle ODN = 90°$. Consequently, $\angle NOD = \angle MCO$, and the triangle ODN is similar to the triangle OCM. We obtain the proportion $ND : ON = OM : MC$, where $ON = OM = r$ and $MC = \frac{r}{2}$ (by hypothesis). Hence, $ND = 2r$, and $AD = AN + ND = r + 2r = 3r$.

Answer: $S = \frac{9r^2}{2}$.

565. The triangle OMC is similar to the triangle OND (Fig. 50) (see the preceding problem). Since $\frac{OD}{OC} = \frac{4}{2} = 2$, then $\frac{ND}{OM} = 2$ and $\frac{ON}{MC} = 2$, i.e. $ND = 2OM = 2r$ and $MC = \frac{ON}{2} = \frac{1}{2}r$. From the right-angled triangle OND we find $r^2 + (2r)^2 = 4^2$, whence,

$$r = \frac{4}{\sqrt{5}} \text{ (cm)}$$

Now we find $AD = AN + ND = r + 2r = 3r = \frac{12}{\sqrt{5}}$ cm and $BC = \frac{6}{\sqrt{5}}$ cm. The altitude MN of the trapezoid is equal to

$$2r = \frac{8}{\sqrt{5}} \text{ cm}$$

Answer: $S = 14.4$ cm^2.

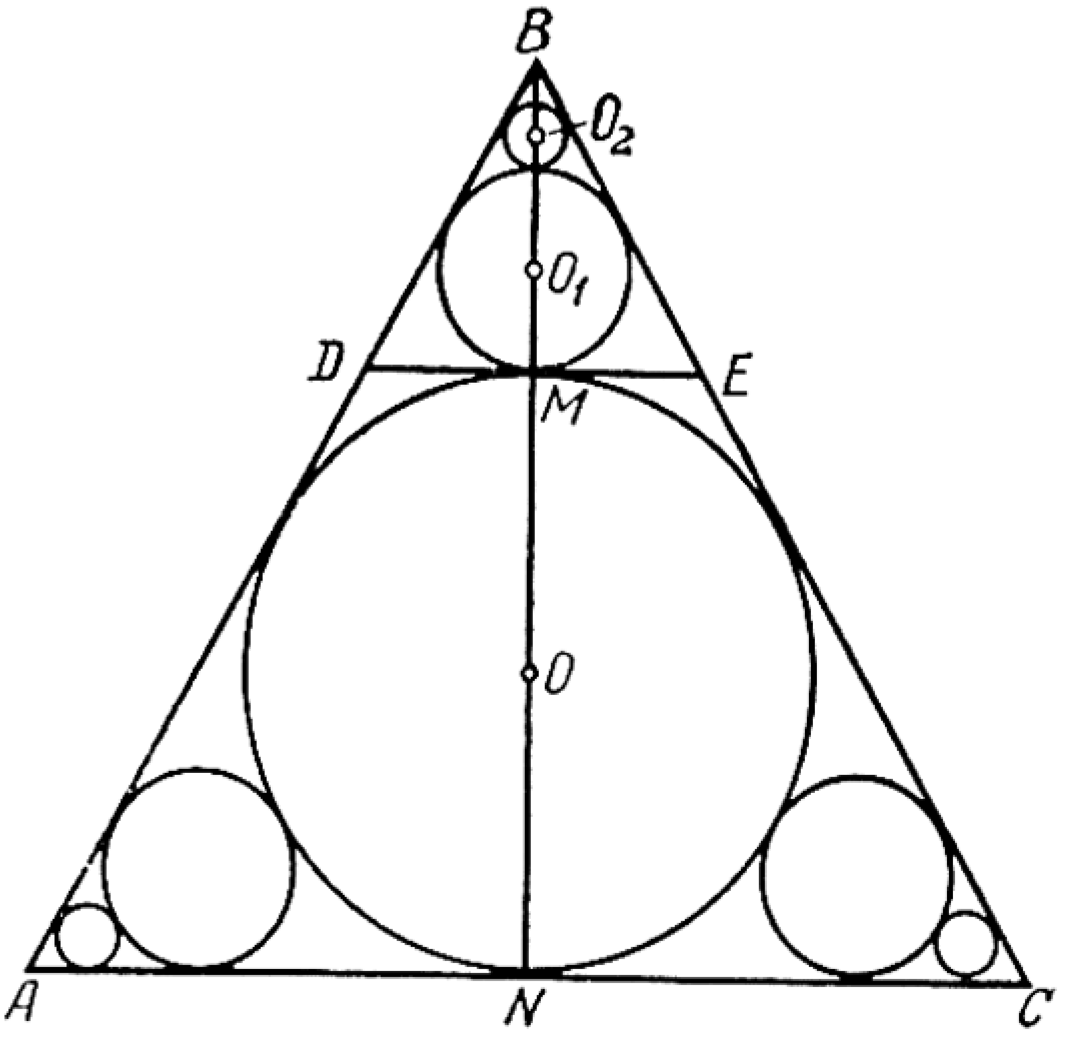


Fig. 51

566. The centre O of the first circle (Fig. 51) divides the altitude $BN = h$ in the ratio $BO : ON = 2 : 1$. Consequently, the diameter MN is equal to $\frac{2}{3}h$ and hence, $BM = \frac{1}{3}h$. The second circle is inscribed in the triangle BDE, whose altitude is equal to one third of the altitude h of the triangle ABC. Hence, the radius $r_1 = O_1M$ is one third of the radius $r = ON$. Therefore, if S is the area of the circle $O\left[S = \pi\left(\frac{a\sqrt{3}}{6}\right)^2 = \frac{\pi a^2}{12}\right]$, then the area of the circle O_1 will be $S_1 = \frac{1}{3^2}S$. And since there are three such circles, their total area Q_1 will be

$$Q_1 = \frac{1}{3}S$$

Reasoning in the same way, we find the total area of the next three circles

$$Q_2 = \frac{1}{3^2} Q_1 = \frac{1}{3^3} S \text{ and so on.}$$

Thus, we obtain an infinite sequence of addends

$$S + Q_1 + Q_2 + Q_3 + \ldots = S + \frac{1}{3} S + \frac{1}{3^3} S + \frac{1}{3^5} S + \ldots$$

The terms of this sequence, beginning with the term $\frac{1}{3} S$ (the addend S is treated separately), form an infinitely decreasing geometric progression $\left(a_1 = \frac{1}{3} S;\ q = \frac{1}{3^2}\right)$. The sum of this progression is equal to

$$\frac{a_1}{1-q} = \frac{\frac{1}{3} S}{\frac{8}{9}} = \frac{3}{8} S$$

To get the required area the addend S should be added to the above sums.

Answer: the required area is equal to $\frac{11}{8} S = \frac{11}{96} \pi a^2$.

567. To find the area of the trapezoid $BMNC$ (Fig. 52) it is required to find the base BM and altitude MN, since CN is known. First determine $CD = x$. We have

$$x(BC + x) = AD^2$$

or

$$x(5 + x) = 150$$

Hence,

$$CD = x = 10 \text{ (cm)}$$

From the similarity of the triangles BMD and CND it follows that $\frac{BM}{BD} = \frac{CN}{CD}$ or $\frac{BM}{15} = \frac{6}{10}$, whence $BM = 9$ (cm). The altitude MN is found from the proportion $\frac{MN}{BC} = \frac{ND}{CD}$, where $ND = \sqrt{CD^2 - CN^2}$. We get $MN = 4$ cm.

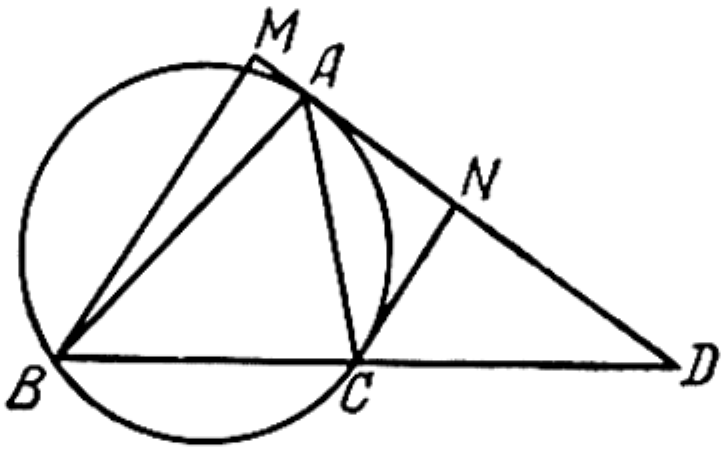


Fig. 52

Answer: $S = 30$ cm².

568. Let O_1, O_2 and O_3 be the centres of equal inscribed circles and let r be their radius (Fig. 53). Since AO_1 and CO_2 are the bisectors of the angles A and C, each being equal to 60°, then $\angle O_1AD = 30°$; hence, $AD = EC = r\sqrt{3}$. Furthermore, $DE = O_1O_2 = 2r$. Therefore $2r(1 + \sqrt{3}) = a$.

Answer: $r = \frac{a}{2(\sqrt{3}+1)} = \frac{a(\sqrt{3}-1)}{4}$.

569. The required area LMN (hatched in Fig. 53) is obtained by subtracting the total area of the three sectors O_1ML, O_2LN and O_3NM (which is equal to the area of a semi-circle of radius r) from the area of the triangle $O_1O_2O_3$, whose side is equal to $2r=\frac{a(\sqrt{3}-1)}{2}$ (see the preceding problem); therefore

$$S_{O_1O_2O_3}=r^2\sqrt{3}=\frac{a^2\sqrt{3}(\sqrt{3}-1)^2}{16}$$

The total area of the three sectors is equal to

$$\frac{\pi r^2}{2}=\frac{\pi a^2(\sqrt{3}-1)^2}{32}=\frac{\pi a^2(2-\sqrt{3})}{16}$$

Answer: $S=r^2\left(\sqrt{3}-\frac{\pi}{2}\right)=\frac{a^2(2-\sqrt{3})(2\sqrt{3}-\pi)}{16}$.

570. Solved in the same way as the preceding problem (Fig. 54).

Answer: $S=\frac{a^2(4-\pi)}{16}$.

Alternate solution. The required figure $KLMN$ is equal to the one which is hatched in Fig. 54. The latter is obtained by subtracting two semi-circles from the square B_1C_1MK.

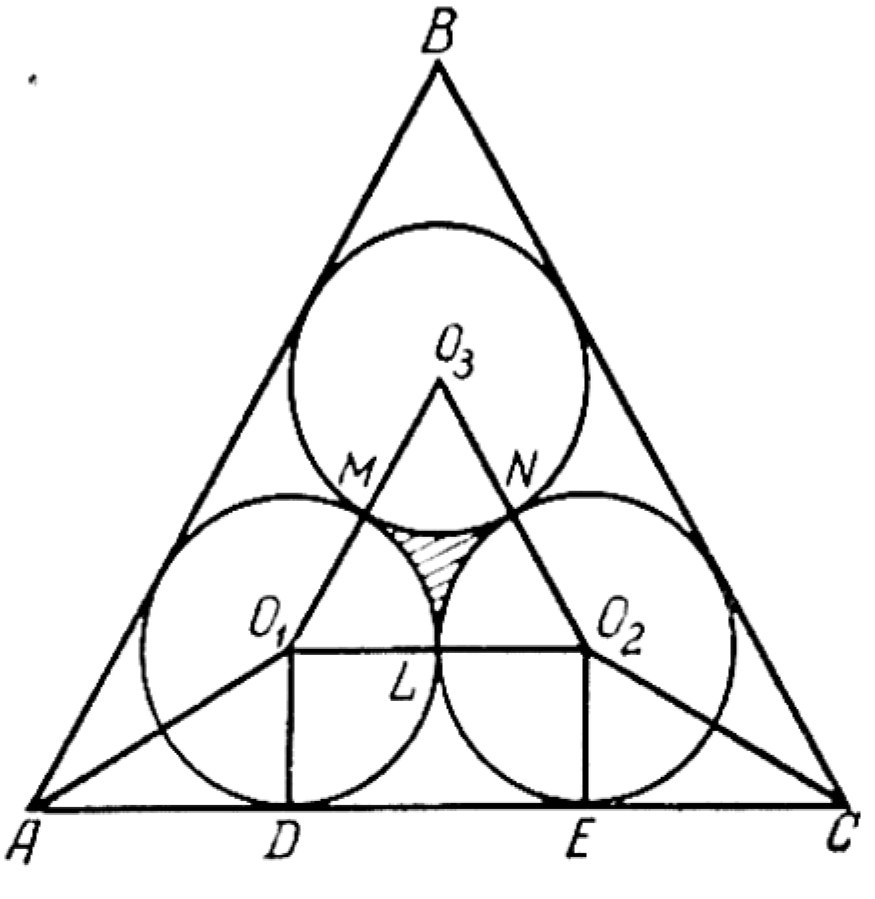


Fig. 53

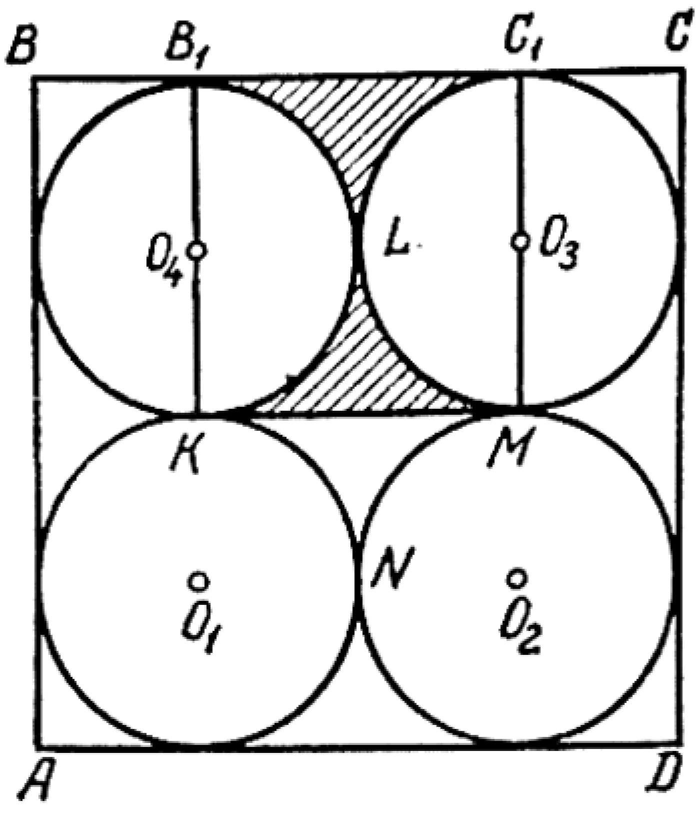


Fig. 54

571. Determine the radius R of the circular arc of the segment, whose perimeter is equal to the sum of the lengths of the arc $\overset{\frown}{ACB}$ and chord AB (Fig. 55). We get $\frac{2}{3}\pi R+R\sqrt{3}=p$, whence

$$R=\frac{3p}{2\pi+3\sqrt{3}}$$

The area S of the segment is equal to the area of the sector less the area of the triangle OAB, so that

$$S=\frac{1}{3}\pi R^2-\frac{R^2\sqrt{3}}{4}$$

Answer: $S=\dfrac{3p^2(4\pi-3\sqrt{3})}{4(2\pi+3\sqrt{3})^2}$.

572. To find the sides AB and BC of the triangle ABC (Fig. 56), it is sufficient to determine $EB=BG=x$, since $AE=AD=6$ cm and $CG=CD=8$ cm. For this purpose let us compare the following two expressions for the area of the triangle:

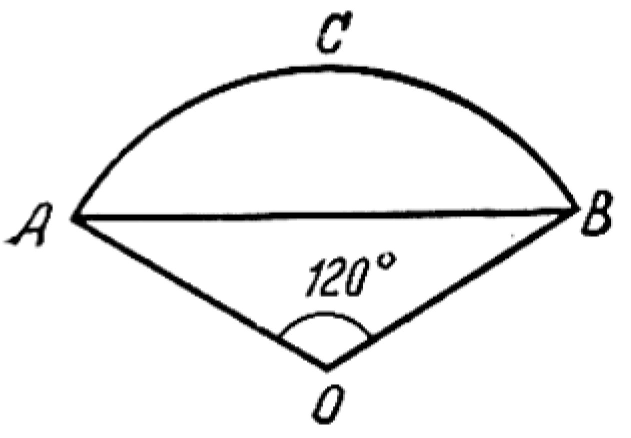


Fig. 55

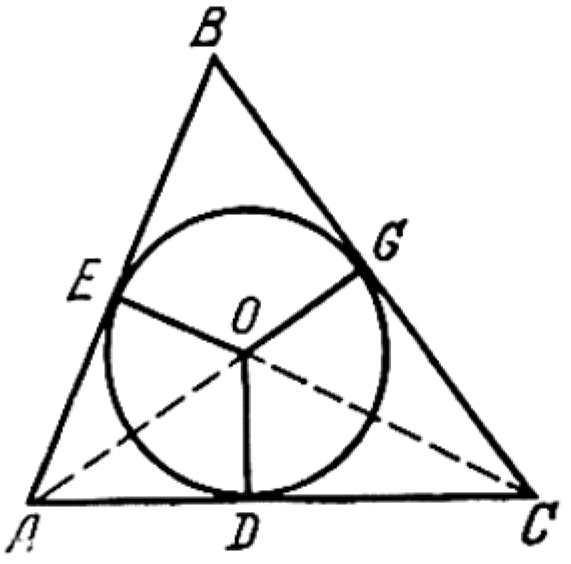


Fig. 56

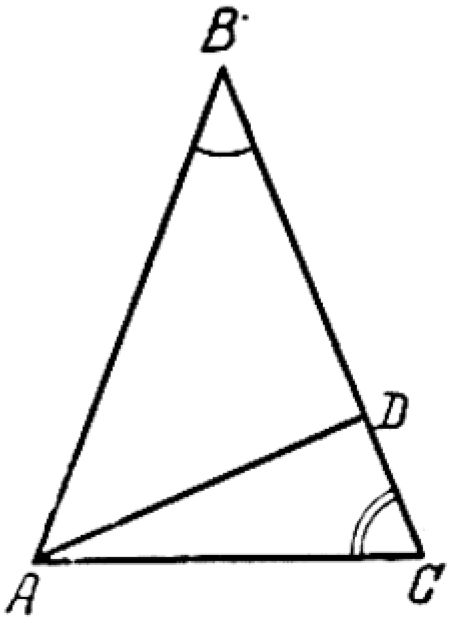


Fig. 57

$$S=rp \quad \text{and} \quad S=\sqrt{p(p-a)(p-b)(p-c)}$$

where p is the semiperimeter of the triangle, i.e.

$$\frac{1}{2}(EA+AD+DC+CG+GB+BE)=\frac{1}{2}(28+2x)=14+x$$

We get the equation

$$4(14+x)=\sqrt{(14+x)\cdot x\cdot 6\cdot 8}$$

Hence, $x=7$ (cm).

Answer: $AB=13$ cm; $BC=15$ cm.

573. Let $CD:DB=m:n$ (Fig. 57). Then $BD:BC=n:(m+n)$. Consequently, $\cos B=\dfrac{BD}{AB}=\dfrac{BD}{BC}=\dfrac{n}{m+n}$. Since $B=180°-2C$, $\cos 2C=\cos(180°-B)=-\dfrac{n}{m+n}$. Hence

$$\cos C=\sqrt{\frac{1+\cos 2C}{2}}=\sqrt{\frac{m}{2(m+n)}}$$

Answer: $B=\arccos\dfrac{n}{m+n}$

$$C=\arccos\sqrt{\frac{m}{2(m+n)}}\left[=\frac{1}{2}\arccos\left(-\frac{n}{m+n}\right)\right]$$

574. The circle is divided into four pairwise equal arcs: $AB = BC$ and $CD = DA$ (Fig. 58). Let the arc $\breve{BC}$ be less than 90° we do not consider the simplest case $m : n = 1$, when all the arcs are equal to 90° each). Find the central angle

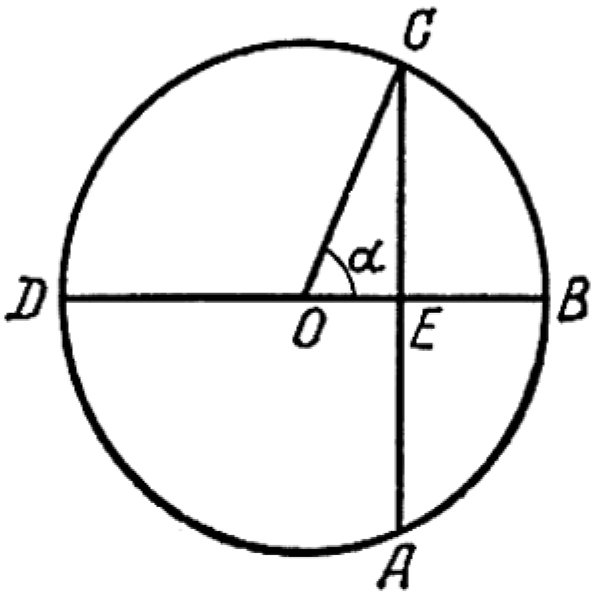


Fig. 58

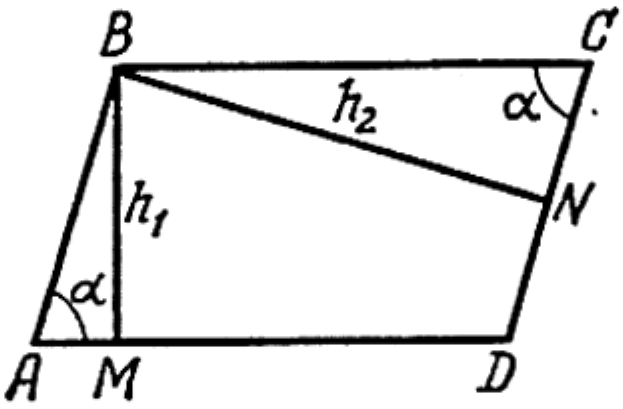


Fig. 59

$\alpha = \angle BOC$ measured by the arc BC. By hypothesis, $DE : EB = m : n$. Taking the quantity $\frac{DE}{m}$ for the unit of length, we have $DE = m$ and $EB = n$. Hence,

$$\frac{DB}{2} = \frac{m+n}{2}$$

and

$$OE = DE - DO = m - \frac{m+n}{2} = \frac{m-n}{2}$$

Hence,

$$\cos \alpha = \frac{OE}{OC} = \frac{m-n}{m+n}$$

and

$$\alpha = \arccos \frac{m-n}{m+n}$$

The arc CD contains $180° - \arccos \frac{m-n}{m+n}$ (degrees), i.e.

$$\pi - \arccos \frac{m-n}{m+n} \text{ (radians)}$$

Answer: the arc smaller than $\frac{\pi}{2}$ is equal to $\arccos \frac{m-n}{m+n}$ $(m > n)$; the arc larger than $\frac{\pi}{2}$ is equal to $\pi - \arccos \frac{m-n}{m+n} = \arccos \frac{n-m}{m+n}$.

575. Let α (Fig. 59) be the angle of the parallelogram. Then

$$h_1 = BM = AB \cdot \sin \alpha$$

and

$$h_2 = BN = BC \cdot \sin \alpha$$

Hence, $h_1+h_2=(AB+BC)\sin\alpha=p\sin\alpha$, whence $\sin\alpha=\frac{h_1+h_2}{p}$. If α is an acute (or right) angle, then $\alpha=\arcsin\frac{h_1+h_2}{p}$. Then the obtuse (or right) angle of the parallelogram will be $\pi-\arcsin\frac{h_1+h_2}{p}$.

Note. The problem has no solution if $h_1+h_2>p$. If $h_1+h_2\leqslant p$, the problem is solvable (at $h_1+h_2=p$ we have a rectangle).

Answer: one of the angles is equal to $\arcsin\frac{h_1+h_2}{p}$, the other, to

$$\pi-\arcsin\frac{h_1+h_2}{p}.$$

576. By hypothesis, $BD:BE=40:41$ (Fig. 60). Let us take $\frac{1}{40}$ part of BD for the unit of length. Then $BD=40$, $BE=41$. Since the triangle ABC is a right one and BE is the median of the right angle, $AE=BE=41$. BDE is a right-angled triangle, therefore

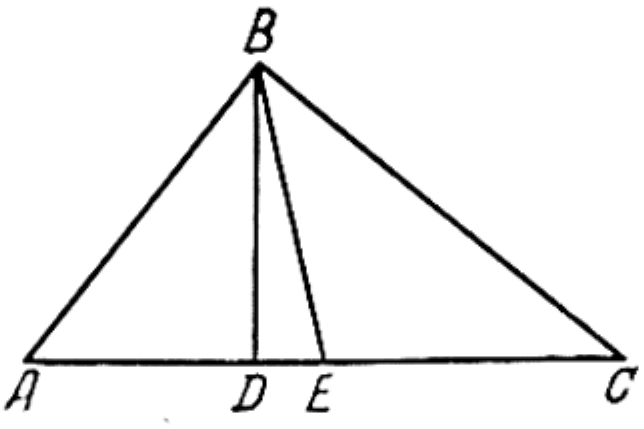


Fig. 60

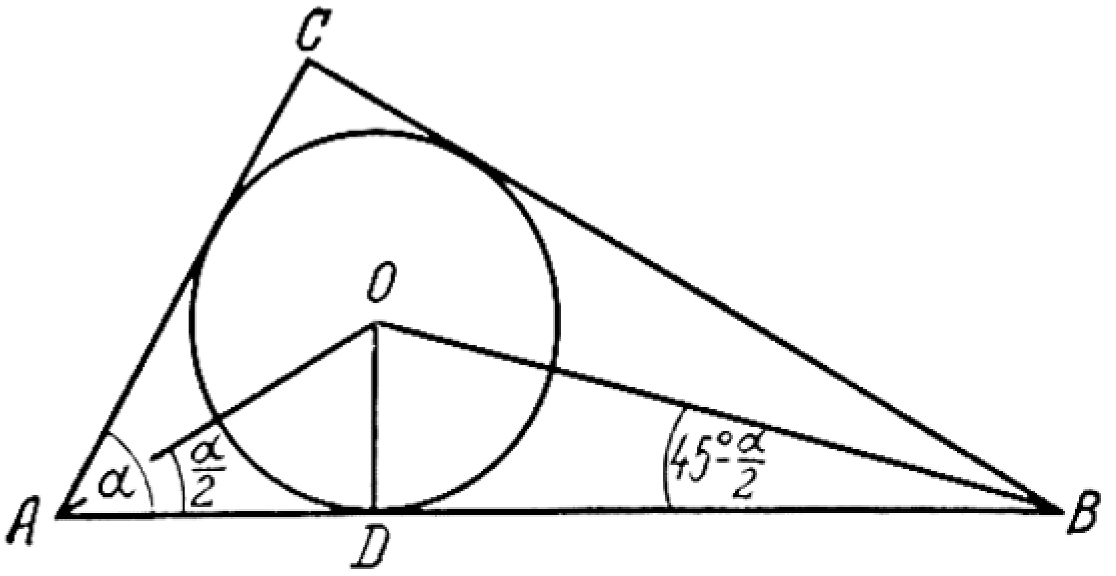


Fig. 61

$$DE=\sqrt{BE^2-BD^2}=9$$

Consequently, $AD=AE-DE=32$. From the similarity of the triangles ABD and ABC we find $\frac{AB}{BC}=\frac{AD}{BD}=\frac{32}{40}=\frac{4}{5}$.

Answer: $\frac{AB}{BC}=\frac{4}{5}$.

577. Since AO (Fig. 61) is the bisector of the angle $\alpha=\angle CAD$, $\angle BAO=\frac{\alpha}{2}$. In the same way we get $\angle ABO=\frac{1}{2}(90^\circ-\alpha)=45^\circ-\frac{\alpha}{2}$. From the triangles AOD and BOD we have

$$AD=OD\cot\frac{\alpha}{2}\quad\text{and}\quad DB=OD\cdot\cot\left(45^\circ-\frac{\alpha}{2}\right)$$

Consequently,

$$c=AB=AD+DB=OD\left[\cot\frac{\alpha}{2}+\cot\left(45^\circ-\frac{\alpha}{2}\right)\right]$$

wherefrom we find $r=\dfrac{c}{\cot\dfrac{\alpha}{2}+\cot\left(45^\circ-\dfrac{\alpha}{2}\right)}$. The denominator may be reduced to a form convenient for taking logarithms:

$$\cot\frac{\alpha}{2}+\cot\left(45^\circ-\frac{\alpha}{2}\right)=\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}+\frac{\cos\left(45^\circ-\frac{\alpha}{2}\right)}{\sin\left(45^\circ-\frac{\alpha}{2}\right)}=$$

$$=\frac{\cos\frac{\alpha}{2}\sin\left(45^\circ-\frac{\alpha}{2}\right)+\sin\frac{\alpha}{2}\cos\left(45^\circ-\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}\sin\left(45^\circ-\frac{\alpha}{2}\right)}=\frac{\sin 45^\circ}{\sin\frac{\alpha}{2}\sin\left(45^\circ-\frac{\alpha}{2}\right)}$$

Answer: $r=c\sqrt{2}\sin\frac{\alpha}{2}\sin\left(45^\circ-\frac{\alpha}{2}\right)$.

Note. By using the formula $r = S : p$ (S is the area of the triangle; p, semiperimeter), we could get a solution in the equivalent form

$$r=\frac{c\sin\alpha\cos\alpha}{1+\cos\alpha+\sin\alpha}$$

578. Let us denote the sides of the triangle by a, b and c, and let $a = 7$ cm, $b = 24$ cm and $c = 25$ cm. Since $a^2+b^2=c^2$, the given triangle is a right-angled one. Consequently, the radius R of the circumscribed circle is equal to $\frac{c}{2}$. The radius of the inscribed circle is found by the formula $r=\frac{S}{p}$, where S is the area of the triangle and p, semiperimeter.

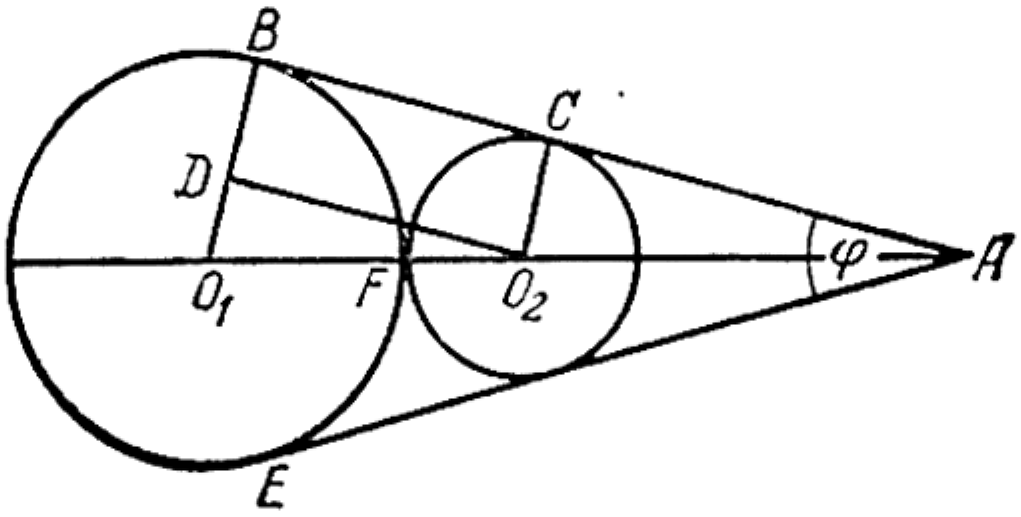


Fig. 62

Answer: $R = 12.5$ cm, $r = 3$ cm.

579. By hypothesis, $\angle BAE=\varphi$ (Fig. 62). Consequently, $\angle BAO_1=\frac{\varphi}{2}$. It is required to determine $R=O_1B$ and $r=O_2C$.

We have

$$R+r=O_1F+FO_2=O_1O_2=d$$

and

$$R-r=O_1B-O_2C=O_1D$$

From the right-angled triangle O_1DO_2, wherein

$$\angle O_1O_2D=\angle BAO_1=\frac{\varphi}{2}$$

we find

$$O_1D=O_1O_2\cdot\sin\frac{\varphi}{2}, \quad \text{i.e. } R-r=d\sin\frac{\varphi}{2}$$

From the two equations obtained we find

$$R=\frac{d\left(1+\sin\frac{\varphi}{2}\right)}{2}$$

and

$$r=\frac{d\left(1-\sin\frac{\varphi}{2}\right)}{2}$$

Substituting $\cos\left(90^\circ-\frac{\varphi}{2}\right)$ for $\sin\frac{\varphi}{2}$ we may transform these expressions

Answer: $R=d\cos^2\left(45^\circ-\frac{\varphi}{4}\right)$

$r=d\sin^2\left(45^\circ-\frac{\varphi}{4}\right)$

580. From Fig. 63 we have

$$\sin\angle BAD=\frac{DE}{AD}=\frac{MN}{AD}=\frac{2r}{a}$$

By hypothesis, $MN\cdot DC=Q$, i.e. $2ra=Q$ and, furthermore, $\pi r^2=S$. These equations enable us to determine r and a separately, but since it is sufficient to know the ratio $\frac{r}{a}$, it is better to divide by terms the second equation by the first one. We get $\frac{\pi r}{2a}=\frac{S}{Q}$, whence $\frac{r}{a}=\frac{2S}{\pi Q}$.

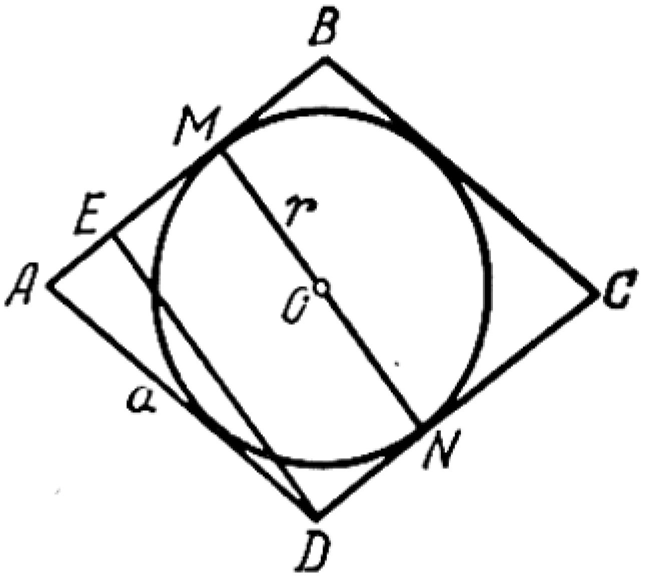


Fig. 63

Answer: $\angle BAD=\arcsin\frac{4S}{\pi Q}$.

581. The area of the inscribed regular $2n$-gon is equal to $nR^2\sin\frac{180^\circ}{n}$. The area of the circumscribed regular n-gon is equal to $nR^2\tan\frac{180^\circ}{n}$. By hypothesis,

$$nR^2\left(\tan\frac{180^\circ}{n}-\sin\frac{180^\circ}{n}\right)=P$$

Hence,

$$R=\frac{\sqrt{P}}{\sqrt{n(\tan\alpha-\sin\alpha)}}$$

where $\alpha=\frac{180°}{n}$. The expression $\tan\alpha-\sin\alpha$ may be transformed in the following way

$$\tan\alpha-\sin\alpha=\tan\alpha(1-\cos\alpha)=2\tan\alpha\sin^2\frac{\alpha}{2}$$

Answer:

$$R=\sqrt{\frac{P}{n\left(\tan\frac{180°}{n}-\sin\frac{180°}{n}\right)}}=\frac{1}{\sin\frac{90°}{n}}\cdot\sqrt{\frac{P\cot\frac{180°}{n}}{2n}}.$$

582. Regular polygons with equal number of sides are similar; therefore (Fig. 64) their areas (S_1 is the area of the inscribed polygon, S_2, the area of the circumscribed one) are in the same ratio as the squares of the radii

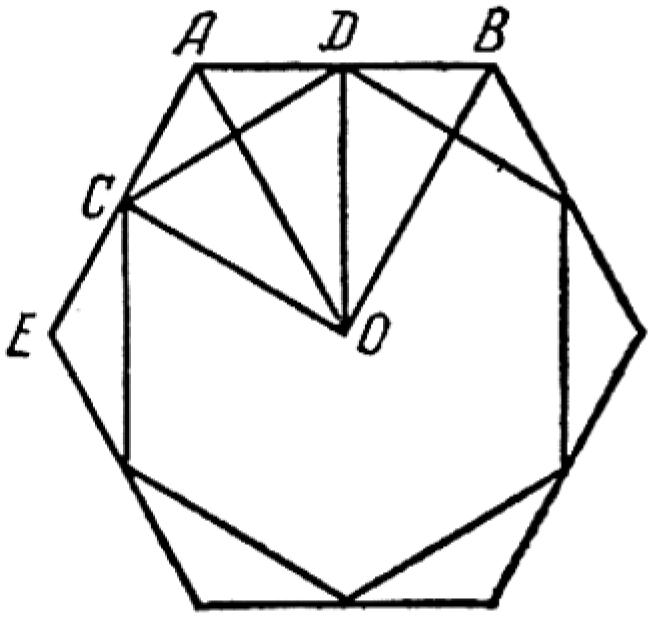


Fig. 64

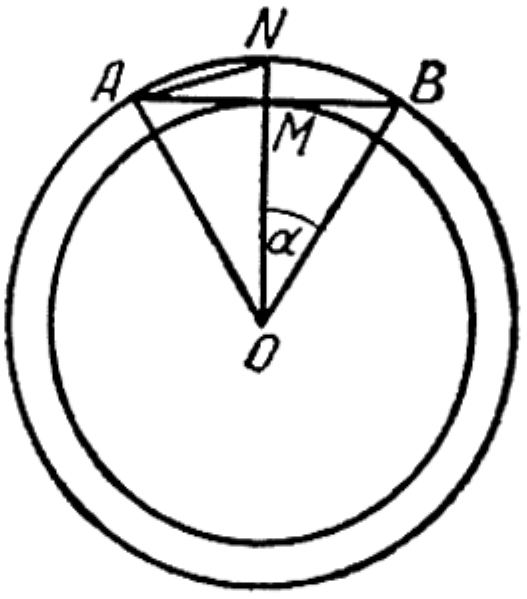


Fig. 65

$$S_1:S_2=OD^2:OA^2$$

But from the triangle OAD we have

$$\frac{OD}{OA}=\cos\angle DOA=\cos\frac{180°}{n}$$

Answer: $S_1:S_2=\cos^2\frac{180°}{n}$.

583. Let $AB=a$ (Fig. 65) be the side of the regular n-gon. Then

$$\angle BON=\alpha=\frac{180°}{n},\quad\text{and}\quad\angle NAM=\frac{\alpha}{2}=\frac{90°}{n}$$

(as the inscribed angle subtended by the arc α). The area of the annulus is

$$Q=\pi(OA^2-OM^2)=\pi\cdot AM^2=\pi\left(\frac{a}{2}\right)^2$$

The width d of the annulus may be found from the triangle NAM.

Answer: $Q=\frac{\pi a^2}{4}$; $d=\frac{a}{2}\tan\frac{90°}{n}$.

584. Denote the required radius by x so that (Fig. 66) $O_2A=O_2B=x$. From the right-angled triangle O_1O_2A, wherein $\angle O_2O_1A=\frac{\alpha}{2}$ and $O_1O_2=O_1B-O_2B=R-x$, we have $O_2A=O_1O_2\sin\frac{\alpha}{2}$, i.e. $x=(R-x)\sin\frac{\alpha}{2}$.

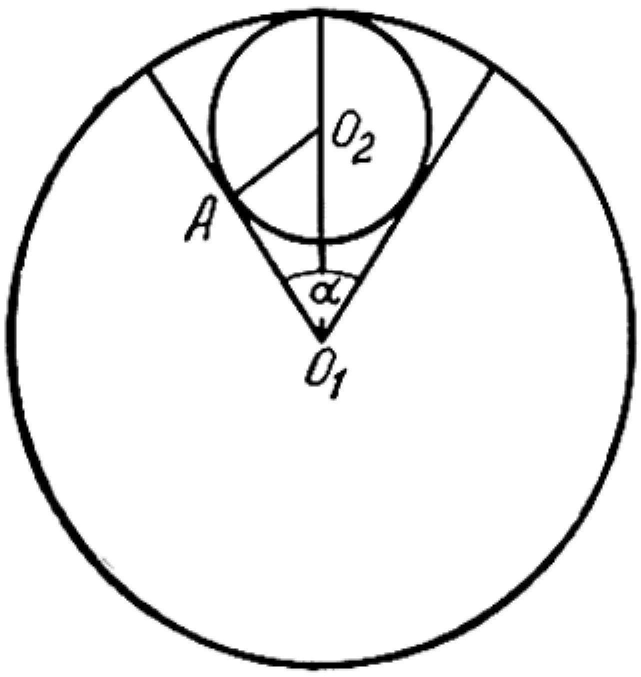


Fig. 66

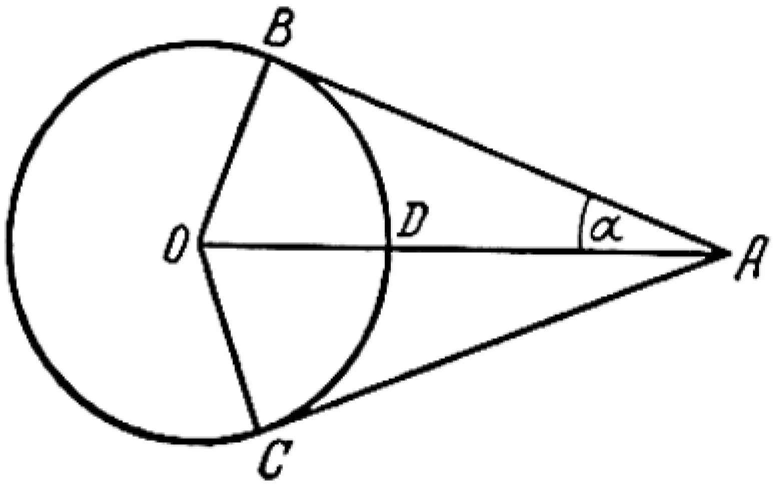


Fig. 67

Answer: $x=\dfrac{R\sin\frac{\alpha}{2}}{1+\sin\frac{\alpha}{2}}=\dfrac{R\sin\frac{\alpha}{2}}{2\cos^2\left(45°-\frac{\alpha}{4}\right)}$.

585. The area S_1 of the quadrilateral $ABOC$ (Fig. 67) is equal to $2\cdot\frac{1}{2}OB\times AB=R^2\cot\alpha$. It is necessary to subtract from it the area S_2 of the sector $COBD$, whose central angle is equal to $(180-2\alpha)°$. We have

$$S_2=\pi R^2\frac{180-2\alpha}{360}=\pi R^2\frac{90-\alpha}{180}$$

(α is measured in degrees).

Answer: $S=S_1-S_2=R^2\left[\cot\alpha-\frac{\pi}{2}+\frac{\pi\alpha}{180}\right]$ where the angle α is measured in degrees, or $S=R^2\left[\cot\alpha'-\frac{\pi}{2}+\alpha'\right]$ where α' is measured in radians.

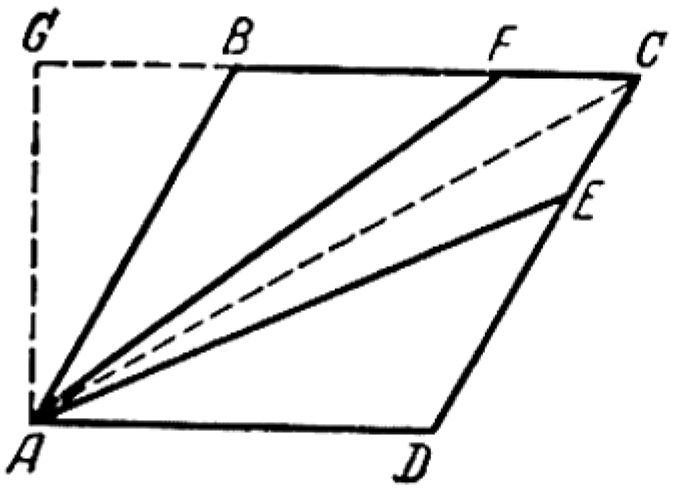


Fig. 68

586. By hypothesis, the area of the triangle ABF (Fig. 68) is equal to one third of the area of the rhombus $ABCD$, i.e. two thirds of the area of the triangle ABC. Since the triangles ABC and ABF have the common altitude AG, we have

$$BF=\frac{2}{3}BC=\frac{2}{3}a$$

Therefore

$$AF^2=AB^2+BF^2-2AB\cdot BF\cos(180^\circ-\alpha)=a^2+\frac{4}{9}a^2+\frac{4}{3}a^2\cos\alpha$$

Answer: $AF=AE=\frac{a}{3}\sqrt{13+12\cos\alpha}$.

587. Extend BM (Fig. 69) to intersect the side OA of the angle AOB at the point R. From the triangle AMR, wherein $\angle AMR=\angle AOB=60^\circ$ (as the angles with mutually perpendicular sides), we find $MR=2AM=2a$. Consequently, $RB=RM+MB=2a+b$. Now, from the triangle ROB, wherein $OR=2OB$, we find $(2OB)^2-OB^2=(2a+b)^2$. Hence,

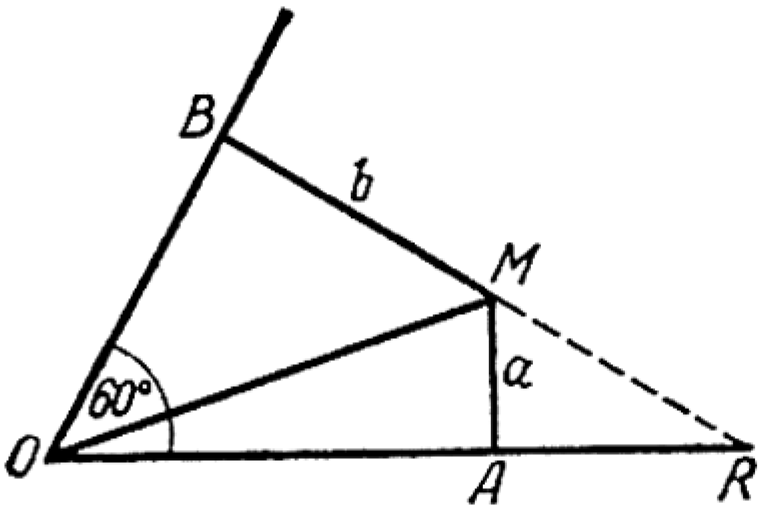


Fig. 69

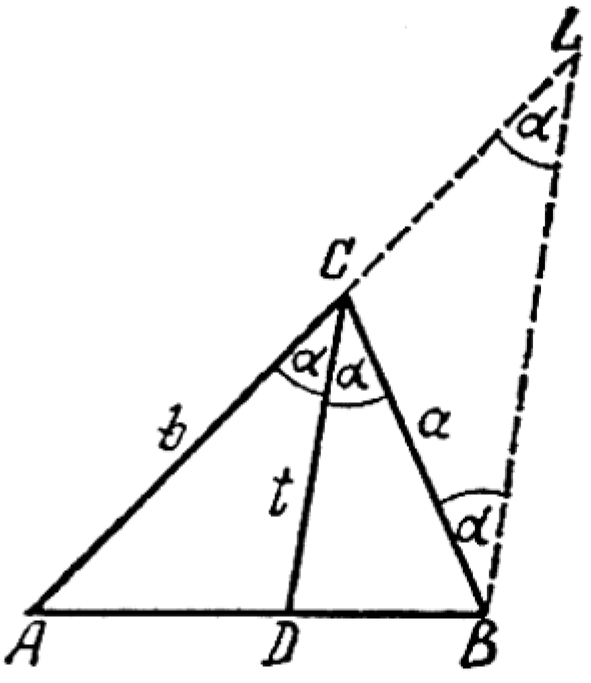


Fig. 70

$$OB=\frac{2a+b}{\sqrt{3}}$$

The sought-for distance OM is determined from the triangle OBM.

Answer: $OM=\frac{2}{\sqrt{3}}\sqrt{a^2+ab+b^2}$.

588. The problem is reduced to finding $\angle ACB=2\alpha$ (Fig. 70). Extending AC and drawing $BL\parallel DC$, let us prove (in the same way as in the theorem on the bisector of an interior angle of a triangle) that $CL=BC=a$. From the similarity of the triangles ADC and ABL we get $BL=\frac{(a+b)t}{b}$, and from the isosceles triangle BCL we have $BL=2a\cos\alpha$. Consequently, $2a\cos\alpha=\frac{(a+b)t}{b}$ wherefrom we find $\cos\alpha$; then we find $\sin\alpha$ and

$$S=\frac{1}{2}at\sin\alpha+\frac{1}{2}bt\sin\alpha=\frac{1}{2}t(a+b)\sin\alpha$$

Alternate solution. The area $\frac{1}{2}ab\sin 2\alpha$ of the triangle ABC is the sum of the areas $\frac{1}{2}bt\sin\alpha$ and $\frac{1}{2}at\sin\alpha$ of the triangles ADC and BCD, respectively. Consequently, $ab \sin \alpha \cos \alpha = \frac{1}{2} bt \sin \alpha + \frac{1}{2} at \sin \alpha$ whence we find $\cos \alpha$.

Answer: $S = \frac{(a+b)t}{4ab} \sqrt{4a^2b^2 - (a+b)^2 t^2}$.

589. Let the rays CD and CE (Fig. 71) divide the angle ACB into three equal parts: $\angle BCD = \angle DCE = \angle ECA = \alpha$. By hypothesis, $AC = CB = a$ and $CE = CD = t$. In the same way as in the preceding problem we find from the triangle BCE: $\cos \alpha = \frac{(a+t)t}{2at} = \frac{t+a}{2a}$; then we find $\sin \alpha$. The required area is the sum of the areas of the triangles ACE; DCE; BCD.

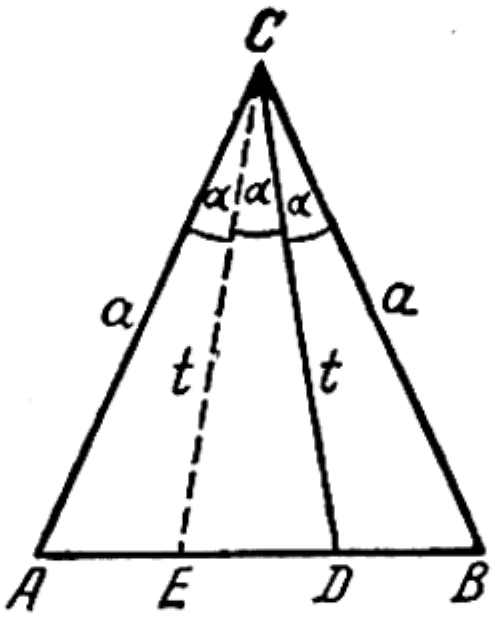


Fig. 71

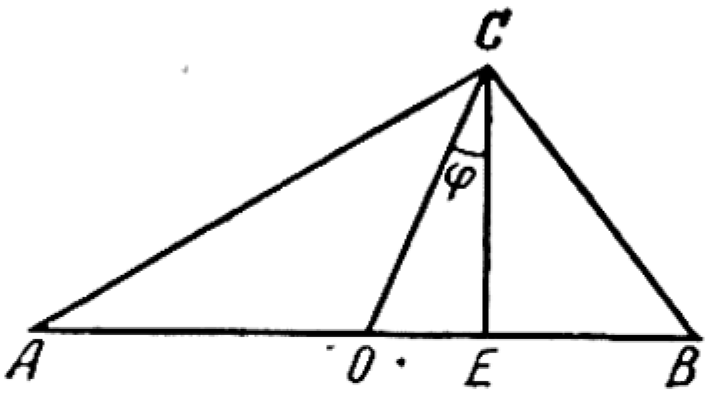


Fig. 72

Answer: $S = \frac{t}{4a} (2a+t) \sqrt{(3a+t)(a-t)}$.

590. In the triangle ABC (Fig. 72) CE is an altitude and CO is a median. Let us denote the required angle OCE by φ, and the angles of the triangle by A, B and C. From the triangles ACE, BCE and OCE we find the following expressions for the segments of the base:

$$AE = EC \cdot \cot A$$

$$BE = EC \cdot \cot B$$

and

$$OE = EC \cdot \tan \varphi$$

Since $AO = OB$, we have

$$AE - BE = (AO + OE) - (OB - OE) = 2OE$$

Substituting the expressions found for these segments, we get

$$EC \cdot \cot A - EC \cdot \cot B = 2EC \cdot \tan \varphi$$

or

$$\cot A - \cot B = 2 \tan \varphi$$

Answer: $\tan \varphi = \frac{1}{2} (\cot A - \cot B)$.

591. The required area S (hatched in Fig. 73) is equal to the three-fold area of the figure $EMFB$. By hypothesis, $OE=\frac{1}{3}AB=\frac{a}{3}$. In the right-angled triangle OED the leg OD (the radius of the inscribed circle) is equal to $\frac{a\sqrt{3}}{6}$; consequently, $OD=OE\frac{\sqrt{3}}{2}$. Hence, $\angle DEO=60°$. Similarly, $\angle KFO=60°$. Since the angle EBF is also equal to 60°, $OE \parallel BF$ and $OF \parallel BE$, and the quadrilateral $OEBF$ is a rhombus with the side $\frac{a}{3}$ and the angle 60° at the vertex O. Subtract the area of the sector EOF, equal to $\frac{1}{6}\pi\left(\frac{a}{3}\right)^2$, from the area of the rhombus $\left(\frac{a}{3}\right)^2\frac{\sqrt{3}}{2}$ and multiply the difference thus obtained by 3.

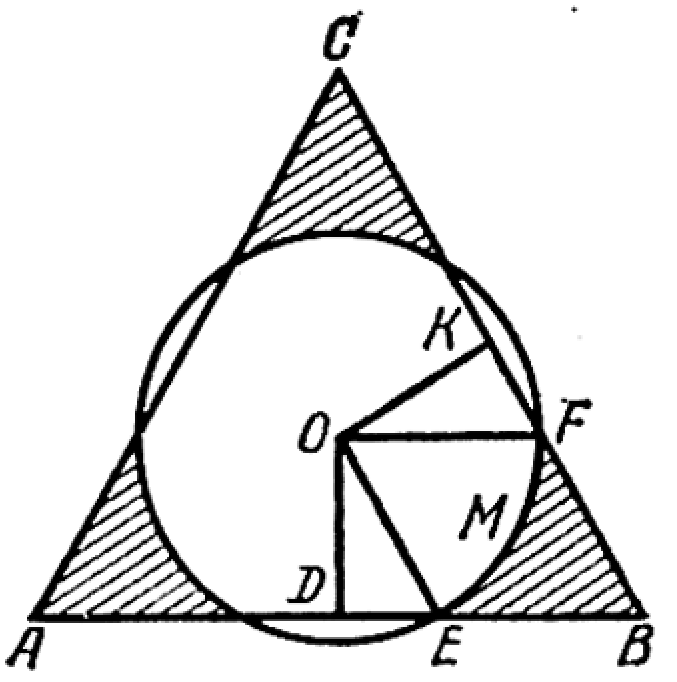


Fig. 73

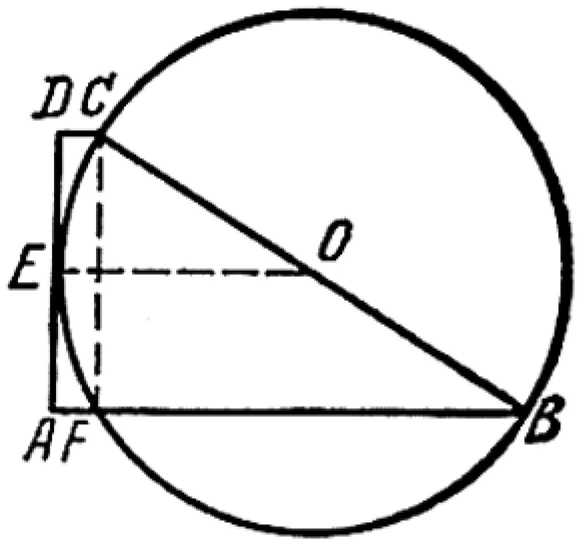


Fig. 74

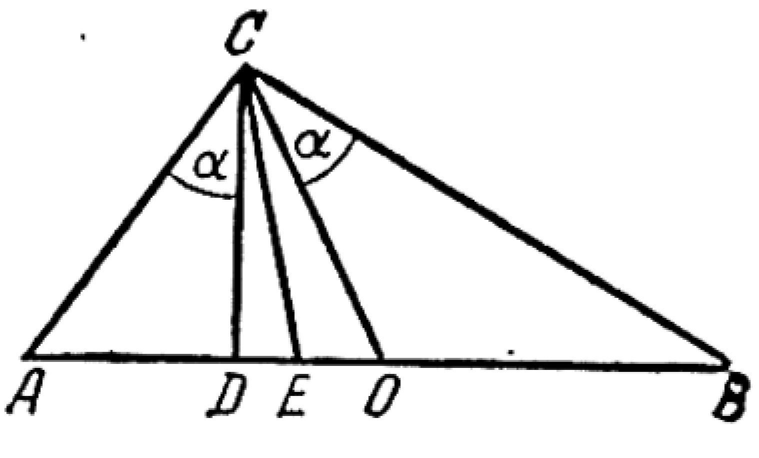


Fig. 75

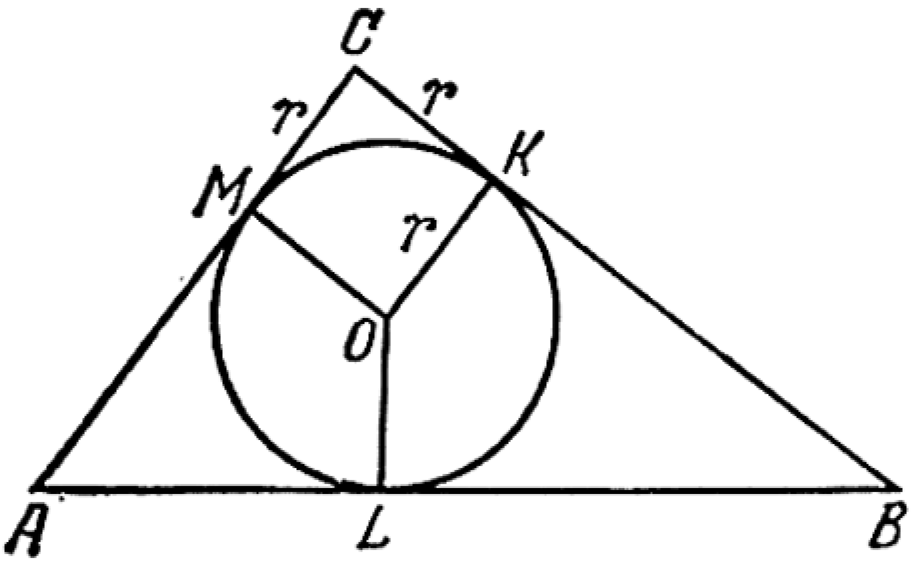


Fig. 76

Answer: $S=\frac{a^2}{18}(3\sqrt{3}-\pi)$.

592. It is required to find $S=\frac{1}{2}AB\cdot DC$ (Fig. 74). The angle CFB is a right one (as an inscribed angle subtended by the diameter). Consequently,

$DC = AF$, and thus, $S = \frac{1}{2} AB \cdot AF$. But, by the property of the secant, we have

$$AB \cdot AF = AE^2 = \left(\frac{h}{2}\right)^2$$

Answer: $S = \frac{h^2}{8}$.

593. Since $\angle DCA = \angle OBC$ (Fig. 75) and $\angle BCO = \angle OBC$ (for the median OC is equal to half the hypotenuse), we have $\angle DCA = \angle BCO$. But, by hypothesis, $\angle ACE = \angle BCE$. Subtracting the former equality from the latter one, we get $\angle DCE = \angle OCE$, i.e. CE bisects the angle DCO.

594. The diameter $2R$ of the circle circumscribed about the right-angled triangle ABC (Fig. 76) is equal to the hypotenuse AB. The diameter $2r$ of the inscribed circle is equal to $MC + CK$ (since $MOKC$ is a square). Hence,

$$AC + BC = (AM + BK) + (MC + CK) = (AL + LB) + (MC + CK) = 2R + 2r.$$

595. In the same way as in the preceding problem, prove that $a + b = 2(r + R)$, i.e. $a + b = 2\left(\frac{2}{5}R + R\right) = \frac{7}{5}c$. Furthermore, $a^2 + b^2 = c^2$.

Hence,

$$a = \frac{3}{5}c,$$

$$b = \frac{4}{5}c \quad \left(\text{or } a = \frac{4}{5}c,\ b = \frac{3}{5}c\right)$$

Answer: $\sin A = \frac{3}{5}$, $\sin B = \frac{4}{5}$.

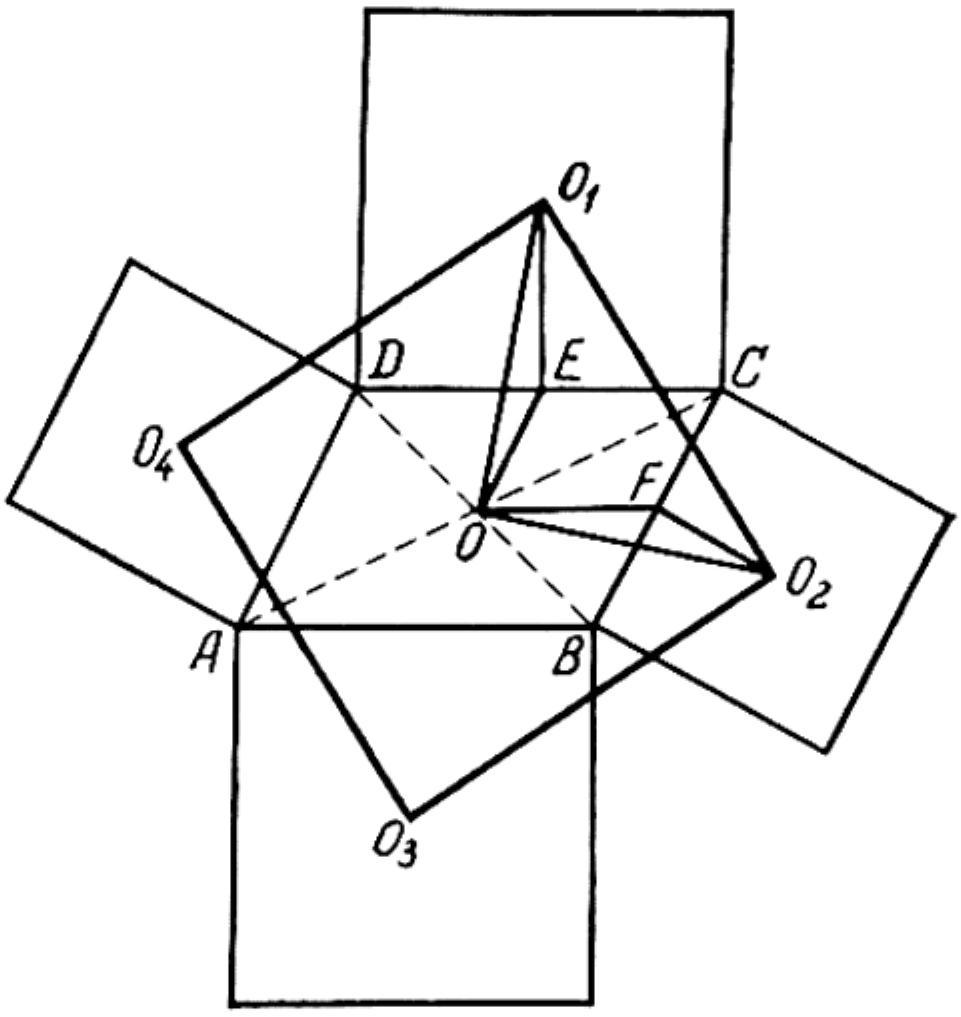


Fig. 77

596. Let us construct (Fig. 77) the triangles OEO_1 and OFO_2 (E and F are the midpoints of the sides of the parallelogram). They are congruent. Indeed, $OE = FC$, and, by hypothesis, $FC = O_2F$. Hence, $OE = O_2F$. Similarly, we prove that $O_1E = OF$. The angles OEO_1 and OFO_2 (both of them are obtuse) are congruent, since their sides are mutually perpendicular. From the congruence of the triangles OEO_1 and OFO_2 it follows that $OO_1 = OO_2$ and $\angle OO_1E = \angle O_2OF$. And since O_1E and OF form a right angle, the straight lines OO_1 and OO_2 also form a right angle. Hence, the triangle O_1O_2O is an isosceles right-angled one. The same refers to the triangles O_2O_3O, O_3O_4O and O_4O_1O, which means that the quadrilateral $O_1O_2O_3O_4$ is a square.

CHAPTER IX

POLYHEDRONS

Notation (for this and next chapters):

$V =$ volume
S or $S_{base} =$ area of the base
$S_{lat} =$ area of the lateral surface
$S_{total} =$ total area
$a =$ side of the base
$r =$ radius of the inscribed circle
$R =$ radius of the circumscribed circle
$H =$ altitude of a solid
$h =$ altitude of the base

If the above quantities are denoted otherwise, this fact is mentioned each time. In the accompanying figures invisible lines are presented by broken lines with short dashes, and auxiliary lines, by broken lines with longer dashes.

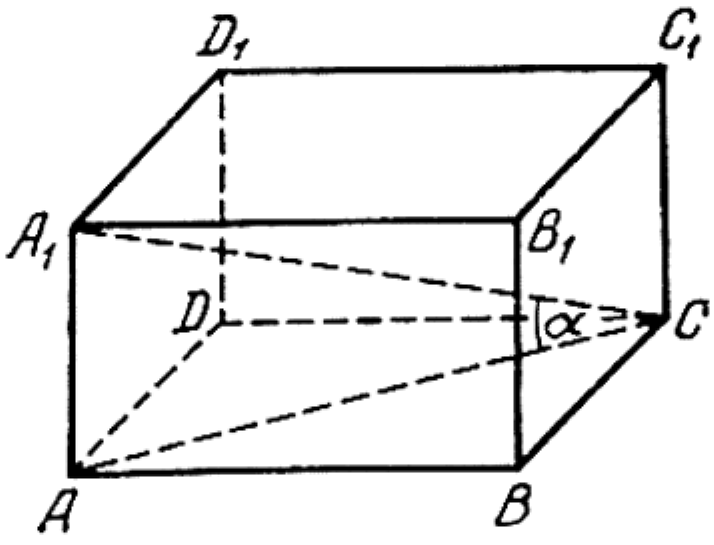


Fig. 78

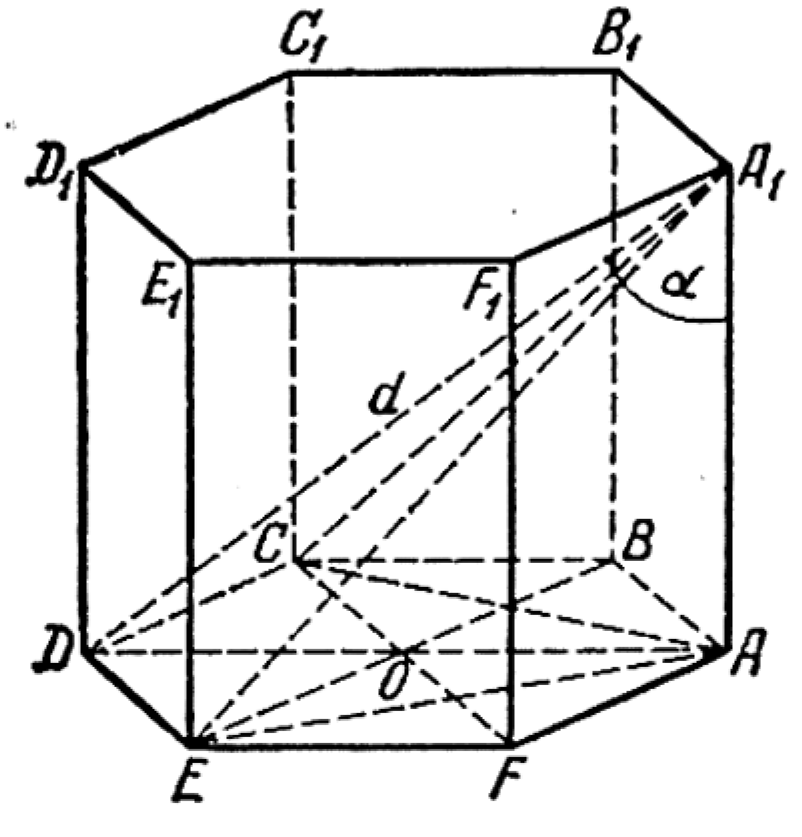


Fig. 79

597. The projection of the diagonal A_1C of the parallelepiped (Fig. 78) on the plane of the base $ABCD$ is AC (the diagonal of the base). Therefore the angle α between A_1C and the plane $ABCD$ is measured by the angle A_1CA. From the triangle AA_1C we find

$$AA_1 = AC \cdot \tan\alpha = \sqrt{a^2+b^2}\tan\alpha$$

Substitute it into the formula $S_{lat} = (2a+2b)\cdot AA_1$.

Answer: $S_{lat} = 2(a+b)\sqrt{a^2+b^2}\tan\alpha$.

598. From each vertex of the prism, say from A_1 (Fig. 79), we can draw three diagonals (A_1E, A_1D, A_1C). They are projected on the plane $ABCDEF$ as the diagonals of the base (AE, AD, AC). Out of the inclined lines A_1E, A_1D, A_1C the greatest is the one having the longest projection. Consequently, the greatest of the three diagonals taken is A_1D (the prism has other diagonals equal to A_1D, but there is none longer than A_1D).

From the triangle A_1AD wherein $\angle DA_1A=\alpha$ and $A_1D=d$, we find $H=AA_1=d\cos\alpha$, $AD=d\sin\alpha$. The area of the equilateral triangle AOB is equal to $\frac{1}{4}\cdot AO^2\cdot\sqrt{3}$. Hence $S_{base}=6\cdot\frac{1}{4}\cdot OA^2\cdot\sqrt{3}=6\cdot\frac{1}{4}\left(\frac{AD}{2}\right)^2\sqrt{3}$. The volume $V=S\cdot H=\frac{3\sqrt{3}}{8}\cdot AD^2\cdot AA_1$.

Answer: $\frac{3\sqrt{3}}{2}d^3\sin^2\alpha\cos\alpha$

Note. For graphical representation of a regular hexagon (the base of the prism) we may construct an arbitrary parallelogram $BCDO$. Extending the lines DO, CO, BO and marking off the segments $OA=OD$, $OF=OC$ and $OE=OB$, we obtain the hexagon $ABCDEF$. The point O represents the centre.

599. (a) *Drawing.* The square serving as the base is represented by an arbitrary parallelogram $ABCD$ (Fig. 80). The point O of intersection of the diagonals represents the centre of the square. Joining the midpoint F of the side AB with the vertex of the pyramid E, we get the slant height EF.

(b) *Solution.* We have

$$V=\frac{1}{3}x^2H$$

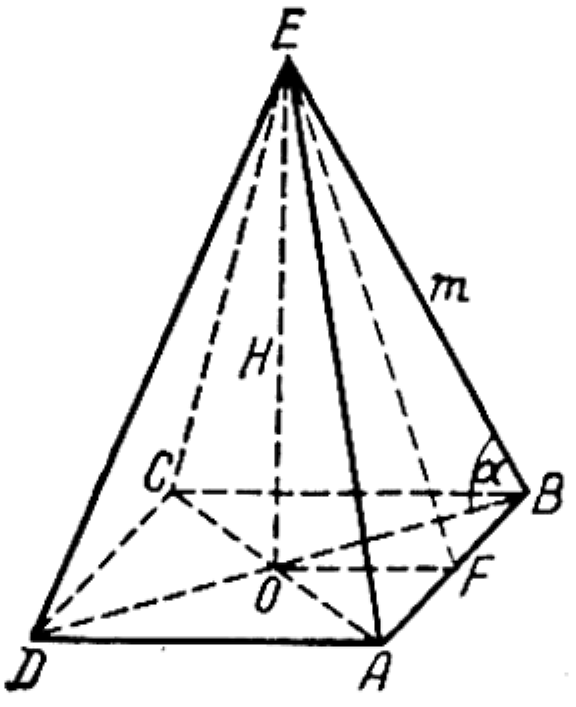


Fig. 80

where x is the side of the base (AB in Fig. 80) and H, the altitude of the pyramid (OE). The angle α is $\angle EBO$ (see solution of Problem 597). From the triangle EBO we find $H=m\sin\alpha$; from the triangle OAB,

$$x=OB\cdot\sqrt{2}=m\sqrt{2}\cdot\cos\alpha$$

Answer: $V=\frac{2}{3}m^3\cos^2\alpha\sin\alpha=\frac{m^3\sin 2\alpha\cdot\cos\alpha}{3}$.

600. Denoting the required lateral edge by m, in the same way as in the preceding problem, we find

$$V=\frac{m^3\sin 2\alpha\cos\alpha}{3}$$

whence we determine m.

Answer: $m=\sqrt[3]{\frac{3V}{\sin 2\alpha\cos\alpha}}$.

601. Let us introduce the following notation: $AB=x$; $EF=y$ (Fig. 80). Then we have $S=2xy$. From the right-angled triangle OEF, wherein $OE=H$, we find $y^2=\left(\frac{x}{2}\right)^2+H^2$. Eliminating y from the found equations, we get

$$x^4+4H^2x^2-S^2=0$$

This equation has two real solutions, but only one of them is positive.

Answer: $x=\sqrt{\sqrt{4H^4+S^2}-2H^2}$ cm.

602.* Joining the midpoints M and N (Fig. 81) of the sides BC and FE, we obtain the graphical representation MN of the diameter of the inscribed circle so that $MN = d$ and $OM = \frac{d}{2}$. Since OM is the altitude of an equilateral triangle with the side a $(= BC = OC = OB)$, $\frac{d}{2} = \frac{a\sqrt{3}}{2}$; whence $a = \frac{d}{\sqrt{3}}$.

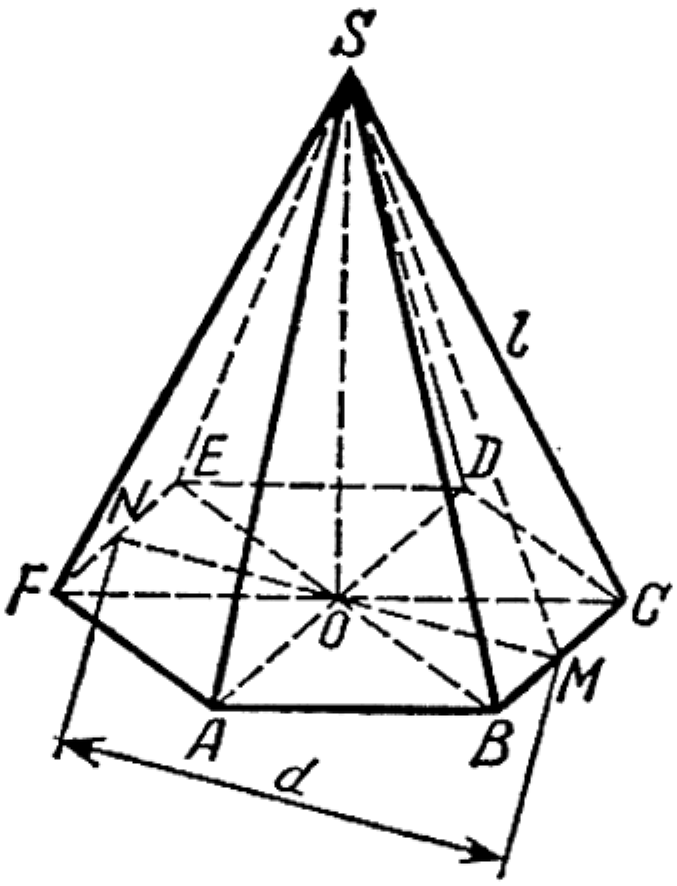


Fig. 81

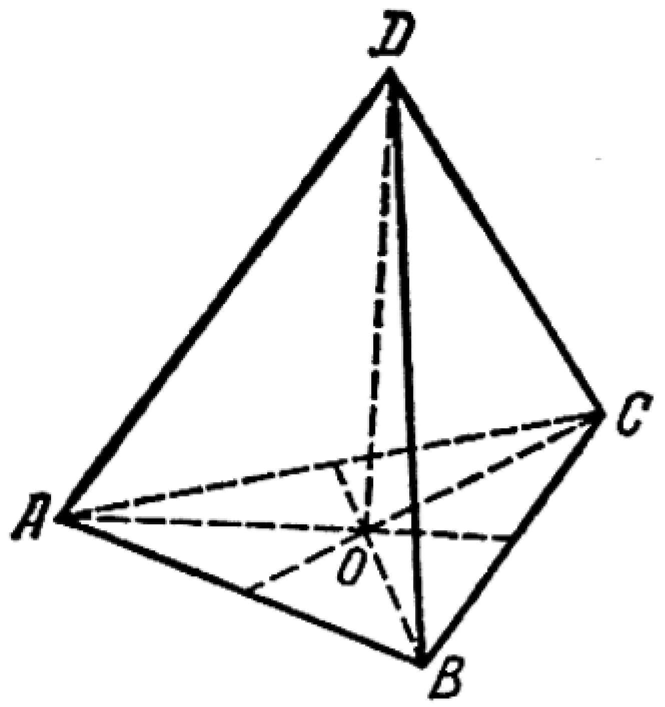


Fig. 82

The altitude $H = OS$ is found from the triangle SCO:

$$H = \sqrt{CS^2 - OC^2} = \sqrt{l^2 - a^2} = \sqrt{l^2 - \frac{d^2}{3}}$$

The slant height $m = SM$ of the pyramid is found from the triangle SCM:

$$m = \sqrt{l^2 - \left(\frac{a}{2}\right)^2} = \frac{1}{2\sqrt{3}} \cdot \sqrt{12l^2 - d^2}$$

Answer: $V = \frac{d^2}{6}\sqrt{3l^2 - d^2}$, $S_{lat} = \frac{d}{2}\sqrt{12l^2 - d^2}$.

603. (a) *Drawing*. The base may be represented by any triangle ABC (Fig. 82). The centre of the base is represented by the point O of intersection of the medians**.

(b) *Solution*. We have $V = \frac{1}{3} \cdot S_{base} \cdot H = \frac{1}{3} \cdot \frac{1}{4} a^2 \sqrt{3}\, H$. The relationship between a and H is found from the triangle AOD, wherein $AD = a$, and AO is the radius R of the circle circumscribed about the base; thus $a = R\sqrt{3}$.

---

* For graphical representation of a regular hexagon see Note to Problem 598 on page 247.

** Then two of these medians, which are of no importance for solving the problem, may be erased, leaving only the point O on the median AE as is done in Fig. 85 on page 250.

We have $H^2=AD^2-AO^2=a^2-\frac{a^2}{3}=\frac{2}{3}a^2$. Substituting $a^2=\frac{3}{2}H^2$ into the expression of V, we get $V=\frac{\sqrt{3}}{8}H^3$.

Answer: $H=2\sqrt[3]{\frac{V}{\sqrt{3}}}$.

604. (a) *Drawing*. As distinct from a *rectangular* parallelepiped, all the faces of which are rectangles, the base of a *right* parallelepiped is a parallelogram, only the four lateral faces being the rectangles. But in drawing a rectangular parallelepiped (see Fig. 78 on page 246) we are forced to represent the base also in the form of a parallelogram. Therefore the drawing of a right parallelepiped does not essentially differ from that of a rectangular parallelepiped, which creates additional difficulties for reading such drawings: it is necessary to remember that the acute angle of the parallelogram shown in the drawing corresponds to the actual acute angle of the figure represented. For the sake of clarity it is recommended to make this angle *too* acute, as in Fig. 83, and mark it *obligatory* with a letter (in the given case—with the Greek letter α).

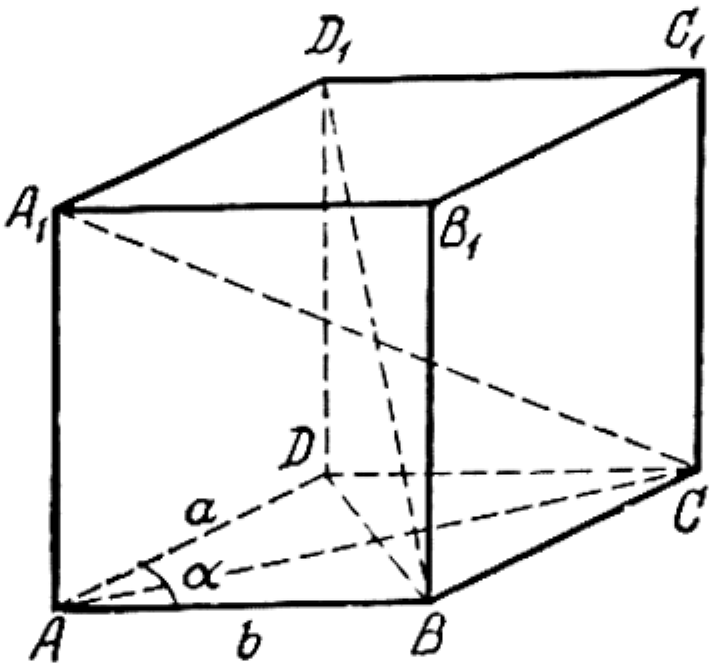


Fig. 83

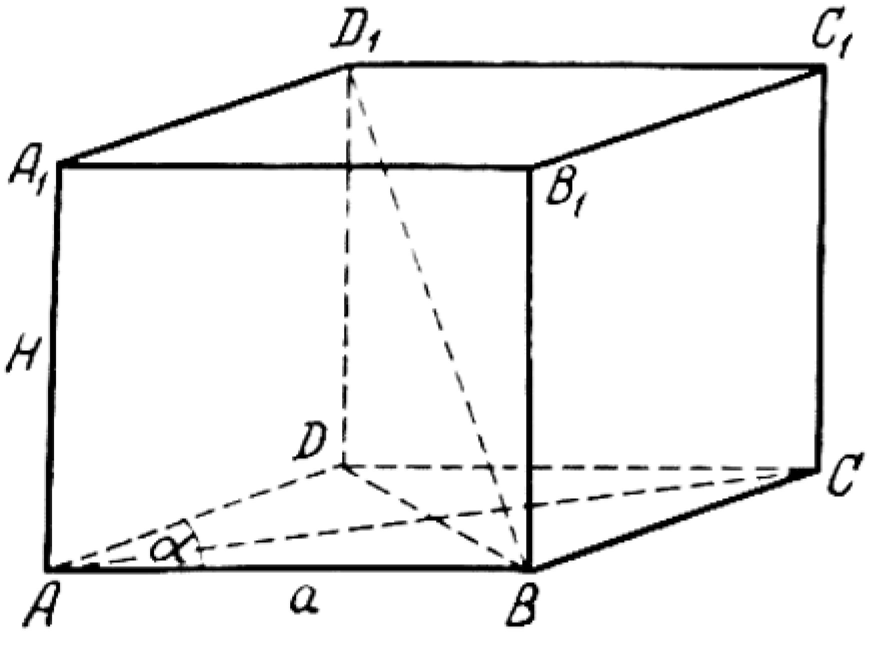


Fig. 84

(b) *Solution*. In a right parallelepiped the diagonals (four in number) are equal pairwise: $A_1C = AC_1$ and $BD_1 = B_1D$ (in Fig. 83 AC_1 and DB_1 are not shown). Let $\angle DAB = \alpha$ be an acute angle of the base $ABCD$; then $\angle ABC = 180° - \alpha$ is an obtuse one and $AC > BD$. Hence, BD_1 is the smaller diagonal of the parallelepiped (since $BD_1^2 = H^2 + BD^2$, whereas $A_1C^2 = H^2 + AC^2$; hence, $BD_1^2 < A_1C^2$). From the condition $BD_1 = AC$ we may find H. Namely, from the triangle BDD_1 we have

$$H^2 = BD_1^2 - BD^2 = AC^2 - BD^2$$

From the triangle ABD we find

$$BD^2 = a^2 + b^2 - 2ab\cos\alpha$$

and from the triangle ABC we find

$$AC^2 = a^2 + b^2 - 2ab\cos(180° - \alpha)$$

Consequently, $H^2 = 4ab\cos\alpha$.

Answer: $V = 2\sin\alpha\sqrt{(ab)^3\cos\alpha}$.

605. Let us denote the larger side of the base (AB in Fig. 84) by a and the smaller one (BC), by b. By hypothesis, $a + b = 9$ cm. To find a, b, and the acute angle α, let us compute the diagonals of the base. As has been proved in the solution of the preceding problem, the smaller diagonal $[BD_1 = \sqrt{33}$ (cm)] of the parallelepiped is projected on the plane of the base as the diagonal BD. Therefore

$$BD^2 = BD_1^2 - DD_1^2 = (\sqrt{33})^2 - 4^2 = 17 \text{ (cm}^2\text{)}.$$

In the same way we find $AC^2 = 65$ (cm²). And so we get the following two equations:

$$a^2 + b^2 - 2ab\cos\alpha = 17; \quad a^2 + b^2 + 2ab\cos\alpha = 65$$

Adding them, we find $a^2 + b^2 = 41$, which, together with $a + b = 9$, yields $a = 5$, $b = 4$ (we have denoted the larger side by a). Subtracting, we find $4ab\cos\alpha = 48$, i.e. $\cos\alpha = \frac{48}{4\cdot5\cdot4} = 0.6$. Consequently,

$$S_{base} = ab\sin\alpha = 4\cdot5\cdot0.8 = 16 \text{ cm}^2$$

Answer: $V = 64$ cm³, $S_{total} = 104$ cm².

606. (a) *Drawing*. For constructing the point O see Problem **603** (Fig. 82). To construct the plane angle of the dihedral angle at the edge BC (Fig. 85), join the midpoint E of the segment BC with the points D and A; since CDB and CAB are actually the isosceles triangles, DE and AE are perpendicular to BC, i.e. $\angle DEA = \varphi$ is the required plane angle. The altitude of the pyramid $DO = h$ lies in the plane DEA.

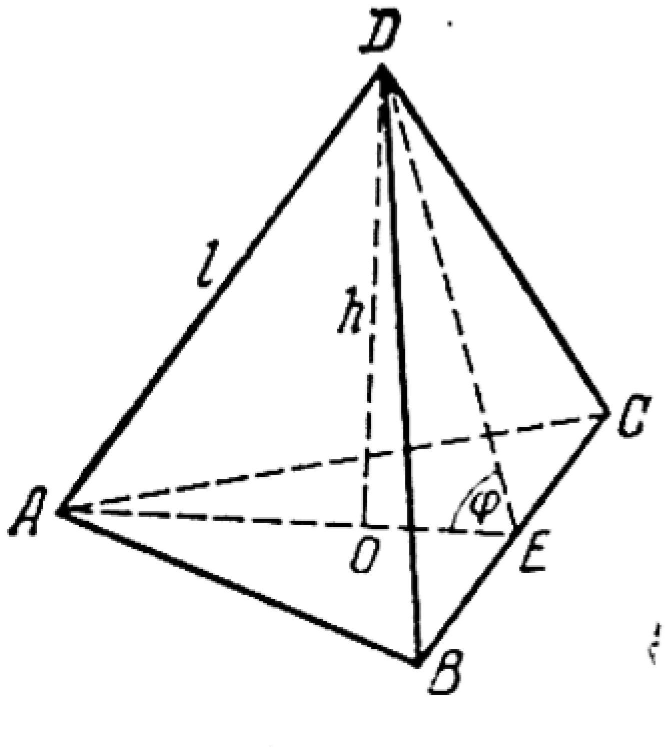


Fig. 85

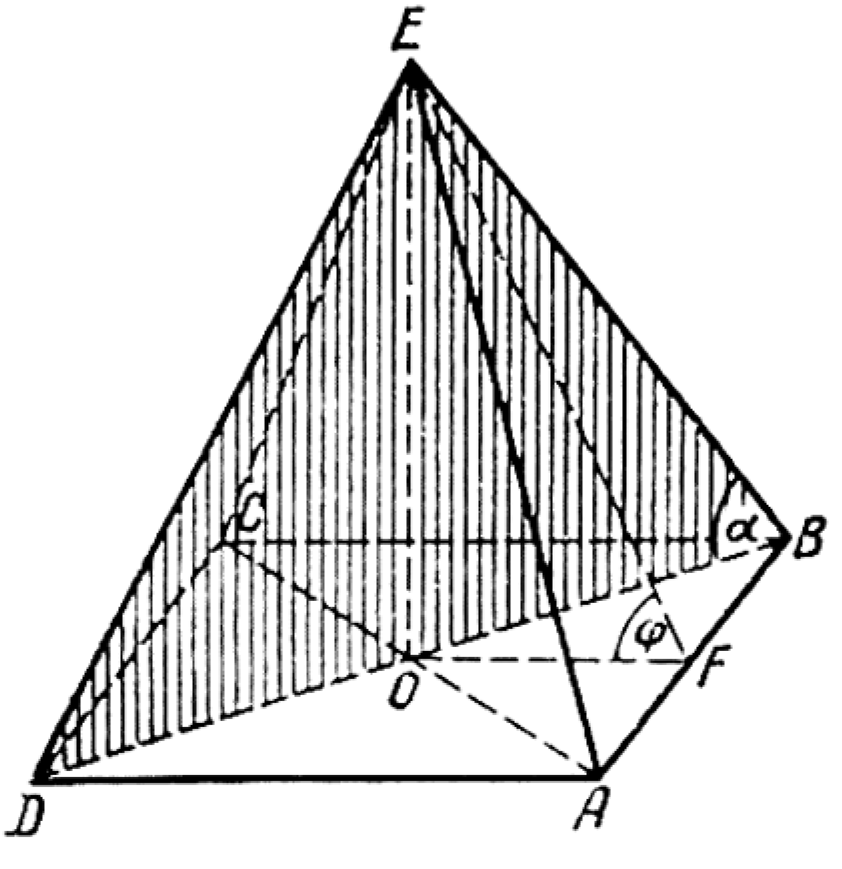
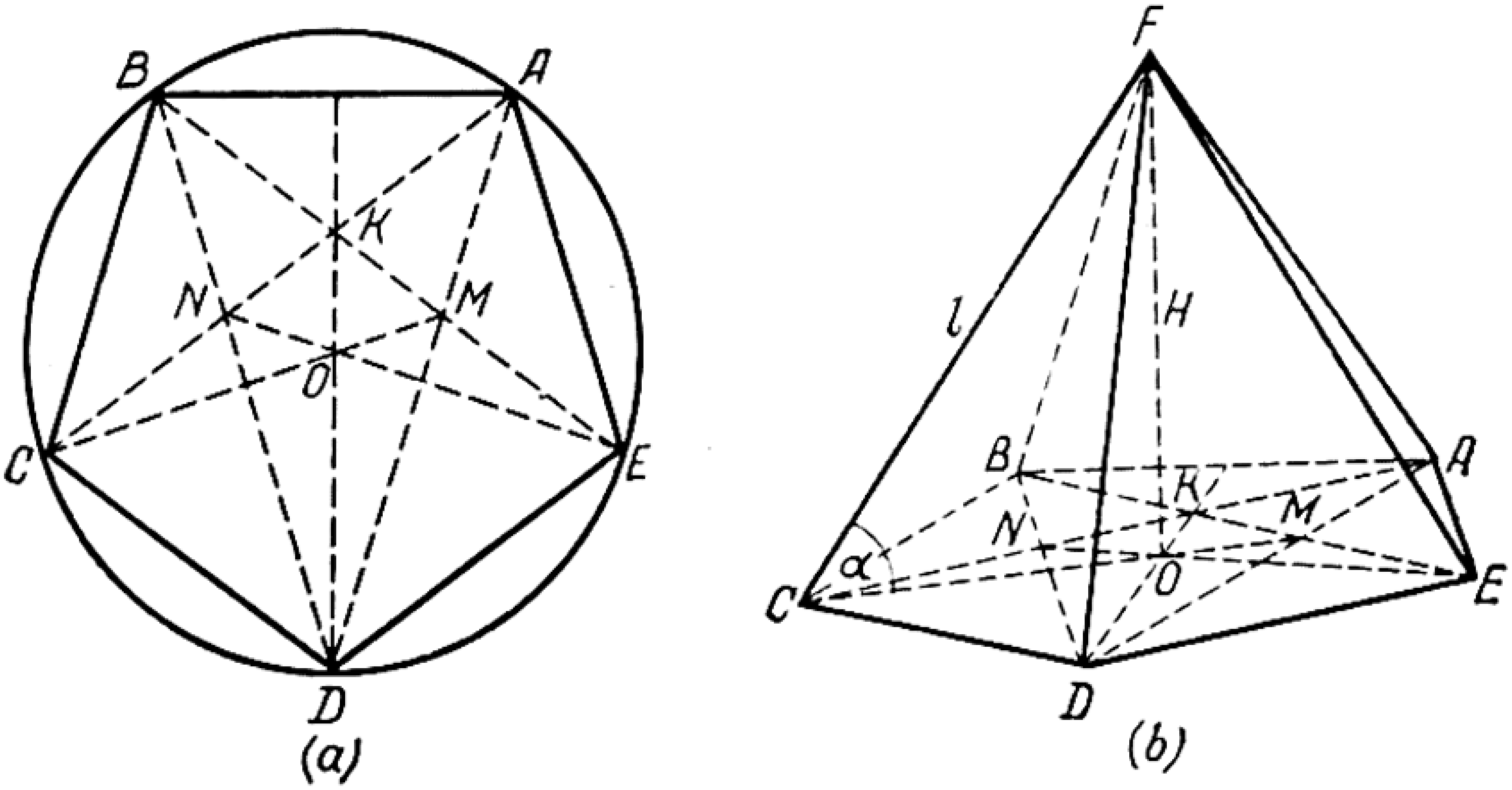


Fig. 86

(b) *Solution*. We have $\tan\varphi = \frac{OD}{OE}$, where $OD = h$, and $OE = \frac{1}{2}\cdot AO$ (the medians are divided in the ratio 1 : 2). AO is found from the triangle AOD, wherein $AD = l$.

Answer: $\varphi = \arctan\frac{2h}{\sqrt{l^2 - h^2}}$.

607. The angle α is measured by the angle OBE (Fig. 86), because OB is the projection of the edge BE on the plane of the base. To construct the plane angle φ of the dihedral angle at the edge AB, join the midpoint F of the side AB with O and E (see the explanation to Problem 606). Since $S_{base} = a^2 = \frac{d^2}{2}$, to compute V we have to find $H = OE$ and $d = BD$. From the triangle OBE we find $H = \frac{d}{2} \tan\alpha$, and by hypothesis, $\frac{d}{2} H = S$. Multiplying



(a) (b)

Fig. 87

these equations and then dividing them termwise, we find

$$H^2 = S \tan\alpha \quad \text{and} \quad \left(\frac{d}{2}\right)^2 = S \cot\alpha$$

Hence,

$$V = \frac{1}{3} \cdot S_{base} \cdot H = \frac{2}{3} \cdot S^{\frac{3}{2}} \cot^{\frac{1}{2}}\alpha$$

The angle φ is determined from the triangle OFE, wherein

$$OF = \frac{a}{2} = \frac{d}{2\sqrt{2}}$$

$$\tan\varphi = \frac{OE}{OF} = H : \frac{1}{\sqrt{2}} \frac{d}{2} = \sqrt{S \tan\alpha} : \frac{1}{\sqrt{2}} \sqrt{S \cot\alpha} = \sqrt{2} \tan\alpha$$

Answer: $V = \frac{2}{3} S^{\frac{3}{2}} \cot^{\frac{1}{2}}\alpha$; $\tan\varphi = \sqrt{2}\tan\alpha$.

608. (a) *Drawing*. The base of the pyramid is a regular pentagon (from the equation $180° (n-2) = 540°$ we find $n = 5$). And in the regular pentagon $ABCDE$ (Fig. 87*a*) each diagonal (say, AD) is divided by each of the other diagonals (for instance BE) in the extreme and mean ratio, so that $DM = \frac{\sqrt{5}-1}{2} AD \approx$

$\approx 0.6\,AD$. Furthermore, each diagonal is parallel to one of the sides (for instance $AD \parallel BC$). The centre O is the point of intersection of CM and EN. Therefore, the drawing of the regular pentagon may be constructed in the following way.

Construct an arbitrary triangle ABD (Fig. 87*b*). Divide the sides AD and BD by the points M and N in the extreme and mean ratio—approximately in the ratio

$$AM : MD = 2 : 3$$

for this purpose it is sufficient to divide one side and then to draw $MN \parallel AB$. Draw $AE \parallel BD$ to intersect the extension of the line BM at the point E. Point C is constructed likewise. The centre is represented by the point O which is the point of intersection of CM and EN.

(b) *Solution.* From the triangle COF, wherein $\angle OCF = \alpha$ and $CF = l$, we find $H = OF = l \sin \alpha$; $OC = l \cos \alpha$. The area of the base $S = 5 \cdot \frac{1}{2} \cdot OC \cdot OD \times \sin \angle COD =$

$$= \frac{5}{2} \cdot OC^2 \cdot \sin 72° = \frac{5}{2}\, l^2 \cos^2 \alpha \sin 72°.$$

Answer: $V = \frac{1}{3}\, SH = \frac{5}{6}\, l^3 \sin 72° \times \cos^2 \alpha \sin \alpha$.

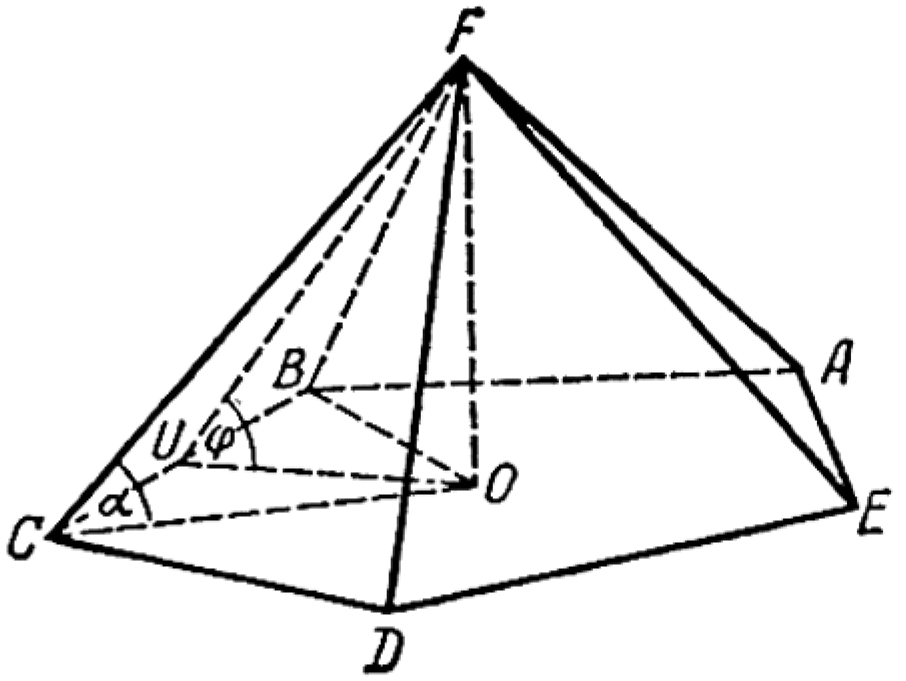


Fig. 88

609.* The angle α is determined from the triangle COF (Fig. 88), wherein $FC = CB = a$ (by hypothesis, the triangle CBF is an equilateral one). And the side OC (the radius of the circumscribed circle) is expressed in terms of a from the triangle COU, wherein the angle COU is equal to 36° and $CU = \frac{a}{2}$. We have $OC = \frac{a}{2 \sin 36°}$, hence, $\cos \alpha = \frac{OC}{CF} = \frac{1}{2 \sin 36°}$.

The angle φ is determined from the triangle OUF, wherein $FU = \frac{a\sqrt{3}}{2}$ (as the altitude of an equilateral triangle with the side a), and $OU = \frac{a \cot 36°}{2}$ (from the triangle COU). We have

$$\cos \varphi = \frac{OU}{FU} = \frac{a \cot 36°}{2} : \frac{a\sqrt{3}}{2} = \frac{\cot 36°}{\sqrt{3}}$$

Answer: $\alpha = \arccos \frac{1}{2 \sin 36°}$, $\varphi = \arccos \frac{\cot 36°}{\sqrt{3}}$.

610. We have (see Fig. 88): $BC = a$, $OU = \frac{a}{2} \cot \frac{180°}{n}$. The area of the base

$$S = \frac{na}{2} \cdot \frac{a}{2} \cot \frac{180°}{n} = \frac{na^2}{4} \cot \frac{180°}{n}$$

---

[*] For graphical representation of a regular pentagon see the preceding problem.

From the formula $V=\frac{1}{3}SH$ we find

$$H=\frac{3V}{S}=\frac{12V}{na^2}\tan\frac{180^\circ}{n}$$

Denoting the required angle OCF by α, we have

$$\tan\alpha=\frac{H}{OC}$$

where

$$OC=\frac{a}{2\sin\frac{180^\circ}{n}}$$

Answer: $\alpha=\arctan\dfrac{24V\sin\frac{180^\circ}{n}\tan\frac{180^\circ}{n}}{na^3}$.

Preliminary Notes to Problems 611 *through* 616

If all the lateral edges of a pyramid form equal angles with the base, then (1) all the lateral edges are equal; (2) a circle can be circumscribed about the base; (3) the altitude of the pyramid passes through the centre of this circle.

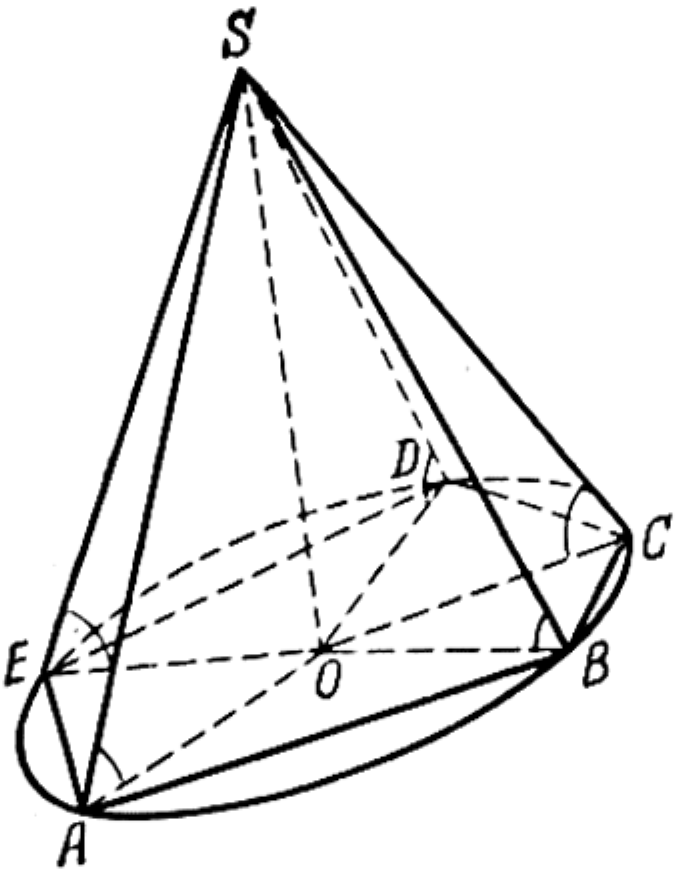


Fig. 89

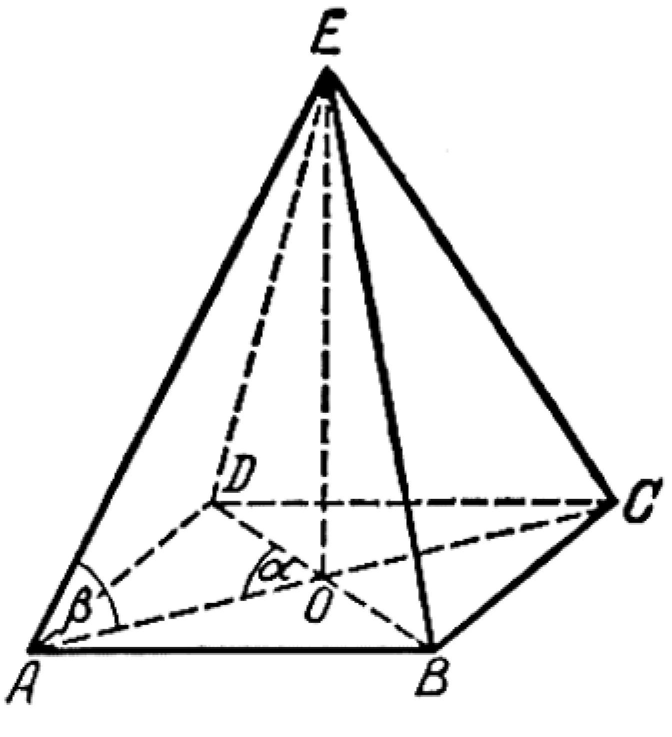


Fig. 90

Proof. Let the edges SA, SB, SC and so on (Fig. 89) form equal angles with the plane $ABCDE$. Consider the right-angled triangles AOS and BOS (OS is the altitude of the pyramid). They have a common altitude, and the acute angles OAS and OBS are equal to each other (since they measure the angles of inclination of the edges SA and SB to the base, respectively). Consequently, $AS = BS$. Likewise, we prove that $BS = CS$ and so on. From the same triangles AOS and BOS we find $AO = OB$. Likewise, we prove that $OB = OC$ and so on. Hence, the circle of radius OA and with O as the centre will pass through the points B, C, and so on.

611. As has been proved, the altitude EO passes through the centre of the circumscribed circle, i.e. through the point O of intersection of the diagonals (Fig. 90). The area of any parallelogram is equal to half the product of the diagonals and the sine of the angle contained between them. Therefore $S_{base} =$

$=\frac{1}{2}b^2\sin\alpha$. From the triangle AOE we find:

$$H=AO\cdot\tan\beta=\frac{b}{2}\tan\beta$$

Answer: $V=\frac{1}{12}b^3\sin\alpha\tan\beta$.

612. (a) *Drawing.* According to the Preliminary Notes, the altitude of the pyramid must pass through the centre of the circle circumscribed about the isosceles triangle ABC (Fig. 91). Since the angle $\alpha=\angle CAB$ at the vertex remains arbitrary, the centre O may be represented by any point of the line-segment AE (E is the midpoint of BC) and even of its extension (in the latter case the actual angle α is an obtuse one).

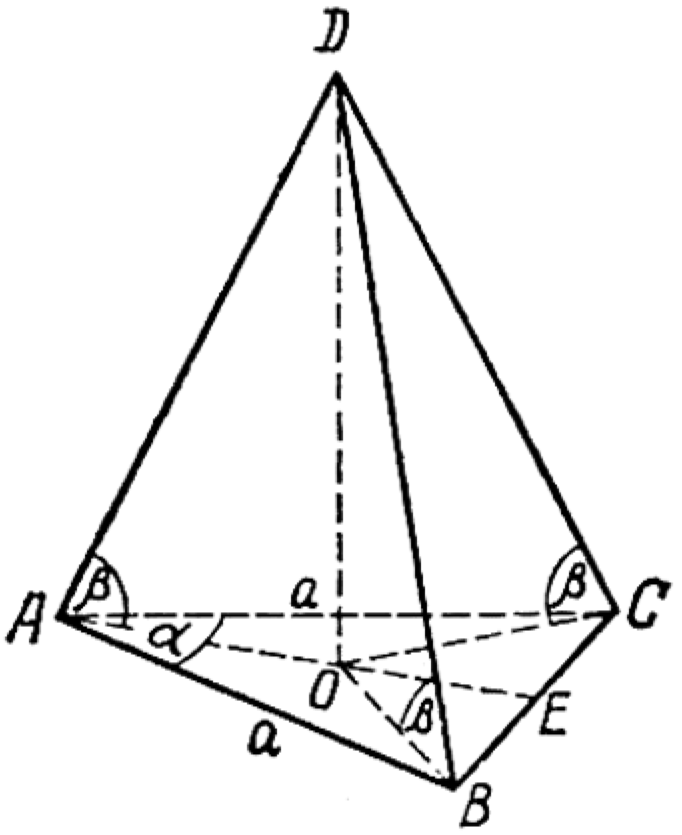


Fig. 91

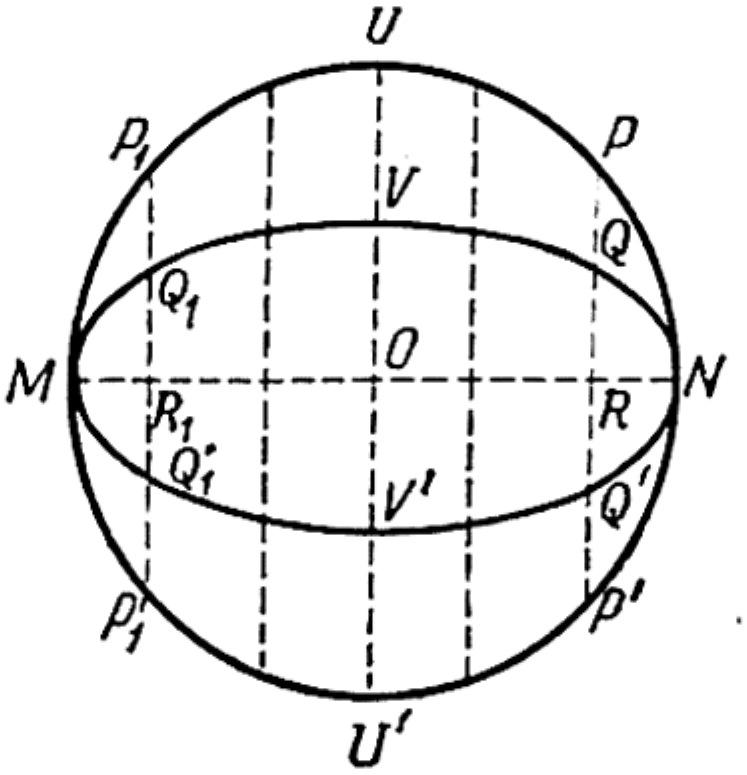


Fig. 92

(b) *Solution.* The altitude DO is determined from the triangle AOD, wherein $\angle OAD=\beta$, and $AO=R$ is the radius of the circumscribed circle. According to the law of sines the side BC is equal to the product of the diameter $2R$ of the circumscribed circle by the sine of the opposite angle α, so that $R=\frac{BC}{2\sin\alpha}$. The quantity $\frac{BC}{2}=BE$ is found from the triangle ABE $\left(\frac{BC}{2}=a\sin\frac{\alpha}{2}\right)$. Hence

$$H=R\tan\beta=\frac{a\sin\frac{\alpha}{2}\tan\beta}{\sin\alpha}$$

The area of the base

$$S=\frac{1}{2}a^2\sin\alpha$$

Answer: $V=\frac{a^2\sin\frac{\alpha}{2}\tan\beta}{6}$.

613. (a) *Drawing.* In the parallel projection a circle is represented as an ellipse. The ellipse may be constructed in the following way. Draw the diameter MN of the circle (Fig. 92) and from an arbitrary point P of the circle draw the straight line PP' perpendicular to MN. Let R be the point of intersection of PP' and MN. Shorten the line-segment RP in some ratio (say, to half its length) and lay off the shortened segment RQ on the same line PP' to both sides of R ($RQ = RQ'$). Proceed in the same way with a number of points on the circle to get a number of points for the ellipse under construction.

The ellipse is symmetrical about MN (*the major axis*) and about the straight line UU' drawn through the centre O perpendicular to MN (VV' is *the minor axis of the ellipse*.) The point O is called *the centre* of the ellipse.

To depict a circle circumscribed about a rectangle it is convenient first to draw an ellipse $ABCD$ representing the circumscribed circle (Fig. 93). It is good practice to arrange the major axis of the ellipse in an *inclined* position*.

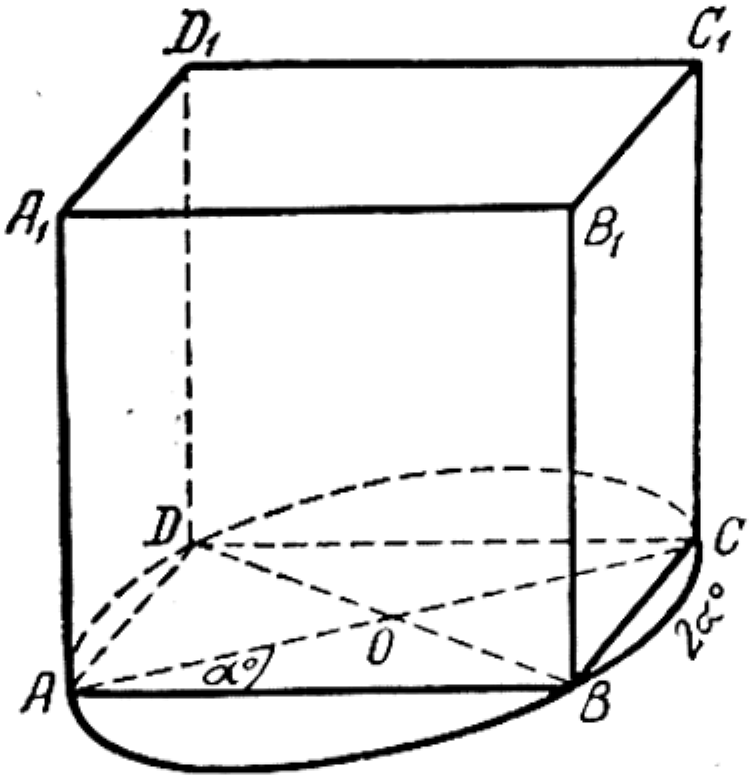


Fig. 93

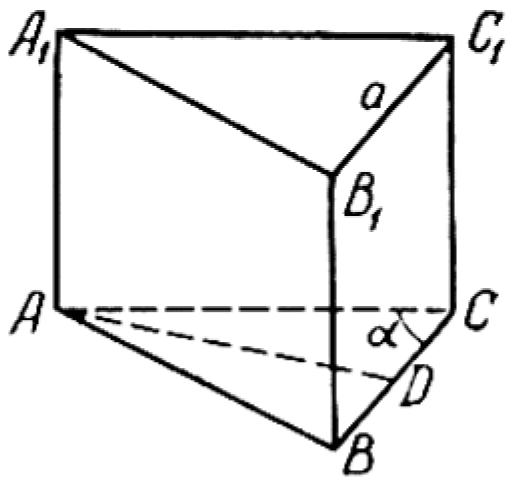


Fig. 94

One side of the rectangle may be represented by an arbitrary chord AB of the ellipse. It is advisible to draw the chord horizontally. Draw straight lines BD and AC through the centre of the ellipse. The quadrilateral $ABCD$ thus obtained is the graphical representation of the rectangle.

(b) *Solution*. The inscribed angle CAB contains $\alpha°$, since it is subtended by the arc BC containing $(2\alpha)°$. From the triangle BAC we have $AB = 2R\cos\alpha$; $BC = 2R\sin\alpha$, and so

$$S = 2(AB + BC)H = 4R(\cos\alpha + \sin\alpha)H$$

Hence,

$$H = \frac{S}{4R(\cos\alpha + \sin\alpha)}$$

We now find $V = AB \cdot BC \cdot H$. The condition that the arc $(2\alpha)°$ is subtended by a *smaller* side of the rectangle is an unnecessary one.

Answer: $V = \dfrac{SR\cos\alpha\sin\alpha}{\cos\alpha + \sin\alpha} = \dfrac{SR\sin 2\alpha}{\sqrt{8}\cos(45° - \alpha)}$.

614. The area of the base $S = \frac{1}{4}a^2\tan\alpha$ (Fig. 94). By hypothesis,

$$S_{lat} = 2S = \frac{1}{2}a^2\tan\alpha$$

---

* In Fig. 93 the major axis of the ellipse coincides with the diagonal AC of the rectangle. This simplifies the drawing, but is not obligatory.

On the other hand,

$$S_{lat}=\left(a+2\cdot\frac{\frac{a}{2}}{\cos\alpha}\right)H=\frac{2a\cos^2\frac{\alpha}{2}}{\cos\alpha}\,H$$

Equating the two expressions for S_{lat} we find

$$H=\frac{a}{4}\cdot\frac{\sin\alpha}{\cos^2\frac{\alpha}{2}}=\frac{a}{2}\tan\frac{\alpha}{2}$$

Answer: $V=\frac{a^3}{8}\tan\alpha\tan\frac{\alpha}{2}$.

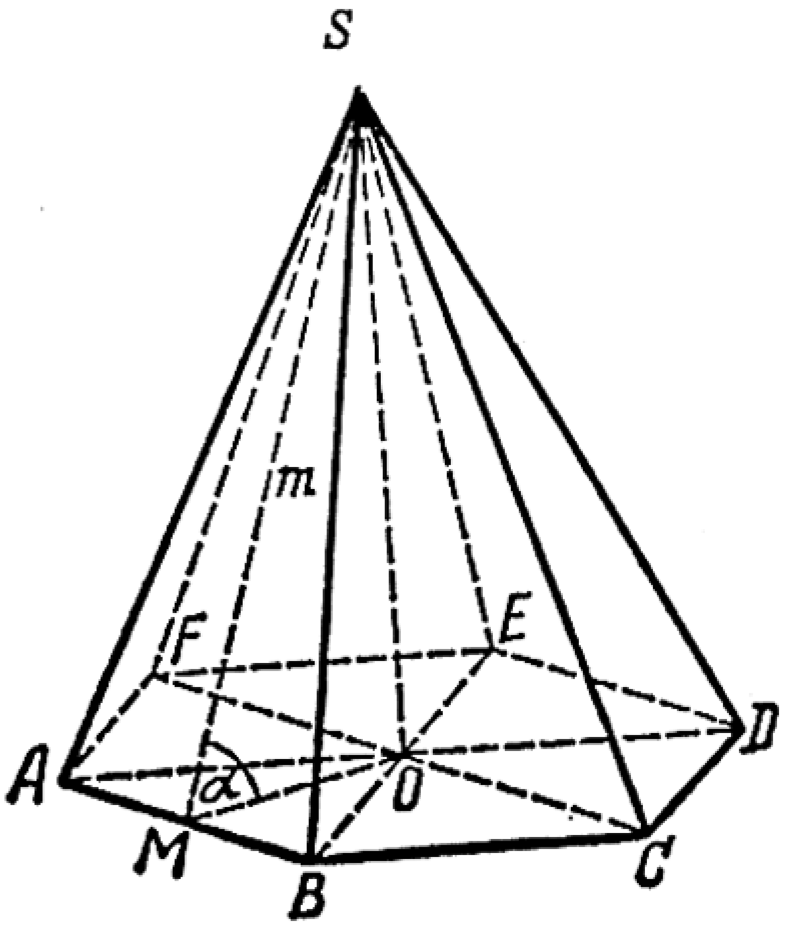


Fig. 95

615.* Join the midpoint M of the side AB with O and S (Fig. 95). The angle OMS is the plane angle of the dihedral angle α (see explanation to Problem 606). Hence,

$$OM=SM\cos\alpha=m\cos\alpha$$

From the triangle AOM, wherein $\angle AOM=30°$, we find

$$AM=\frac{a}{2}=\frac{\sqrt{3}}{3}\cdot OM=\frac{\sqrt{3}}{3}\,m\cos\alpha$$

Then we find

$$S_{base}=6\left(\frac{a}{2}\right)^2\sqrt{3}$$

and

$$S_{lat}=6\,\frac{a}{2}\cdot m$$

Substituting the found expression for $\frac{a}{2}$, we get

$$S_{total}=S_{base}+S_{lat}=2\sqrt{3}\,m^2\cos\alpha\,(1+\cos\alpha)$$

Answer: $S_{total}=4\sqrt{3}\,m^2\cos\alpha\cos^2\frac{\alpha}{2}$.

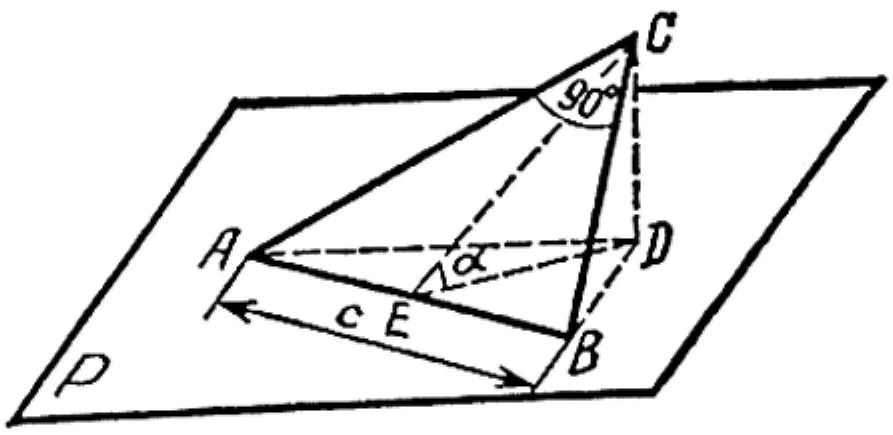


Fig. 96

616. By hypothesis, the inclined lines AC and CB (Fig. 96) are equal to each other. Hence, their projections are also equal: $AD = DB$. The angle DEC (E is the midpoint of AB) is the plane angle of the dihedral angle α.

---

* For graphical representation of a regular hexagon see Note to Problem **598.**

Since the triangle ACB at the vertex C is a right-angled one, $CE = AE = \frac{c}{2}$. Hence, $ED = \frac{c}{2}\cos\alpha$. Finally,

$$AD = BD = \sqrt{AE^2 + ED^2} = \frac{c}{2}\sqrt{1+\cos^2\alpha}$$

Answer: $S_{ABD} = \frac{c^2\cos\alpha}{4}$

$$AB + BD + AD = c\left(1 + \sqrt{1+\cos^2\alpha}\right)$$

Preliminary Notes to Problems 617-704

If all the lateral faces of a pyramid are inclined to the base at one and the same angle α, and the altitude passes through some point O on the base of the pyramid, then:

(1) the slant heights of all the faces are equal;

(2) a circle can be inscribed in the base of the pyramid with point O as the centre;

(3) $S_{base} = S_{lat}\cos\alpha$.

Proof. (1) Draw (Fig. 97) the slant height FM of the lateral face BFC and join M with O. The line-segment OM is the projection of FM on the plane $ABCDE$. Consequently, it is perpendicular to BC ("the theorem on three perpendiculars"). Hence, the angle OMF is the plane angle of the dihedral angle α. From the triangle OMF we have $FM = \frac{OF}{\sin\alpha}$; $OM = OF\cdot\cot\alpha$. If we draw FL, FN and the slant heights of other lateral faces, we find likewise that all of them are equal to $\frac{OF}{\sin\alpha}$.

(2) The line-segments OL, OM, etc. are perpendicular respectively to the sides AB, BC, etc. and are equal to $OF\cdot\cot\alpha$. Therefore, if a circle of radius OM is drawn from O as the centre, it will be inscribed in the base $ABCDE$.

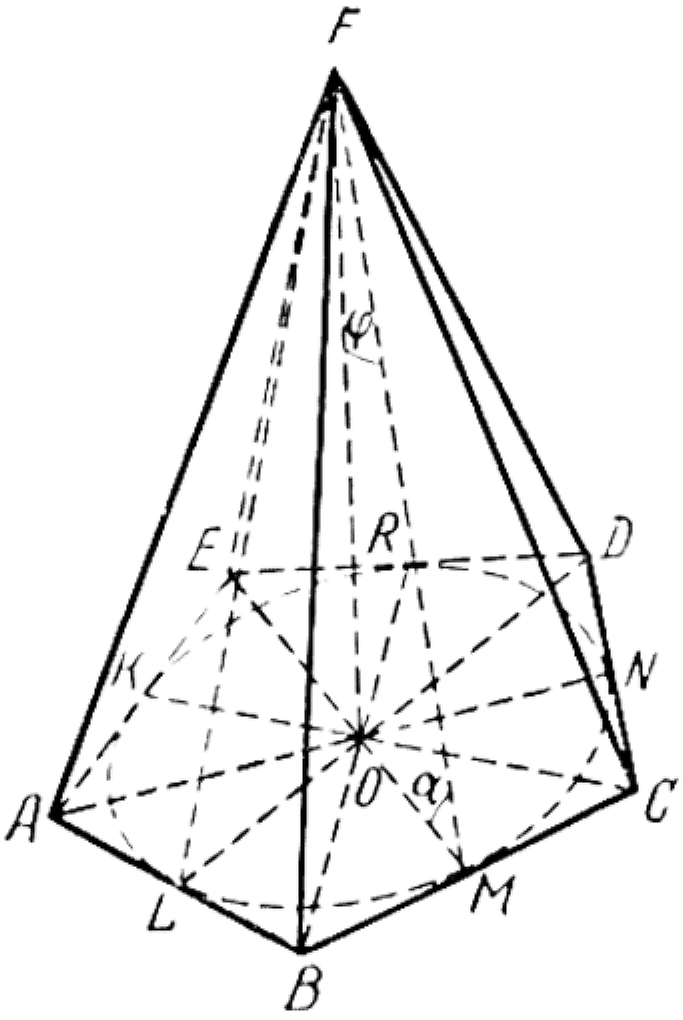


Fig. 97

(3) As has been proved the point O, which is the foot of the altitude of the pyramid is the centre of the inscribed circle.

(4) $S_{OBC} = \frac{1}{2}BC\cdot OM = \frac{1}{2}BC\,(FM\cdot\cos\alpha) = \left(\frac{1}{2}BC\cdot FM\right)\cos\alpha = S_{FBC}\cos\alpha$

Likewise we find that $S_{OAB} = S_{FAB}\cos\alpha$, and so on. Adding these equalities, we get $S_{base} = S_{lat}\cos\alpha$.

617. The altitude FO of any pyramid (Fig. 97) is projected on the lateral face BFC as a line-segment lying on the straight line FM. Therefore, $\angle OFM = \varphi$. Hence, $\alpha = 90° - \varphi$, i.e. all the faces are inclined to the base at one and the same angle. As has been proved

$$S_{lat} = \frac{Q}{\cos\alpha} = \frac{Q}{\sin\varphi}$$

Answer: $S_{lat}=\frac{Q}{\sin\varphi}$

$$S_{total}=Q\left(1+\frac{1}{\sin\varphi}\right)=\frac{2Q\cos^2\left(45^\circ-\frac{\varphi}{2}\right)}{\sin\varphi}\cdot$$

618. From the triangle DOE (Fig. 98)* we find

$$H=OE\cdot\tan\varphi=\frac{1}{3}\cdot CE\cdot\tan\varphi=\frac{1}{3}\cdot\frac{a\sqrt{3}}{2}\cdot\tan\varphi$$

We have

$$S_{base}=\frac{1}{4}a^2\sqrt{3}\quad\text{and}\quad S_{lat}=\frac{S_{base}}{\cos\varphi}$$

(see the preliminary note to the preceding problem).

Answer: $V=\frac{a^3\tan\varphi}{24}$

$$S_{total}=\frac{a^2\sqrt{3}(1+\cos\varphi)}{4\cos\varphi}=\frac{a^2\sqrt{3}\cos^2\frac{\varphi}{2}}{2\cos\varphi}$$

Note. The general expression for the total surface area of a pyramid, whose faces are inclined to the base at one and the same angle φ may be written as follows:

$$S_{total}=S_{base}+S_{lat}=$$

$$=S_{base}\left(1+\frac{1}{\cos\varphi}\right)=\frac{2S_{base}\cos^2\frac{\varphi}{2}}{\cos\varphi}$$

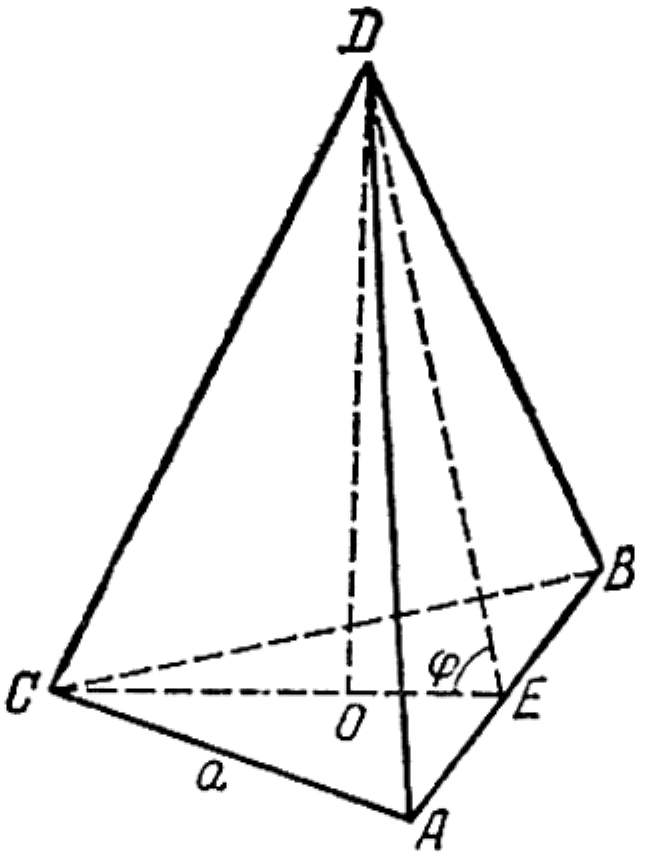


Fig. 98

619. Make use of the formula $S_{total}=\frac{a^2\sqrt{3}\cos^2\frac{\varphi}{2}}{2\cos\varphi}$, found in the preceding problem.

Answer: $a=\frac{1}{\cos\frac{\alpha}{2}}\sqrt{\frac{2S\cos\alpha}{\sqrt{3}}}$.

620. (a) *Drawing*. The straight line, joining the points of tangency L and N of the opposite sides of the rhombus (Fig. 99*a*) passes through the centre of the circle. Therefore, first draw an ellipse (Fig. 99*b*), representing the circle**, and then the straight lines NL and KM passing through the centre O. To complete the parallelogram $ABCD$ representing the rhombus draw straight lines tangent to the ellipse at the points N, L, K, M.

(b) *Solution*. To determine S_{base} find the altitude DF and the side AB of the rhombus. From Fig. 99*a* we find $DF=2OK=2r$; from the triangle AFD, wherein $\angle A=\alpha$, we have

$$a=AD=\frac{DF}{\sin\alpha}=\frac{2r}{\sin\alpha}$$

---

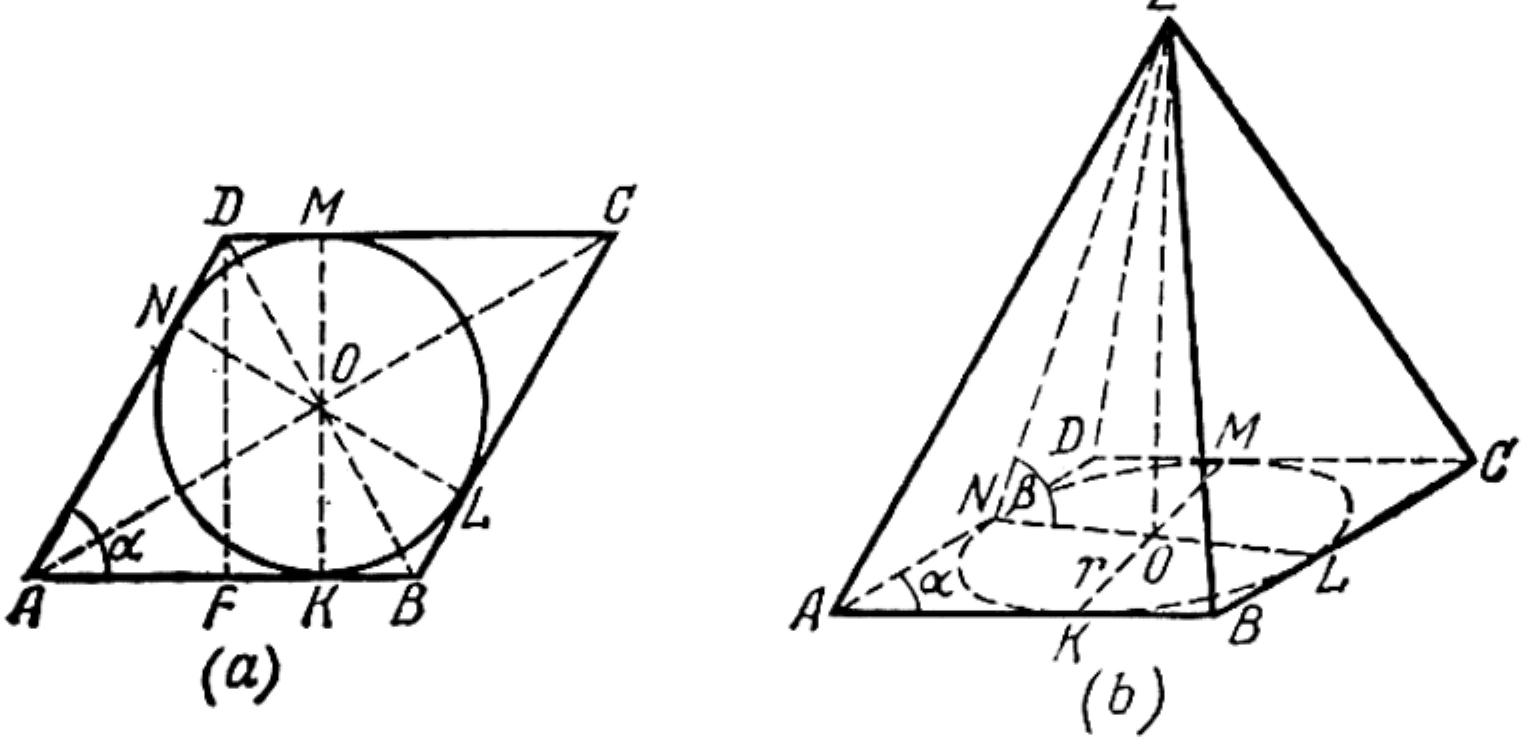
* For representation and construction see Fig. 82.

** For construction of an ellipse see Problem 613.

Then we find

$$S_{base} = AB \cdot DF = a \cdot 2r = \frac{4r^2}{\sin \alpha}$$

H is determined from the triangle ONE (Fig. 99b), wherein $ON = r$ and $\angle ONE = \beta$. For determining S_{total} make use of the Note to the preceding problem.



(a) (b)

Fig. 99

Answer: $V = \dfrac{4r^3 \tan \beta}{3 \sin \alpha}$; $S_{total} = \dfrac{8r^2 \cos^2 \dfrac{\beta}{2}}{\sin \alpha \cos \beta}$.

621. Use the Note to Problem 618.

Answer: $\varphi = \arccos \dfrac{S}{\sigma}$.

622. (a) *Drawing**. The section figure is the parallelogram A_1D_1CB (Fig. 100). To depict the plane angle formed by the cutting plane A_1D_1CB and the plane of the base draw the straight line DM representing the altitude of the rhombus $ABCD$. Since DM and DD_1 are actually perpendicular to the edge AD, the plane DD_1NM is perpendicular to AD, and, hence, to BC. This plane intersects the cutting plane along the straight line MD_1, and thus $\angle D_1MD = \beta$.

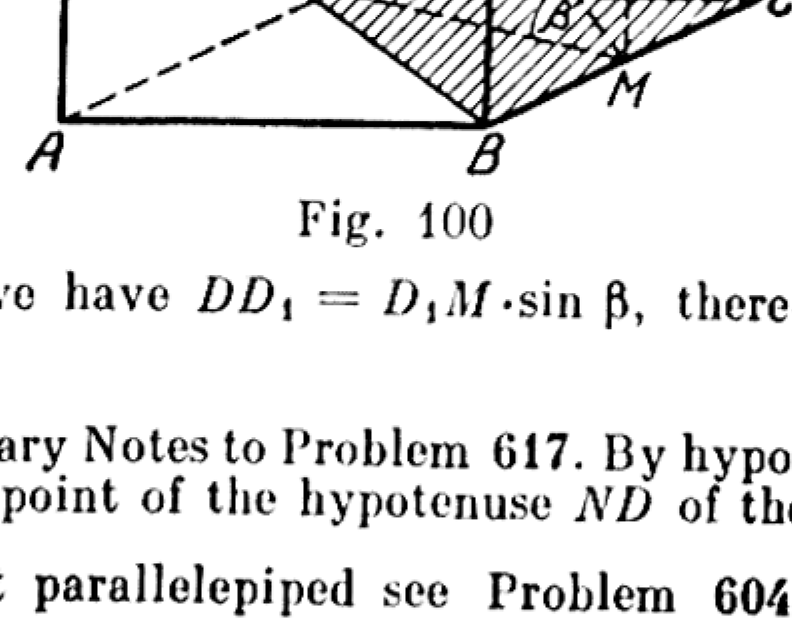


Fig. 100

(b) *Solution*. The lateral surface consists of four equal rectangles (since the base is a rhombus). The area of the lateral face A_1D_1DA is $S_1 = A_1D_1 \cdot DD_1$, and the area of the section figure is $Q = A_1D_1 \cdot D_1M$. From the triangle DMD_1 we have $DD_1 = D_1M \cdot \sin \beta$, therefore $S_1 = Q \sin \beta$.

Answer: $S_{lat} = 4Q \sin \beta$.

623. Take into consideration the Preliminary Notes to Problem 617. By hypothesis, $EO = d$ (Fig. 101). Point E (the midpoint of the hypotenuse ND of the triangle NOD) is the centre of the circle circumscribed about the triangle NOD. Therefore, $ND = 2\cdot ED = 2\cdot EO = 2d$. From the triangle DON, wherein $\angle OND = \varphi$, find the radius $ON = r$ of the circle inscribed in the base: $r = 2d\cos\varphi$. To find S_{base} determine BN (half the base of the isosceles triangle ABC) and AN (its altitude). The centre O of the inscribed circle lies on the bisector of the angle ABC equal to α, i.e. $\angle OBN = \frac{\alpha}{2}$. From the triangle BON we find $BN = r\cot\frac{\alpha}{2}$. From the triangle ABN we find $AN = BN\cdot\tan\alpha$. Consequently,

---

* For graphical representation of a right parallelepiped see Problem 604.

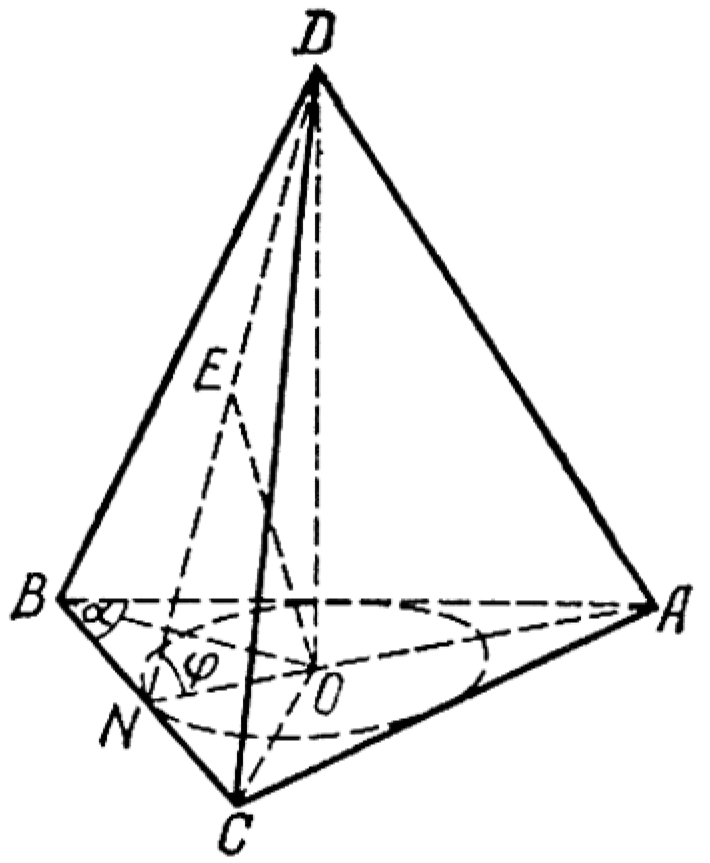


Fig. 101

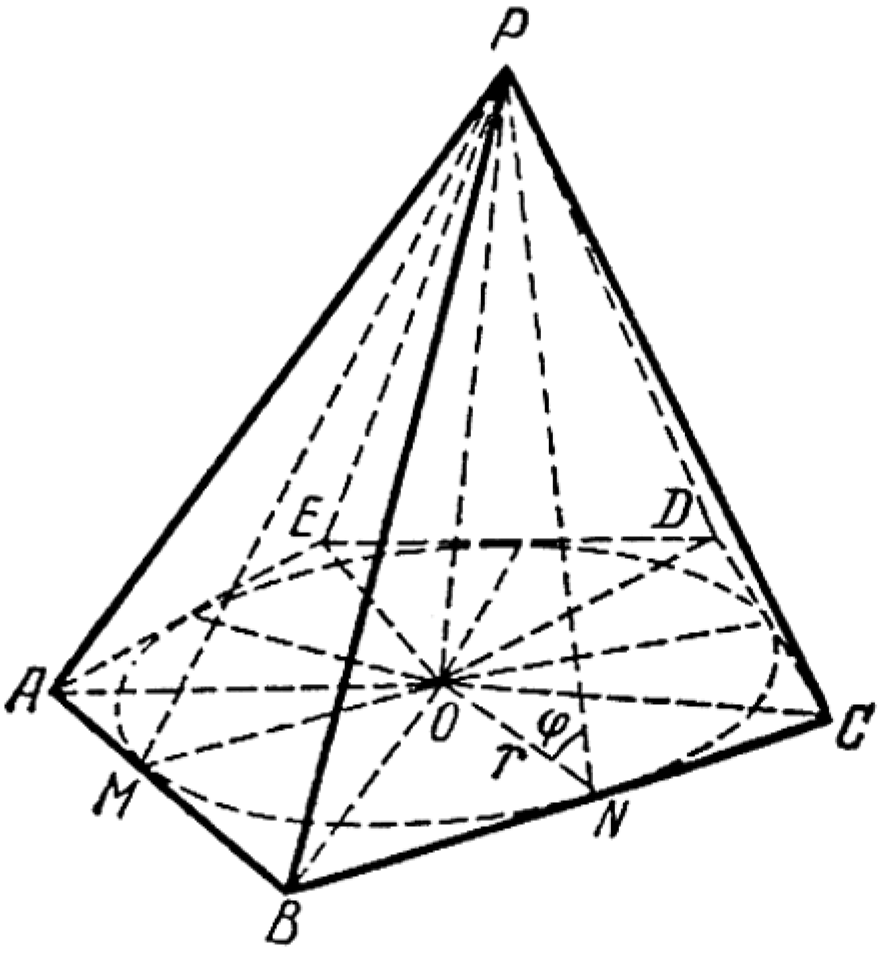


Fig. 102

$$S_{base} = \frac{1}{2}BC\cdot AN = BN\cdot AN = BN^2\cdot\tan\alpha = r^2\cot^2\frac{\alpha}{2}\cdot\tan\alpha =$$

$$= 4d^2\cos^2\varphi\cot^2\frac{\alpha}{2}\tan\alpha$$

wherefrom (see Note to Problem 618) we find:

$$S_{total} = \frac{2S_{base}\cos^2\frac{\varphi}{2}}{\cos\varphi}$$

Answer: $S_{total} = 8d^2\cos\varphi\cos^2\frac{\varphi}{2}\cot^2\frac{\alpha}{2}\tan\alpha$.

624. Take into consideration the Preliminary Notes to Problem 617[*]. The altitude of the pyramid is found from the triangle ONP (Fig. 102): $H = r\tan\varphi$. If a_1, a_2, etc. are the sides of the base, then

$$S_{base} = S_{AOB} + S_{BOC} + \ldots = \frac{1}{2}AB\cdot OM + \frac{1}{2}BC\cdot ON + \ldots =$$

$$= \frac{1}{2}a_1r + \frac{1}{2}a_2r + \ldots = \frac{1}{2}r(a_1 + a_2 + \ldots) = \frac{1}{2}r\cdot 2p = rp$$

---

[*] For construction of the ellipse representing the circle inscribed in the base see Problem 613.

Answer: $V=\dfrac{r^2 p \tan\varphi}{3}$.

625. (a) *Drawing*. Having drawn the regular triangular pyramid $DABC$ (Fig. 103)*, let us construct the triangle $A_1B_1C_1$, whose sides are parallel to the respective sides of the triangle ABC. The triangle $A_1B_1C_1$ depicts the upper base of the frustum of the pyramid. The centre O_1 of the upper base is found at the point of intersection of DO and one of the medians A_1E_1 of the triangle $A_1B_1C_1$. The line-segment A_1M (parallel to OO_1), whose foot lies on the median AE, represents the altitude of the frustum, dropped from the point A_1 (the line-segments DA_1, DB_1, DC_1 and DO_1 may be erased).

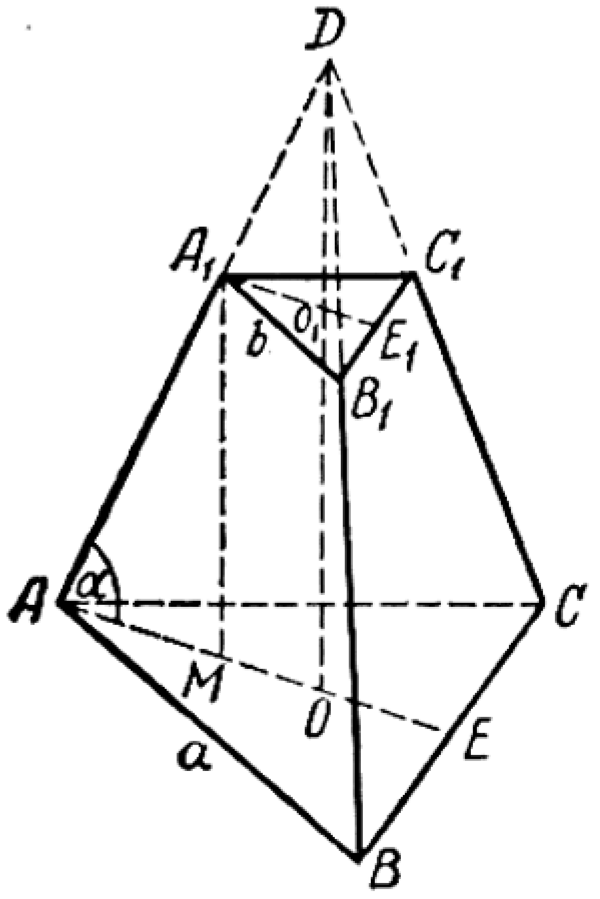


Fig. 103

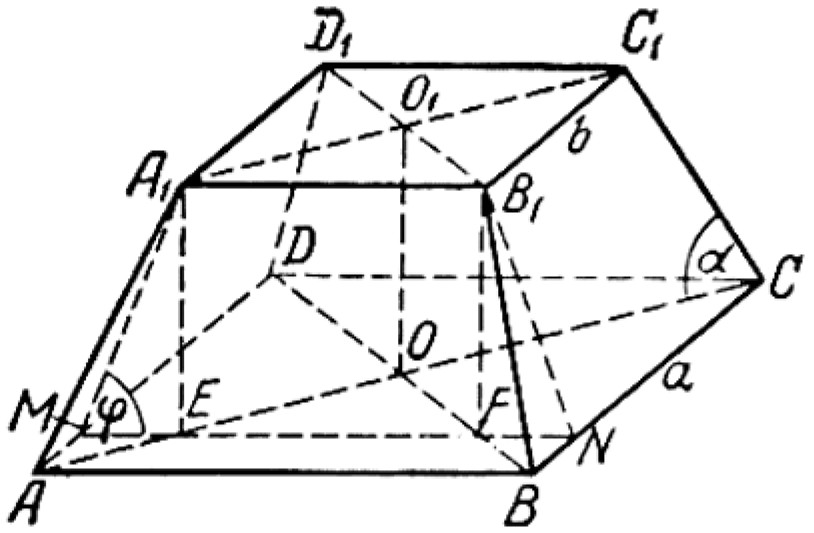


Fig. 104

(b) *Solution*. The volume of the frustum of a pyramid

$$V=\frac{H}{3}\left(Q+q+\sqrt{Qq}\right)$$

where Q and q are the areas of the triangles ABC and $A_1B_1C_1$ respectively, so that $Q=\dfrac{\sqrt{3}}{4}a^2$; $q=\dfrac{\sqrt{3}}{4}b^2$. The altitude $H=A_1M$ is found from the triangle AA_1M, wherein $\angle MAA_1=\alpha$ and $AM=AO-A_1O_1$. But AO and A_1O_1 are the radii of the circles circumscribed about ABC and $A_1B_1C_1$. Therefore, $AO=\dfrac{a}{\sqrt{3}}$ and $A_1O_1=\dfrac{b}{\sqrt{3}}$. Hence,

$$AM=\frac{a-b}{\sqrt{3}}$$

Consequently,

$$H=\frac{a-b}{\sqrt{3}}\tan\alpha$$

Answer: $V=\dfrac{1}{12}\cdot(a^3-b^3)\tan\alpha$.

626. (a) *Drawing*. The frustum of the pyramid is represented as in the preceding problem. To depict the plane angle of the required dihedral angle draw A_1E and B_1F (Fig. 104) parallel to OO_1 to intersect the diagonals AC and BD.

---

* For graphical representation of a regular triangular pyramid see Fig. 82.

Then draw EF parallel to AB to intersect the edges AD and BC at points M and N. The plane MA_1B_1N is perpendicular to the edge AD, since it passes through A_1E and MN which are perpendicular to the edge. Consequently, $\angle EMA_1 = \varphi$ is a plane angle of the dihedral angle at the edge AD.

(b) *Solution.* From the trapezoid MA_1B_1N we obtain $ME=\frac{a-b}{2}$. The altitude of the truncated pyramid is found from the triangle AEA_1, where $AE=\frac{a-b}{\sqrt{2}}$. We have

$$H=A_1E=\frac{a-b}{\sqrt{2}}\tan\alpha$$

The volume is found by the formula $V=\frac{H}{3}(a^2+ab+b^2)$. The required angle $\varphi=\angle EMA_1$ is found from the triangle A_1ME, where $ME=\frac{a-b}{2}$ (from the trapezoid MNB_1A_1). We have

$$\tan\varphi=\frac{A_1E}{ME}=\frac{a-b}{\sqrt{2}}\tan\alpha : \frac{a-b}{2}$$

Answer: $V=\frac{(a^3-b^3)\tan\alpha}{3\sqrt{2}}$; $\varphi=\arctan(\sqrt{2}\tan\alpha)$.

627. See the Preliminary Notes to Problem 611. The altitude of the pyramid must pass through the centre of the circle circumscribed about the base. But in the right-angled triangle ABC (Fig. 105) the centre lies in the midpoint of the hypotenuse AB at the point E. Consequently, AE, BE and CE are respective projections of the lateral edges AD, BD and CD on the plane of the base, and thus $\angle DAE=\angle DBE=\angle DCE=\beta$. The volume of the pyramid is found by the formula $V=\frac{1}{3}\cdot\frac{AC\cdot CB}{2}\cdot DE$. From $\triangle ABC$ we have: $AC=c\cos\alpha$, $BC=c\sin\alpha$; from $\triangle ADE$ we find $DE=\frac{c}{2}\tan\beta$. Let us denote the plane angles at the vertex: $\angle ADB=\theta_1$, $\angle BDC=\theta_2$ and $\angle ADC=\theta_3$. Since these triangles are isosceles ones, their altitudes DE, DM and DN pass through the midpoints of the corresponding sides of the base. From $\triangle ABD$ we have $\angle\theta_1=180°-2\beta$; from $\triangle DBC$ we have $\sin\frac{\theta_2}{2}=\frac{MB}{BD}$ and from $\triangle ADC$ we have $\sin\frac{\theta_3}{2}=\frac{AN}{AD}$. From $\triangle ADE$ we find $AD=DB=\frac{c}{2\cos\beta}$ and from $\triangle ABC$ we find $MB=\frac{BC}{2}=\frac{c}{2}\sin\alpha$ and $AN=\frac{AC}{2}=\frac{c}{2}\cos\alpha$.

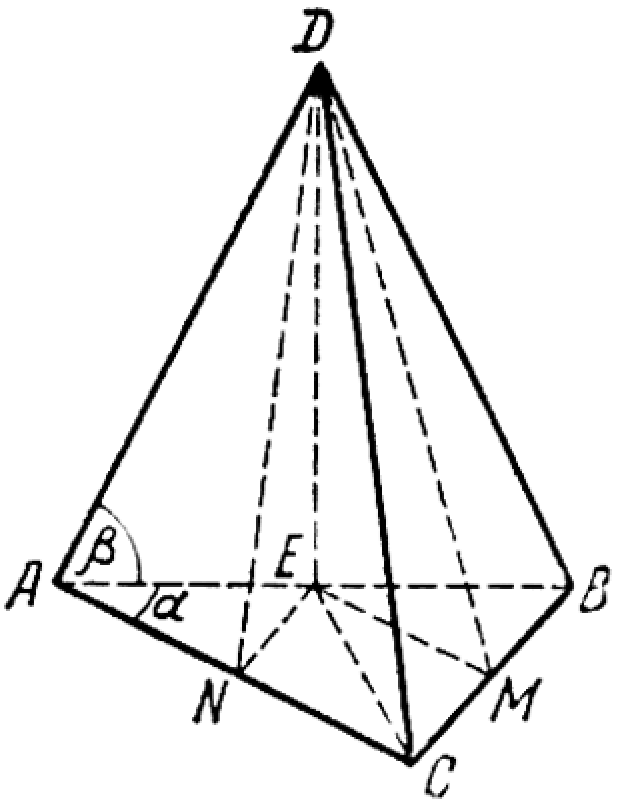


Fig. 105

Answer: $V=\frac{c^3 \sin 2\alpha \tan \beta}{24}$

$\theta_1=180^\circ-2\beta$

$\theta_2=2 \arcsin (\sin \alpha \cos \beta)$

$\theta_3=2 \arcsin (\cos \alpha \cos \beta)$

628. It is required to find the volume of the pyramid C_1ABC (Fig. 106). Since its lateral edges are of the same length, they are inclined to the base at one and the same angle (this theorem is converse to the theorem proved in the Preliminary Notes to Problem 611), and the altitude C_1O passes through the centre O of the circle circumscribed about the triangle ABC. Since this triangle is right-angled one, the point O lies at the midpoint of the hypotenuse AB (see the explanation to the preceding problem). The angle ODC_1 (D is the midpoint of the leg AC) measures the inclination of the lateral face ACC_1A_1 to the base. The legs BC and AC are found from the following two equations:

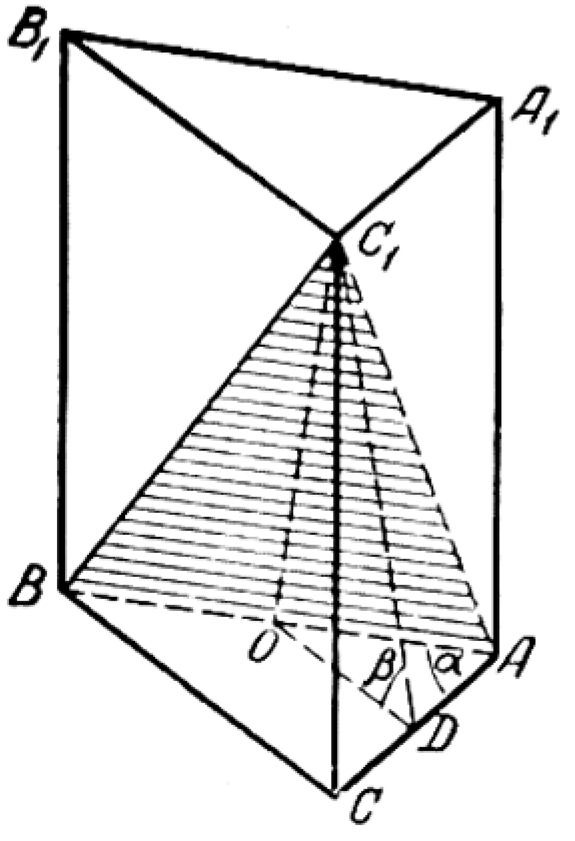


Fig. 106

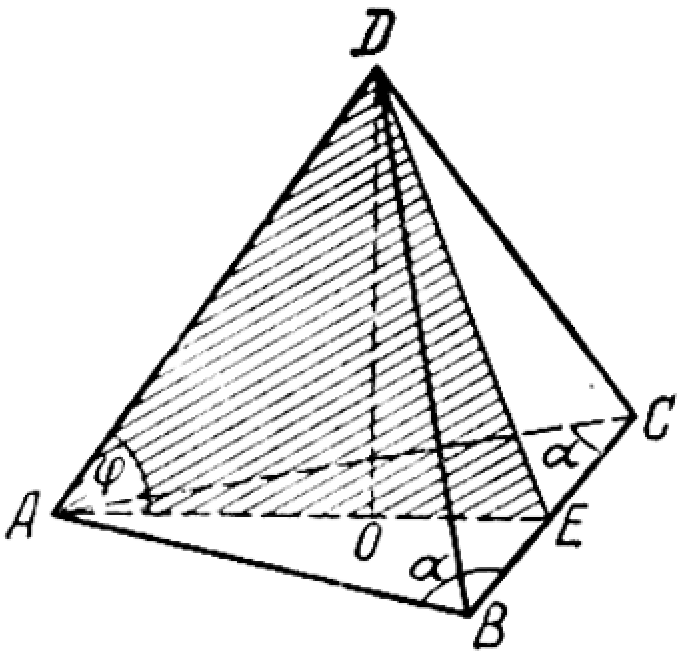


Fig. 107

$$BC+AC=m \quad \text{and} \quad BC=AC\cdot\tan \alpha$$

we get

$$AC=\frac{m}{1+\tan \alpha}=\frac{m \cos \alpha}{\sin \alpha+\cos \alpha}, \quad BC=\frac{m \sin \alpha}{\sin \alpha+\cos \alpha}$$

Then we find $S_{base}=\frac{1}{2} BC\cdot AC$. The altitude H is found from the triangle DOC_1, where $OD=\frac{1}{2} BC$ (as a midline of the triangle).

Answer: $V=\frac{1}{12} \frac{m^3 \sin^2 \alpha \cos \alpha}{(\sin \alpha+\cos \alpha)^3} \tan \beta=\frac{m^3 \sin^2 \alpha \cos \alpha}{24 \sqrt{2} \cos^3 (\alpha-45^\circ)} \tan \beta.$

629. Point O is the centre of the circle circumscribed about the base ABC (Fig. 107) (see the Preliminary Notes to Problem 611). $OA=R$ is the radius of this circle. The volume of the pyramid

$$V=\frac{1}{3}\cdot\frac{BC\cdot AE}{2}\cdot DO=\frac{1}{3}\cdot\frac{AE\cdot DO}{2}\cdot BC=\frac{1}{3}Q\cdot BC$$

$\left(\text{since } \frac{AE\cdot DO}{2}=Q\right)$. The side BC is found by the law of sines:

$$BC=2R\sin(180^\circ-2\alpha)=2R\sin 2\alpha$$

$\triangle ADO \backsim \triangle ABE$ (since $\angle ADO=\angle ABE=\alpha$); we have the proportion $\frac{AO}{AE}=\frac{OD}{BE}$*, wherefrom $AO\cdot BE=AE\cdot OD$.

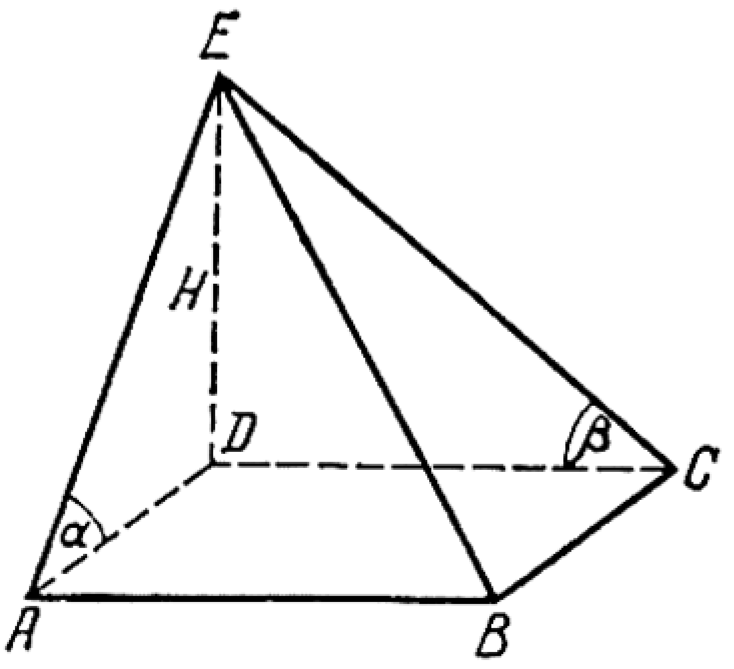


Fig. 108

Substituting

$$AO=R,\quad BE=\frac{BC}{2},\quad AE\cdot OD=2Q$$

we get

$$\frac{R\cdot BC}{2}=2Q$$

Eliminating R from the found formulas, we obtain

$$BC=\sqrt{8Q\sin 2\alpha}$$

Answer: $V=\frac{1}{3}\cdot(2Q)^{\frac{3}{2}}\sin^{\frac{1}{2}}2\alpha$.

630. If the faces ADE and CDE (Fig. 108) are perpendicular to the plane of the base, then the edge DE is the altitude of the pyramid. The angle DAE is a plane angle of the dihedral angle $EABC$, since the plane DAE is perpendicular to the edge AB (prove it!). Consequently, $\angle DAE=\alpha$; likewise, $\angle DCE=\beta$. From the triangles ADE and CDE, where $DE=H$, we find AD and DC and substitute their values into the formula

$$V=\frac{1}{3}AD\cdot DC\cdot H$$

Answer: $V=\frac{1}{3}H^3\cot\alpha\cot\beta$.

631. From the triangle BDE (Fig. 109), where $\angle EBD=\beta$ (prove it!) we find

$$DE=l\sin\beta \quad\text{and}\quad BD=l\cos\beta$$

Hence,

$$AD=\frac{BD}{\sqrt{2}}=\frac{l\cos\beta}{\sqrt{2}}$$

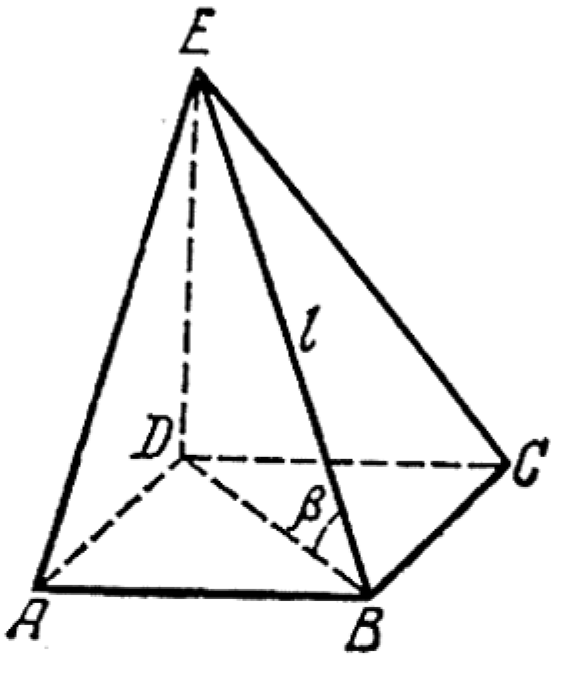


Fig. 109

From the triangle ADE we find $AE=\sqrt{AD^2+DE^2}$. The angle φ of inclination of the edge AE to the plane of the base is $\angle DAE$ (prove it!). From the triangle ADE we find $\tan\varphi=\frac{DE}{AD}$.

---

* As is obvious, Fig. 107 (where $AO<AE$) does not correspond to this relationship. But a drawing depicting the condition of the problem ($\varphi=90^\circ-\alpha$) more accurately would be obscure.

Answer: $DE = l \sin\beta$;

$$\varphi = \arctan(\sqrt{2}\tan\beta), \quad AE = CE = l\sqrt{\frac{1+\sin^2\beta}{2}}.$$

632. The greatest area belongs to the face ADB (Fig. 110), since its height DE is larger than the height DC of the other two lateral faces, the bases of all

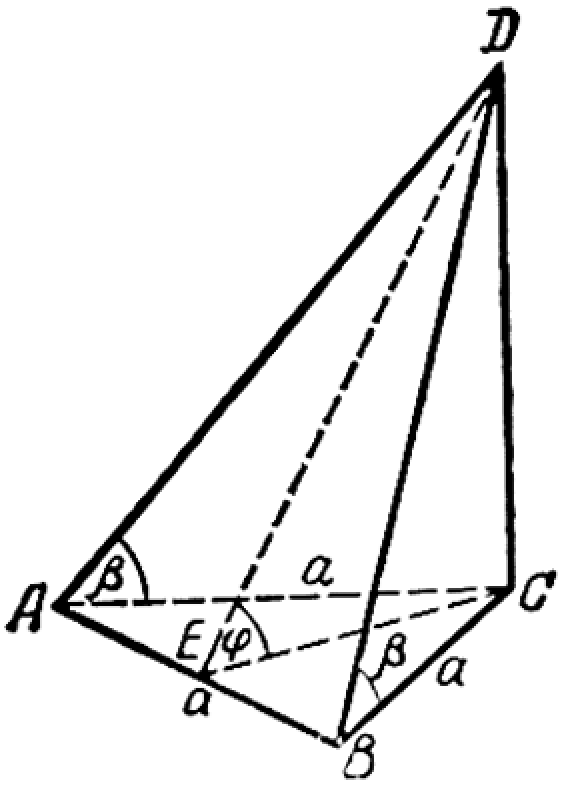


Fig. 110

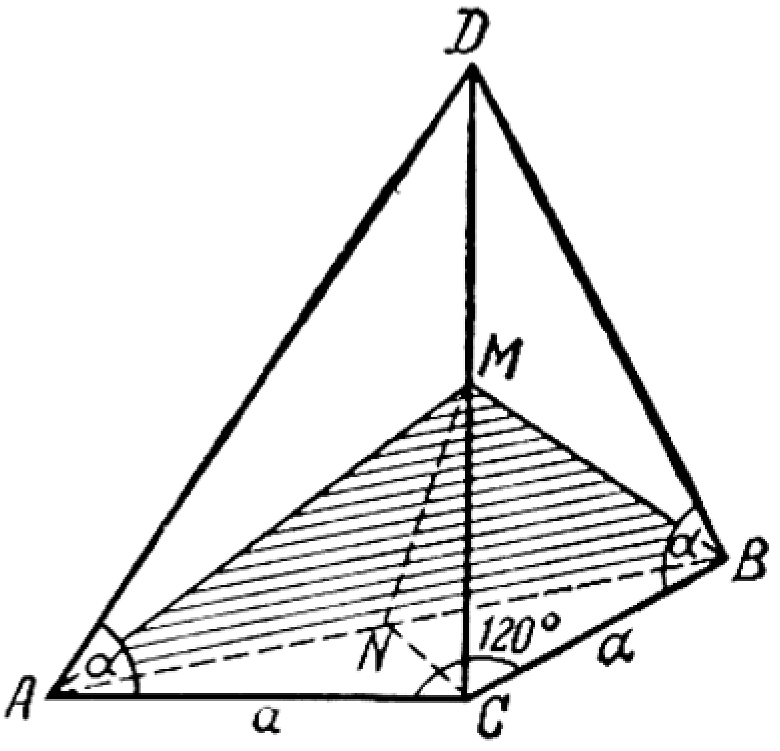


Fig. 111

the faces being equal to a. From the triangle ACD we have

$$AD = \frac{a}{\cos\beta} \quad \text{and} \quad H = a\tan\beta$$

We then find from the triangle ADE

$$DE = \sqrt{AD^2 - AE^2} = \sqrt{\frac{a^2}{\cos^2\beta} - \frac{a^2}{4}}$$

The angle CED is the angle φ of inclination of the face ADB to the plane of the base (prove it!). We have

$$\tan\varphi = \frac{H}{EC}$$

where $EC = \frac{a\sqrt{3}}{2}$.

Answer: $S = \frac{a^2}{4\cos\beta}\sqrt{4-\cos^2\beta}$, $\varphi = \arctan\frac{2\tan\beta}{\sqrt{3}}$.

633. The area S of the section is equal to $\frac{1}{2}\cdot AB\cdot NM$ (Fig. 111). From the right-angled triangle ACN, where $\angle CAN = 30°$, we find

$$AN = \frac{1}{2}\cdot AB = \frac{\sqrt{3}}{2}a \quad \text{and} \quad CN = \frac{1}{2}a$$

From the triangle NCM we have

$$MN=\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{H}{2}\right)^2}$$

where $H=a\tan\alpha$ may be obtained from the triangle ACD.

Answer: $S=\frac{a^2\sqrt{3}}{4\cos\alpha}$.

634. (a) *Drawing**. To depict a section perpendicular to the base ABC (Fig. 112) and bisecting the sides AB and AC of the base draw the midline MN. From the point F, where MN intersects the median AE, draw FK parallel to the altitude OD. NMK is the required section. Indeed, the plane NMK passes through the straight line FK perpendicular to the plane ABC (hence, the plane NMK is perpendicular to the plane ABC). The dihedral angle α is measured by the angle AED (prove it!).

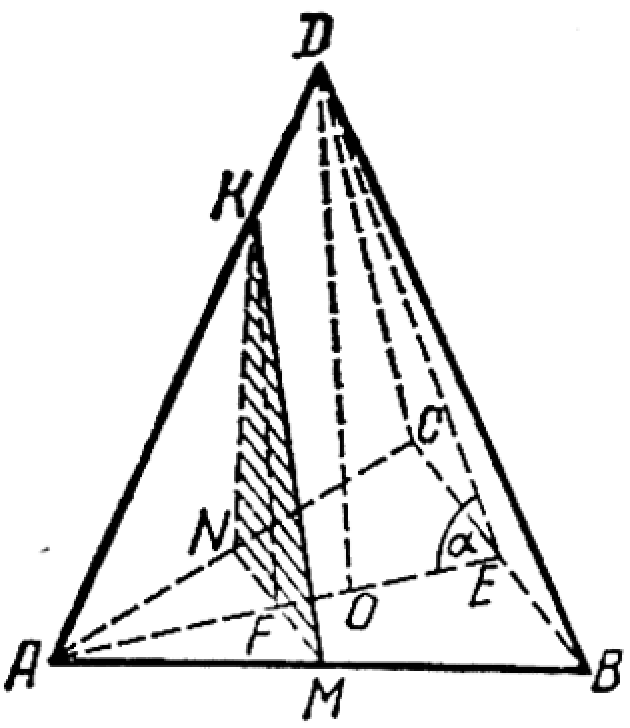


Fig. 112

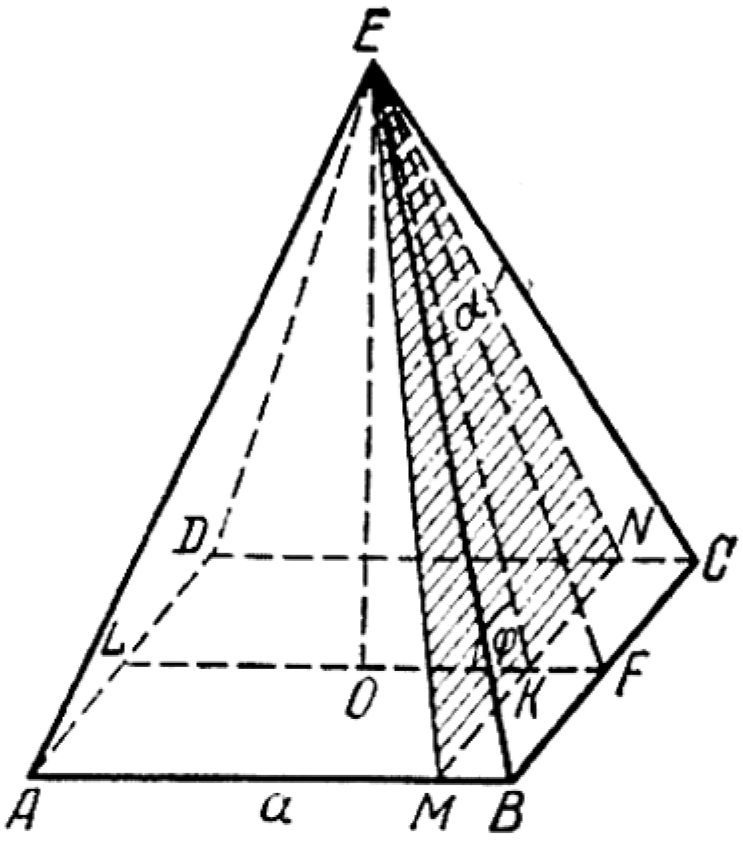


Fig. 113

The plane AED passes through KF, since the points K and F lie in the plane AED.

(b) *Solution*. Let us take the triangle AMN as the base of the pyramid $KANM$. The area S constitutes one fourth of the area of the triangle ABC, i.e. $S=\frac{1}{16}a^2\sqrt{3}$. Let us express the altitude KF through OD making use of the fact that $\triangle AFK$ is similar to $\triangle AOD$. Since AF is equal to $\frac{3}{4}AO$ $\left(\text{for } AF=\frac{1}{2}AE, \text{ and } AO=\frac{2}{3}AE\right)$, $KF=\frac{3}{4}OD$. The line-segment OD is found from the triangle DOE, where $OE=\frac{a\sqrt{3}}{6}$ and $\angle DEO=\alpha$.

Answer: $V=\frac{a^3\tan\alpha}{128}$.

635. The straight line MN (Fig. 113), along which the cutting plane intersects the base, is parallel to BC. To construct the angle φ draw $OF \parallel AB$

---

* For depicting a regular triangular pyramid see Problem 603.

and join the point K, at which OF intersects MN, to E. Then $\angle OKE=\varphi$ (prove all this). The area of the section figure $S=\frac{1}{2}MN\cdot KE$, where $MN=a$ and $KE=\frac{H}{\sin\varphi}$. The altitude H is determined from the triangle EOF, where $OF=\frac{a}{2}$ and $FE=\frac{a}{2}\cot\frac{\alpha}{2}$ (from the triangle EBF). We get

$$H=\sqrt{\left(\frac{a}{2}\cot\frac{\alpha}{2}\right)^2-\left(\frac{a}{2}\right)^2}=\frac{a\sqrt{\sin\alpha}}{2\sin\frac{\alpha}{2}}$$

Answer: $S=\dfrac{a^2\sqrt{\cos\alpha}}{4\sin\frac{\alpha}{2}\sin\varphi}$.

636.* The section figure is a triangle DKN (Fig. 114). As in Problem 634, let us prove that the plane AED is perpendicular to the side BC. Hence, it is perpendicular to the midline KN as well. Consequently, $\angle DME$ is a plane angle of the given dihedral angle α. From the triangle OMD, where $OM=\frac{1}{6}AE=\frac{1}{6}\frac{a\sqrt{3}}{2}$, we find

$$DM=\frac{a\sqrt{3}}{12\cos\alpha}$$

The section area

$$S=\frac{1}{2}\cdot KN\cdot DM=\frac{1}{2}\cdot\frac{a}{2}\cdot\frac{a\sqrt{3}}{12\cos\alpha}=\frac{\sqrt{3}\,a^2}{48\cos\alpha}$$

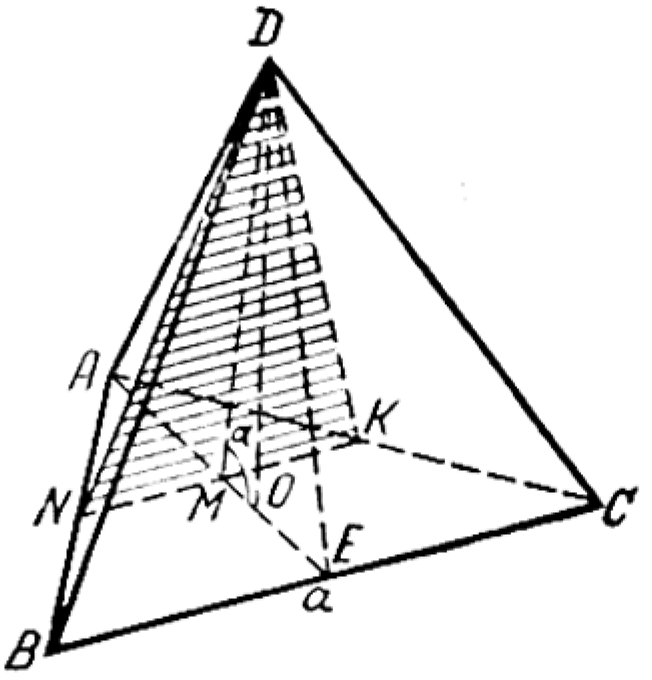


Fig. 114

The area of the base of the pyramid $DAKN$ is one fourth of the area of the base of the pyramid $DABC$, the two pyramids having a common altitude. Therefore the volume V_1 of the pyramid $DAKN$ is equal to $\frac{1}{4}V$, where V is the volume of the pyramid $DABC$. Consequently, the volume of the pyramid $DKNBC$ $V_2=\frac{3}{4}V$. The volume V is equal to $V=\frac{1}{3}\cdot S_{base}\cdot H=\frac{1}{3}\cdot\frac{\sqrt{3}}{4}\times$ $\times\, a^2\cdot\frac{a\sqrt{3}}{12}\tan\alpha$.

Answer: $S=\dfrac{\sqrt{3}\,a^2}{48\cos\alpha}$

$$V_1=\frac{a^3}{192}\tan\alpha$$

$$V_2=\frac{a^3}{64}\tan\alpha$$

---

*For depicting a regular triangular pyramid see Fig. 82.

637. By hypothesis $BE : EA = 2 : 1$ (Fig. 115). The section figure is ΔDEC. Find its area S. The triangle DEC is an isosceles one, since $EC = ED$ as corresponding sides of congruent triangles AEC and AED ($AC = AD$; AE is a common side and $\angle CAE = \angle DAE = 60°$). Draw its altitude EN; then

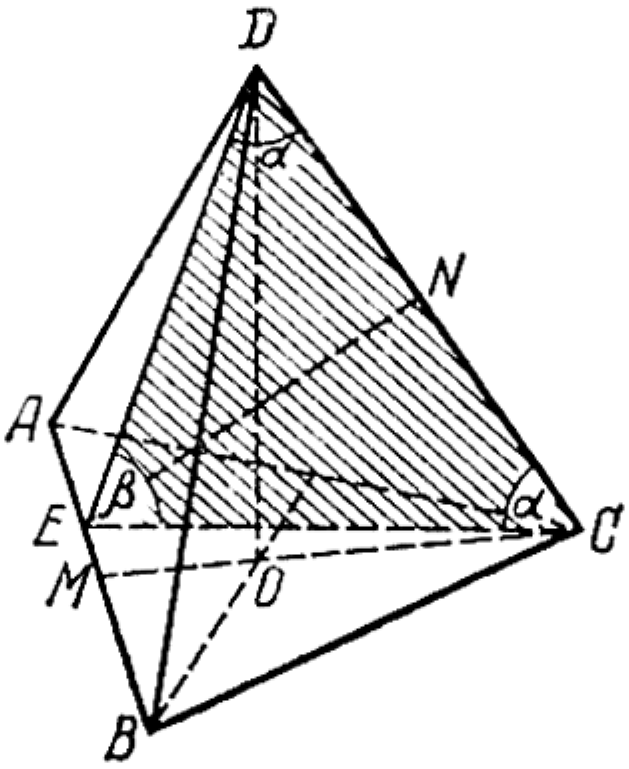


Fig. 115

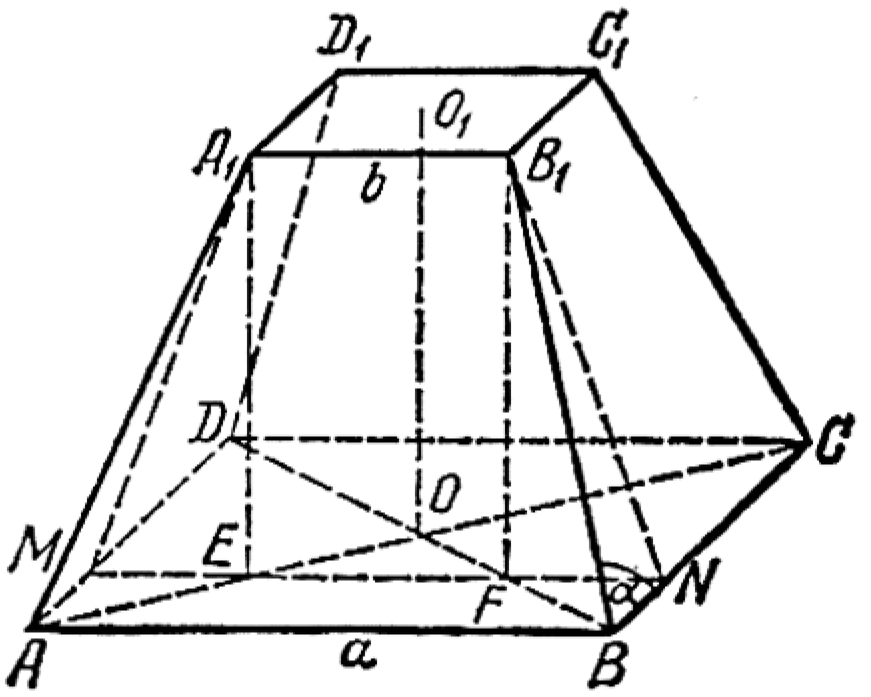


Fig. 116

$S = \frac{a \cdot EN}{2}$. To determine EN first find EC from the triangle ACE (by the law of cosines):

$$EC^2 = AC^2 + AE^2 - 2 \cdot AE \cdot AC \cdot \cos 60° = \frac{7}{9} a^2$$

Now from ΔENC we find

$$EN = \sqrt{EC^2 - NC^2} = \sqrt{\frac{7}{9} a^2 - \frac{a^2}{4}} = \frac{a}{6} \sqrt{19}$$

Denote the section angles $\angle ECD = \angle EDC$ by α. Then $\angle CED = \pi - 2\alpha$. From the triangle CEN we have

$$\cos \alpha = \frac{CN}{EC} = \frac{3}{2\sqrt{7}}$$

Answer: $S = \frac{\sqrt{19}\, a^2}{12}$; $\alpha = \arccos \frac{3}{2\sqrt{7}}$; $\beta = \pi - 2 \arccos \frac{3}{2\sqrt{7}}$.

638.* The lateral face BCC_1B_1 (Fig. 116) is an isosceles trapezoid with the bases $BC = a$ and $B_1C_1 = b$ $(a > b)$ and angle α at the base a. The line-segment B_1N is its altitude. We find $B_1N = \frac{a-b}{2} \tan \alpha$. From the triangle B_1NF, where $FN = \frac{a-b}{2}$, we find

$$H = B_1F = \sqrt{NB_1^2 - FN^2} = \frac{a-b}{2} \sqrt{\tan^2 \alpha - 1}$$

The volume

$$V = \frac{H}{3} (a^2 + b^2 + ab) = \frac{a^3 - b^3}{6} \sqrt{\tan^2 \alpha - 1}$$

---

* For depicting a truncated pyramid see Problems 625 and 626.

Note 1. If the acute angle α is less than 45°, the radicand is negative. But the angle α cannot be less than 45°. Indeed, the sum of the plane angles $BCC_1 = \alpha$ and $DCC_1 = \alpha$ of the trihedral angle C always exceeds the third plane angle BCD; but $\angle BCD = 90°$, therefore $2\alpha > 90°$, i.e. $\alpha > 45°$.

Note 2. The expression $\sqrt{\tan^2\alpha - 1}$ can be transformed to the form

$$\sqrt{\frac{\sin^2\alpha - \cos^2\alpha}{\cos^2\alpha}} = \frac{\sqrt{-\cos 2\alpha}}{\cos\alpha}$$

Since 2α is more than 90° (but less than 180°, since α is an acute angle), $\cos 2\alpha$ is always negative. Hence, the radicand $(-\cos 2\alpha)$ is always positive.

Answer: $V = \dfrac{a^3 - b^3}{6\cos\alpha} \cdot \sqrt{-\cos 2\alpha} = \dfrac{a^3 - b^3}{6\cos\alpha} \sqrt{\cos(180° - 2\alpha)}$.

639. The projection of the diagonal BD_1 (Fig. 117) onto the lateral face BCC_1B_1 is BC_1. Therefore $\angle C_1BD_1 = \alpha$. From the triangle BC_1D_1, where $D_1C_1 = b$, we find $BC_1 = b \cot\alpha$. From the triangle B_1C_1B we have

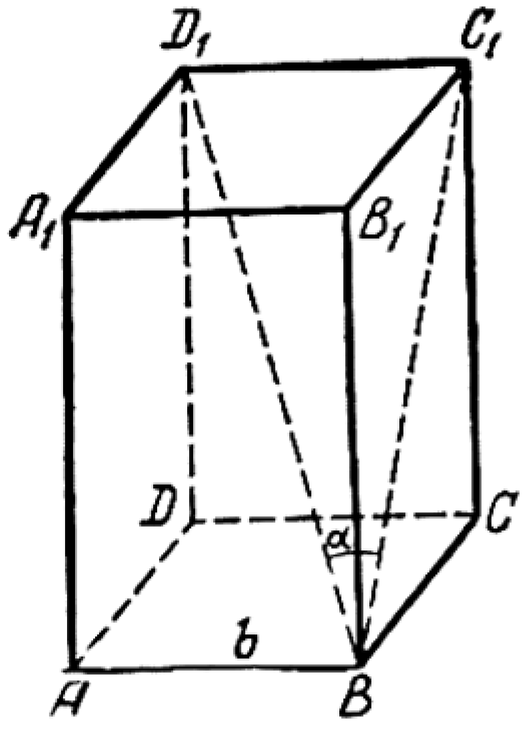


Fig. 117

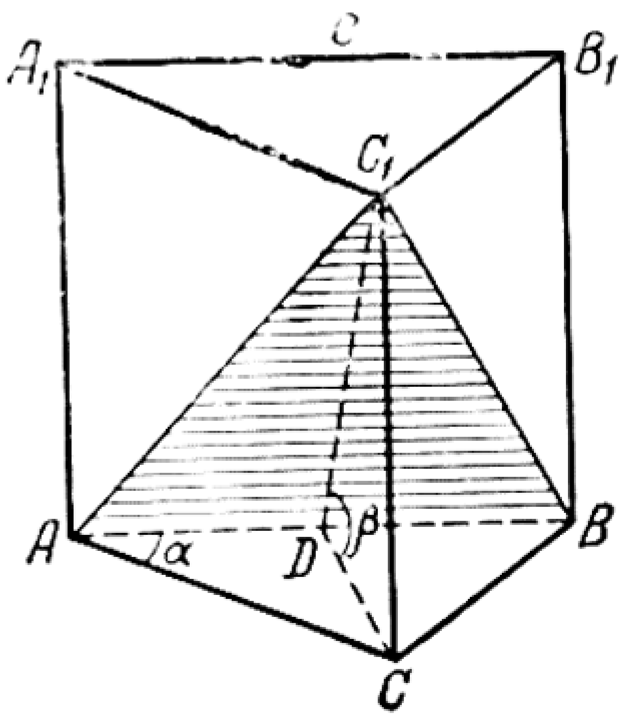


Fig. 118

$$H = \sqrt{BC_1^2 - B_1C_1^2} = \sqrt{b^2\cot^2\alpha - b^2} = \frac{b\sqrt{\cos 2\alpha}}{\sin\alpha}$$

Then

$$V = b^2H = b^3\frac{\sqrt{\cos 2\alpha}}{\sin\alpha}$$

Note. The radicand $\cos 2\alpha$ is always positive here (see Note 2 to Problem 638), since $\alpha < 45°$. Indeed,

$$\tan\alpha = \frac{D_1C_1}{BC_1} = \frac{B_1C_1}{BC_1}$$

But B_1C_1 is a leg, and BC_1 is the hypotenuse of the triangle BB_1C_1. Therefore $\tan\alpha < 1$, i.e. $\alpha < 45°$.

Answer: $V = b^3\dfrac{\sqrt{\cos 2\alpha}}{\sin\alpha}$.

640. If CD (Fig. 118) is the altitude of the triangle ABC dropped onto the hypotenuse $AB = c$ (CD may be drawn inside the angle ACB arbitrarily), then

$\angle CDC_1 = \beta$ (prove it!). We have

$$CD = AB \cdot \sin\alpha \cos\alpha = \frac{1}{2} c \sin 2\alpha$$

and

$$H = CC_1 = CD \cdot \tan\beta$$

Substitute these expressions into the formula

$$V = \frac{1}{3} SH = \frac{1}{3} \cdot \frac{1}{2} c \cdot CD \cdot H$$

Answer: $V = \frac{1}{24} c^3 \sin^2 2\alpha \tan\beta$.

641. One of the portions of the prism is a triangular pyramid B_1ABC (Fig. 119). Its volume $V_1 = \frac{1}{3} V$, where V is the volume of the prism. Hence, the volume V_2 of the other portion (of the quadrangular pyramid $B_1A_1C_1CA$) is equal to $\frac{2}{3} V$. Find V.

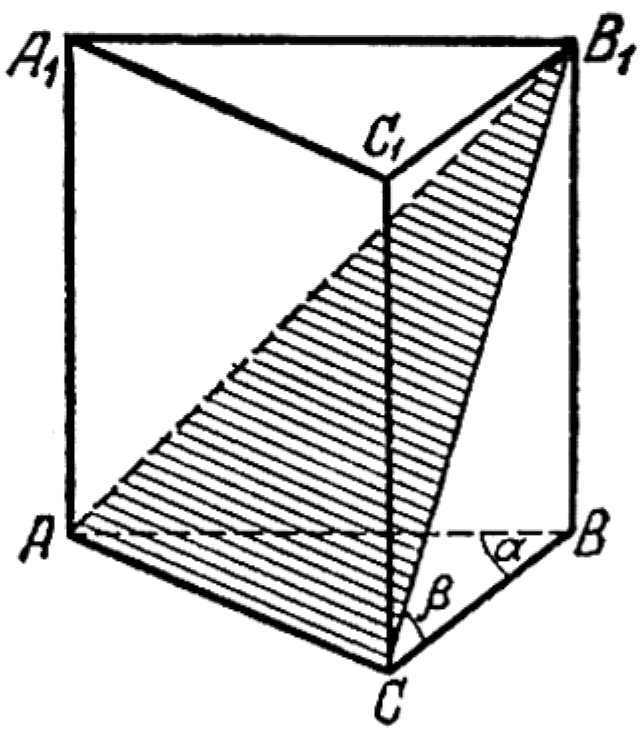


Fig. 119

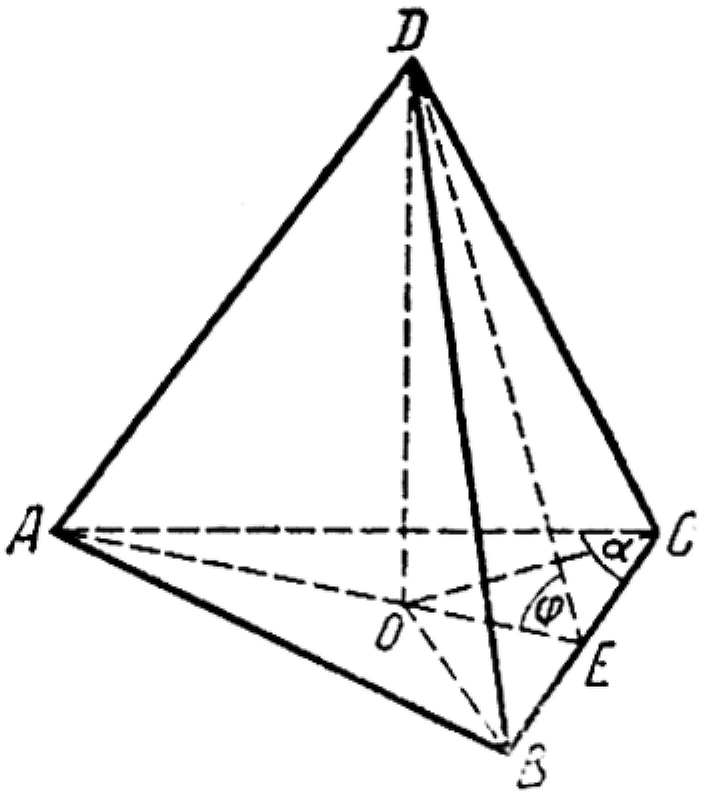


Fig. 120

By hypothesis $BC + AB = m$, and from the triangle ABC we find $BC = AB \cdot \cos\alpha$. Consequently,

$$BC = \frac{m \cos\alpha}{1 + \cos\alpha} = \frac{m \cos\alpha}{2 \cos^2 \frac{\alpha}{2}}$$

The area S of the prism base is equal to

$$S = \frac{1}{2} \cdot AC \cdot BC = \frac{1}{2} \cdot BC^2 \cdot \tan\alpha$$

The altitude $H = BB_1$ is determined from the $\triangle BCB_1$, where $\angle BCB_1 = \beta$ (prove it!). We get $H = BC \cdot \tan\beta$.

Answer: $V_1 = \dfrac{m^3 \cos^3\alpha \tan\alpha \tan\beta}{48 \cos^6 \frac{\alpha}{2}}$; $V_2 = \dfrac{m^3 \cos^3\alpha \tan\alpha \tan\beta}{24 \cos^6 \frac{\alpha}{2}}$.

642. According to the Preliminary Notes to Problem 617, $S_{base} = S \cos\varphi = S \sin\alpha$. On the other hand, $S_{base} = \frac{a^2 \tan\alpha}{4}$. Equating these two expressions, we get $a = 2\sqrt{S\cos\alpha}$. Point O (the centre of the circle inscribed in the triangle ABC, Fig. 120) lies at the point of intersection of the bisectors of the angles of the triangle, hence,

$$\angle OCE = \frac{\alpha}{2}$$

and

$$OE = EC\cdot\tan\frac{\alpha}{2} = \frac{a}{2}\tan\frac{\alpha}{2}$$

from $\triangle DOE$ we find

$$H = OE\cdot\tan\varphi$$

Answer: $V = \frac{1}{3}(S\cos\alpha)^{\frac{3}{2}}\tan\frac{\alpha}{2}$

$$S_{total} = S(1+\cos\varphi) = 2S\cos^2\left(45^\circ - \frac{\alpha}{2}\right).$$

643. In Fig. 121 $OA = OC = R$ are radii of the circle circumscribed about the isosceles triangle ABC ($AB = AC = a$). By virtue of the condition $\alpha > 45^\circ$ the centre O lies inside the triangle ABC (at $\alpha < 45^\circ$ the angle $A = 180^\circ - 2\alpha$ would be obtuse, the centre of the circumscribed circle would lie outside the triangle ABC, and then the plane drawn through the altitude of the pyramid and vertex C would yield no section). The altitude of the pyramid passes through the centre O (see the Preliminary Notes to Problem 611).

From the triangle AOD we have $H = R\tan\beta$. Since by the law of sines $AC = a = 2R\sin\alpha$, $H = \frac{a}{2\sin\alpha}\tan\beta$.

Let us find the base CE of the section figure from the triangle ACE, in which $\angle CAE = 180^\circ - 2\alpha$ and $\angle ACE$ at the base of the isosceles triangle AOC ($AO = OC = R$) is equal to $\angle CAO = \frac{1}{2}\angle CAE = 90^\circ - \alpha$. Hence, $\angle AEC = 3\alpha - 90^\circ$. By the law of sines $\frac{CE}{\sin(180^\circ - 2\alpha)} = \frac{a}{\sin(3\alpha - 90^\circ)}$, whence

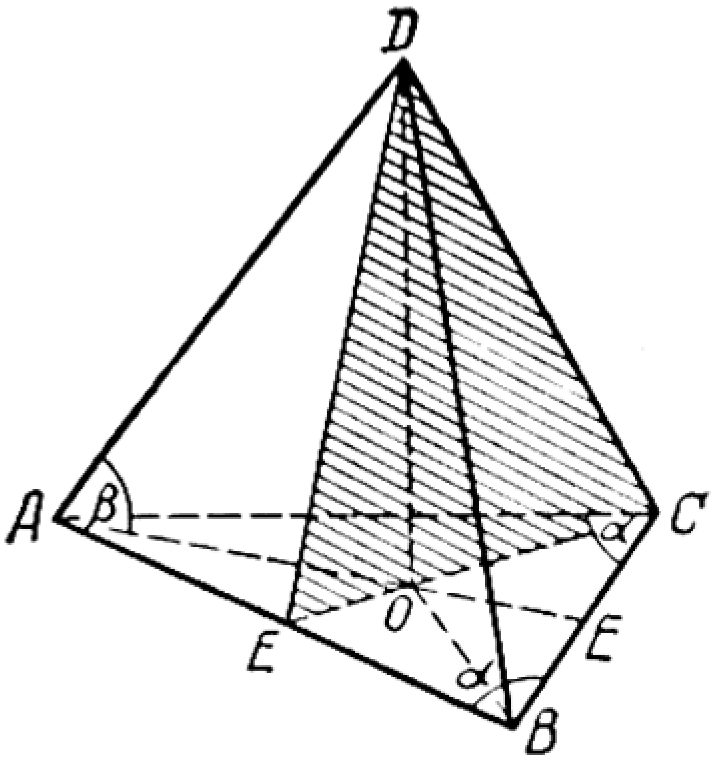


Fig. 121

$$CE = \frac{a\sin(180^\circ - 2\alpha)}{\sin(3\alpha - 90^\circ)} = \frac{a\sin 2\alpha}{\sin(3\alpha - 90^\circ)}$$

Note. We may write $(-\cos 3\alpha)$ in the denominator; but the angle 3α is contained between 135° and 270°, since $45^\circ < \alpha < 90^\circ$. Thus, $(-\cos 3\alpha)$ is a positive number. Therefore, when performing computations with the aid of tables, it is more convenient to deal with the angle $3\alpha - 90^\circ$ contained between 45° and 180°.

Answer: $S = \frac{a^2\cos\alpha\tan\beta}{2\sin(3\alpha - 90^\circ)}$.

644. (1) Find the area Q of the base of the prism (Fig. 122). We have: $Q = S_1 + S_2$, where S_1 is the area of the right-angled triangle ABC, and S_2 is the area of the right-angled triangle ADC,

$$S_1 = \frac{AB \cdot BC}{2} = \frac{l \sin \alpha \cdot l \cos \alpha}{2} = \frac{l^2 \sin 2\alpha}{4}$$

and

$$S_2 = \frac{l^2 \sin 2\beta}{4}$$

Hence

$$Q = \frac{l^2}{4} (\sin 2\alpha + \sin 2\beta) = \frac{l^2 \sin (\alpha + \beta) \cos (\alpha - \beta)}{2}$$

(2) Find the altitude H of the prism from the condition $S = BD \cdot H$. Since in the quadrilateral $ABCD$ the sum of the angles at the vertices B and D is equal to 180°, it can be inscribed in a circle of diameter equal to the diagonal AC because the latter subtends the inscribed right angles. From the triangle BCD, inscribed in the circle, we find (by the law of sines)

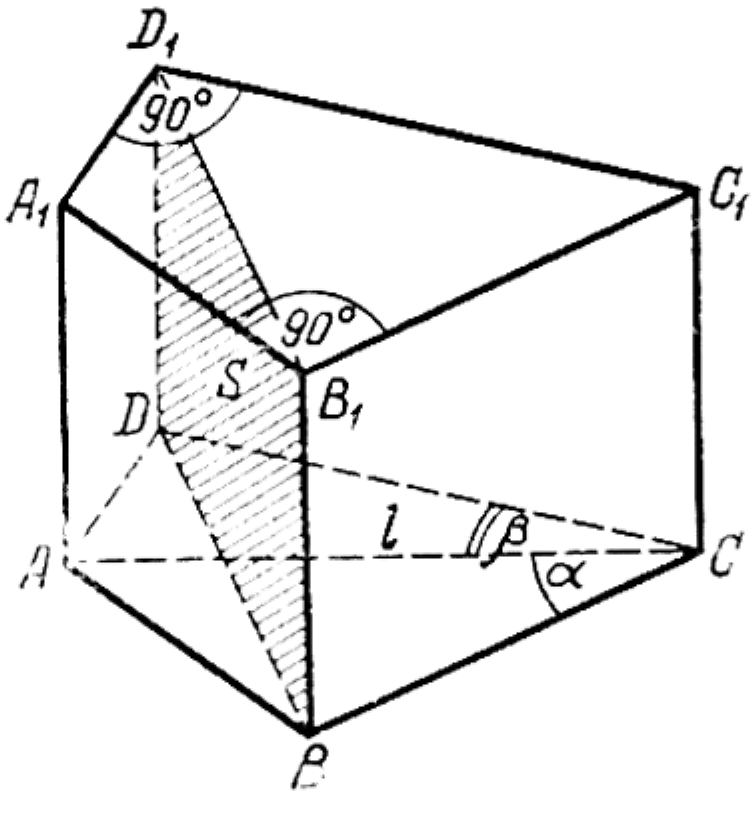


Fig. 122

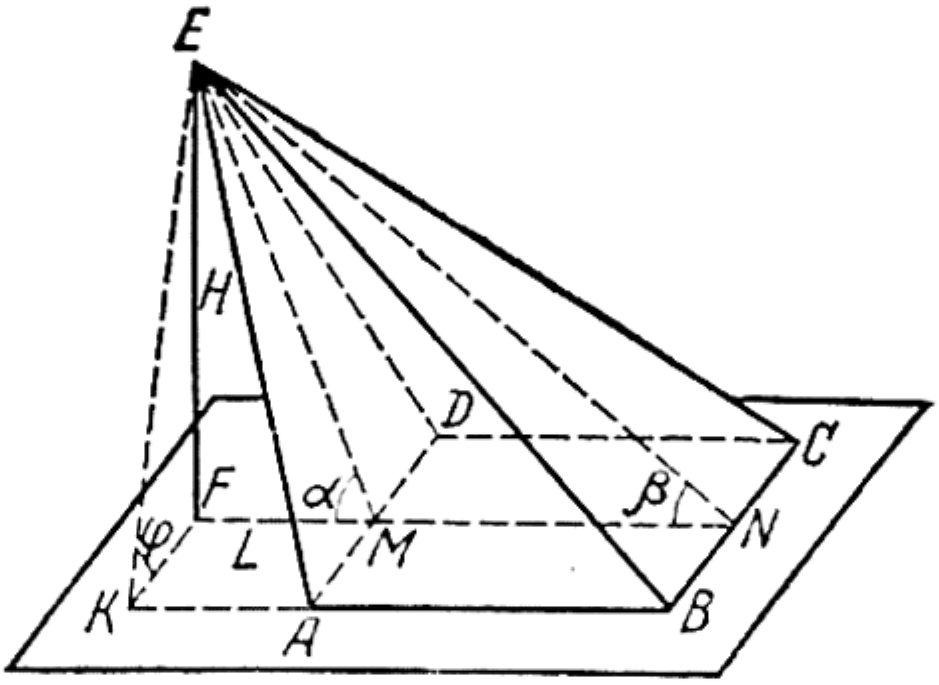


Fig. 123

$$BD = AC \cdot \sin \angle DCB = l \sin (\alpha + \beta)$$

Hence,

$$H = \frac{S}{BD} = \frac{S}{l \sin (\alpha + \beta)}$$

Answer: $V = \frac{1}{2} S \cdot l \cos (\alpha - \beta)$.

645. The faces ADE and BCE (Fig. 123) are isosceles triangles. The plane EMN (M and N are the midpoints of the edges AD and BC) is perpendicular to BC and AD and passes through the altitude EF of the pyramid (prove it!). By hypothesis the exterior angle $\alpha = \angle EML$ of the triangle EMN is an acute one. Therefore, the altitude EF intersects the extension of MN.

To determine V find the side AB of the square $ABCD$. We have

$$AB = MN = NF - MF = H (\cot \beta - \cot \alpha)$$

Hence,

$$V=\frac{1}{3}AB^2\cdot H=\frac{1}{3}H^3(\cot\beta-\cot\alpha)^2$$

Let us now construct the plane angle φ of the dihedral angle, at which the face ABE is inclined to the base. To this end intersect the dihedral angle by a plane EFK, which is perpendicular to the edge AB. To depict it draw $FK \parallel AD$ to intersect the extension of the edge AB (prove it!). From the triangle EFK we find

$$\tan\varphi=\frac{H}{FK}=\frac{2H}{AB}=\frac{2}{\cot\beta-\cot\alpha}.$$

Answer: $V=\frac{1}{3}H^3(\cot\beta-\cot\alpha)^2=$

$$=\frac{1}{3}H^3\frac{\sin^2(\alpha-\beta)}{\sin^2\alpha\sin^2\beta}$$

$$\varphi=\arctan\frac{2}{\cot\beta-\cot\alpha}=$$

$$=\arctan\frac{2\sin\alpha\sin\beta}{\sin(\alpha-\beta)}.$$

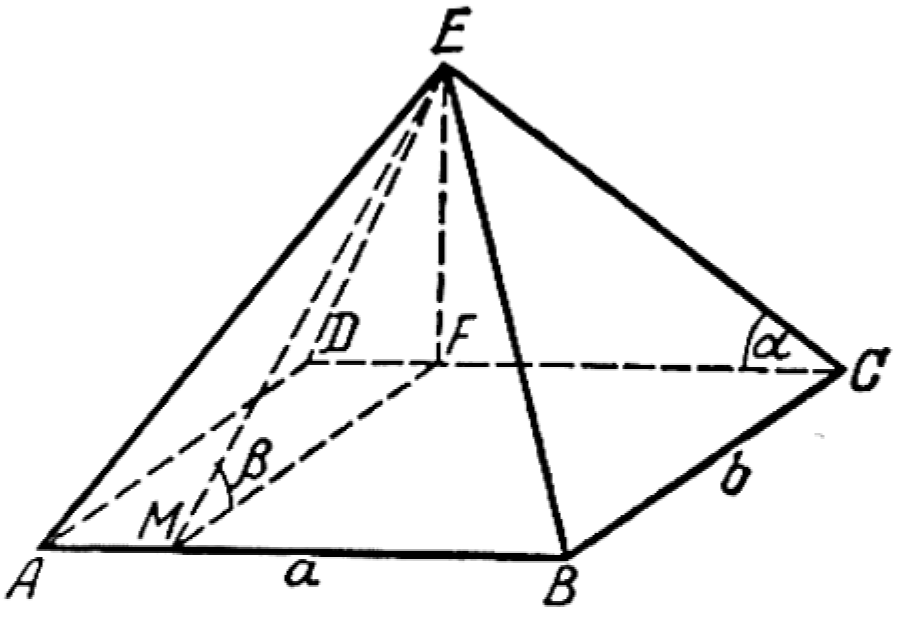


Fig. 124

646. The altitude EF of the pyramid (Fig. 124) lies in the face CED which is perpendicular to the base. The plane, drawn through EF and perpendicular to the edge AB, intersects the base of the pyramid along $MF \parallel BC$ and the lateral face AEB—along ME perpendicular to AB ($\angle EMF=\beta$). Since AD and BC are perpendicular to the plane DEC, $\angle BCE=90^\circ$ and $\angle ADE=90^\circ$ (all this should be proved).

Let us find the altitude $H=EF$. By hypothesis $EF+EM=m$; furthermore $EM=\frac{EF}{\sin\beta}$. Therefore $EF\left(1+\frac{1}{\sin\beta}\right)=m$, whence

$$H=EF=m:\left(1+\frac{1}{\sin\beta}\right)=m:\left(1+\frac{1}{\cos\alpha}\right)=\frac{m\cos\alpha}{2\cos^2\frac{\alpha}{2}}$$

Then from the right-angled triangle DEC we find

$$a=DC=\frac{EC}{\cos\alpha}=\frac{H}{\sin\alpha\cos\alpha}$$

Finally, we find

$$b=BC=MF=H\cot\beta=H\tan\alpha$$

Hence

$$V=\frac{1}{3}Hab=\frac{1}{3}H^3\frac{\tan\alpha}{\sin\alpha\cos\alpha}=\frac{H^3}{3\cos^2\alpha}$$

The sum S_1+S_2 of the areas of the lateral faces BEC and AED is equal to

$$\frac{1}{2}BC\cdot EC+\frac{1}{2}AD\cdot ED=\frac{1}{2}b(EC+ED)=\frac{1}{2}b\left(\frac{H}{\sin\alpha}+\frac{H}{\cos\alpha}\right)$$

Answer: $V=\dfrac{m^3\cos\alpha}{24\cos^6\dfrac{\alpha}{2}}$

$$S_1+S_2=\frac{m^2(\sin\alpha+\cos\alpha)}{8\cos^4\dfrac{\alpha}{2}}=\frac{m^2\cos(45^\circ-\alpha)}{4\sqrt{2}\cos^4\dfrac{\alpha}{2}}.$$

647. (a) *Drawing*. Construct the altitude EF (Fig. 125), joining E to the midpoint F of the side DC. Join the vertex E to the midpoint M of the side AB. Then $\varphi=\angle FEM$ represents the angle between the faces ABE and DCE (prove it!).

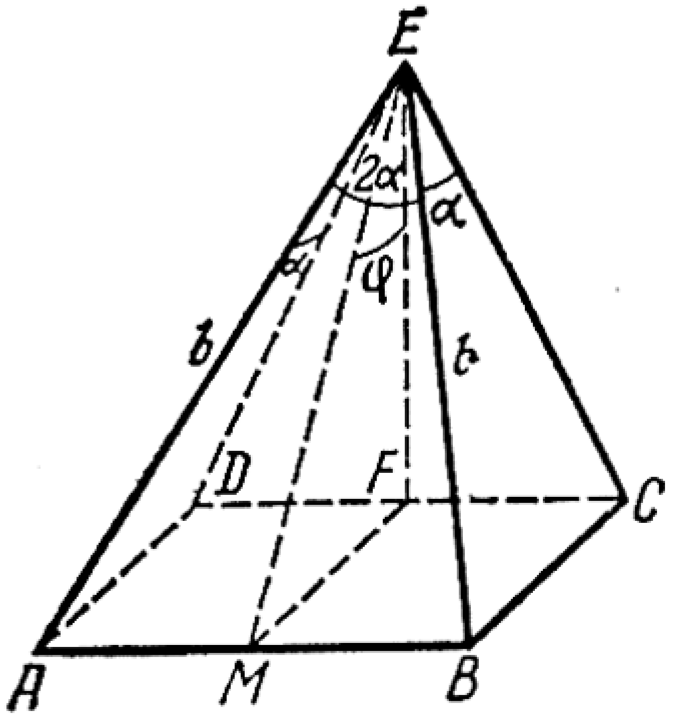


Fig. 125

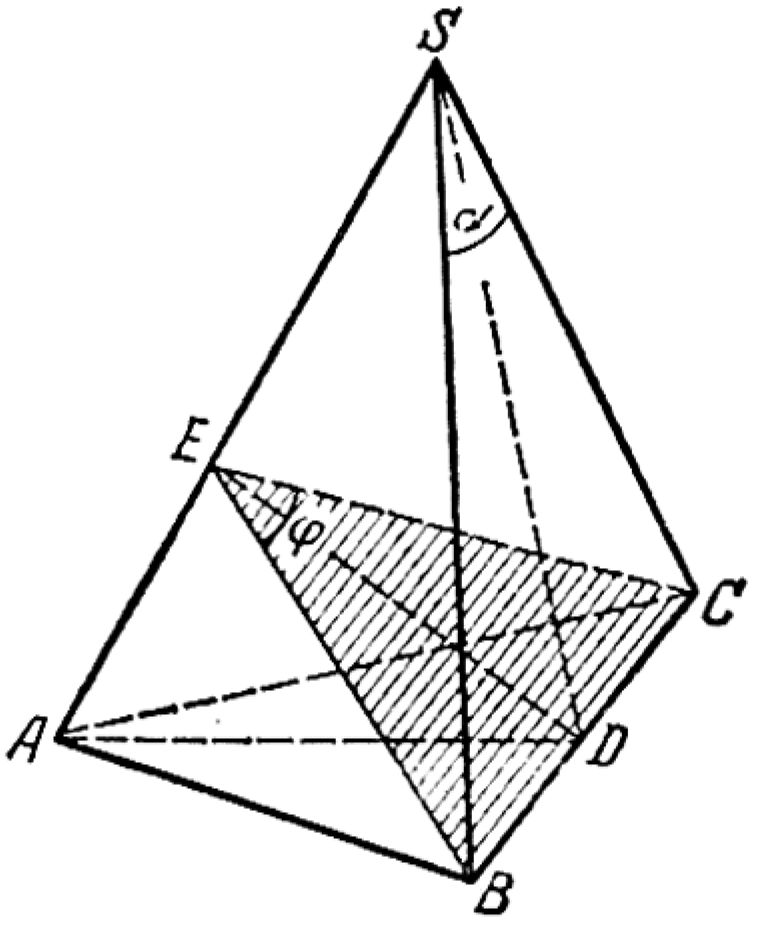


Fig. 126

(b) *Solution*. The triangle BCE is a right-angled one, in it $\angle BEC=\alpha$ (prove it!). Hence, $BC=b\sin\alpha$. From the triangle ABE we have $AB=2b\sin\alpha$ and $ME=b\cos\alpha$. From the triangle MFE, where $MF=BC=b\sin\alpha$, we find

$$FE=\sqrt{ME^2-MF^2}=b\sqrt{\cos^2\alpha-\sin^2\alpha}=b\sqrt{\cos 2\alpha}$$

Note. The radicand $\cos 2\alpha$ is always positive here, since $2\alpha<90^\circ$. Indeed, the sum of two face angles of the trihedral angle at the vertex B $\left(\angle ABE=\dfrac{180^\circ-2\alpha}{2}\text{ and }\angle CBE=90^\circ-\alpha\right)$ exceeds the third one ($\angle ABC=90^\circ$), i.e. $\dfrac{180^\circ-2\alpha}{2}+(90^\circ-\alpha)>90^\circ$, since $2\alpha<90^\circ$.

It is best of all to find the angle φ by its sine.

Answer: $V=\dfrac{2}{3}b^3\sin^2\alpha\sqrt{\cos 2\alpha}$; $\varphi=\arcsin(\tan\alpha)$.

648. The plane BCE (Fig. 126) is drawn through the side BC perpendicular to the edge AS. The dihedral angles between the lateral faces (all of them being

of the same value) are measured by the angle $BEC = \varphi$. The triangle BEC is an isosceles one.

To determine the area S of the section figure and the angle φ it is sufficient to find DE (D is the midpoint of BC). For this purpose we consecutively find BS (from the triangle BSD, where $BD = \frac{a}{2}$ and $\angle BSD = \frac{\alpha}{2}$), then BE (from the triangle BSE, where $\angle BSE = \alpha$), and, finally, $DE = \sqrt{BE^2 - BD^2}$. We get

$$DE = a\sqrt{\cos^2\frac{\alpha}{2} - \frac{1}{4}}$$

Now we find

$$S = \frac{a}{2}\cdot DE = \frac{a^2}{2}\sqrt{\cos^2\frac{\alpha}{2} - \frac{1}{4}}$$

and

$$\sin\frac{\varphi}{2} = \frac{BD}{EB} = \frac{1}{2\cos\frac{\alpha}{2}}$$

Note 1. The sum of face angles at the vertex S is always less then 360°. Therefore $0 < \alpha < 120°$. At this condition $2\cos\frac{\alpha}{2} > 1$, i.e. $\frac{1}{2\cos\frac{\alpha}{2}} < 1$ and, hence, the equation $\sin\frac{\varphi}{2} = \frac{1}{2\cos\frac{\alpha}{2}}$ always has a solution.

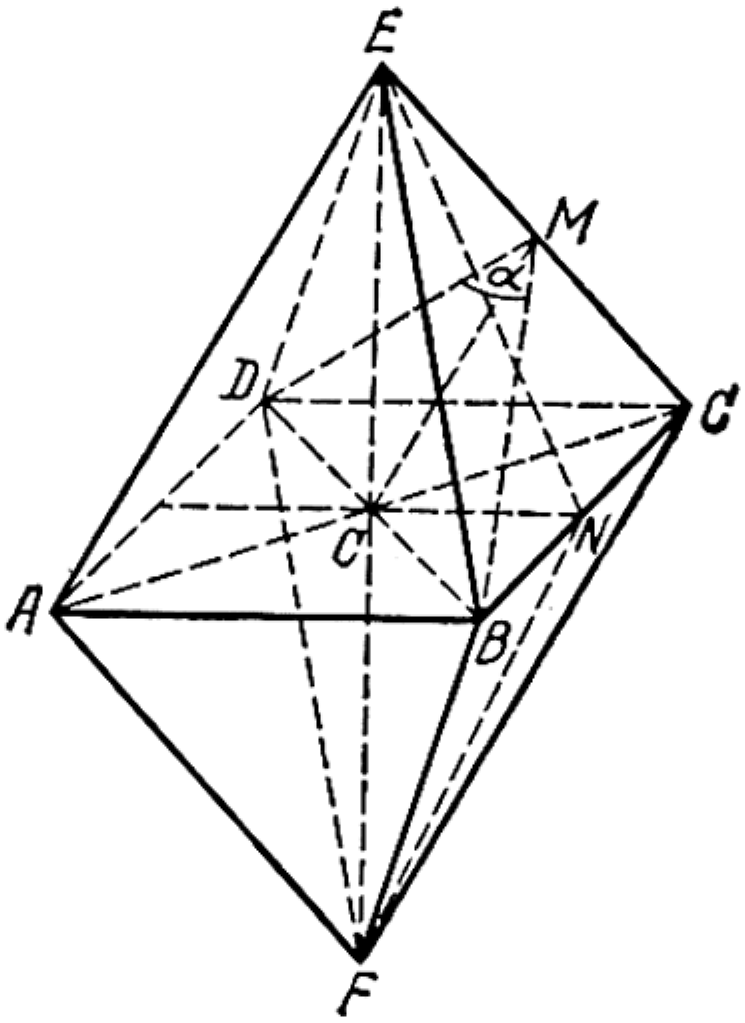


Fig. 127

Note 2. If $\alpha > 90°$, i.e. the angle ASB at the vertex of the lateral face is an obtuse one, then the altitude BE of the triangle ASB intersects the extension of the base, and the plane BEC gives no section of the pyramid. Nevertheless the formula

$$S = \frac{a^2}{2}\sqrt{\cos^2\frac{\alpha}{2} - \frac{1}{4}}$$

yields a definite value of S even with an obtuse angle α (less than 120°, see Note 1).

Answer: $\varphi = 2\arcsin\left(\frac{1}{2}\sec\frac{\alpha}{2}\right)$

$$S = \frac{a^2}{2}\sqrt{\cos^2\frac{\alpha}{2} - \frac{1}{4}} = \frac{a^2}{2}\sqrt{\sin\left(60° + \frac{\alpha}{2}\right)\sin\left(60° - \frac{\alpha}{2}\right)}$$

649. All eight faces of the octahedron are equilateral triangles, thus $NE = \frac{a\sqrt{3}}{2}$ (Fig. 127). The quadrilateral $ABCD$ is a square. The plane it is contained in divides the octahedron into two equal regular pyramids so that

$V=2\cdot\frac{1}{3}a^2\cdot OE$ where

$$OE=\sqrt{EN^2-ON^2}=\sqrt{\frac{3a^2}{4}-\frac{a^2}{4}}=\frac{a\sqrt{2}}{2}$$

All the dihedral angles of the octahedron are equal. The angle $\alpha=\angle BMD$ (M is the midpoint of CE) measures the dihedral angle at the edge CE (prove it!). From the triangle OMB we find

$$\sin\frac{\alpha}{2}=\frac{OB}{BM}=\frac{a\sqrt{2}}{2}:\frac{a\sqrt{3}}{2}=\sqrt{\frac{2}{3}}$$

Answer: $V=\frac{\sqrt{2}\,a^3}{3}$; $\alpha=2\arcsin\sqrt{\frac{2}{3}}$.

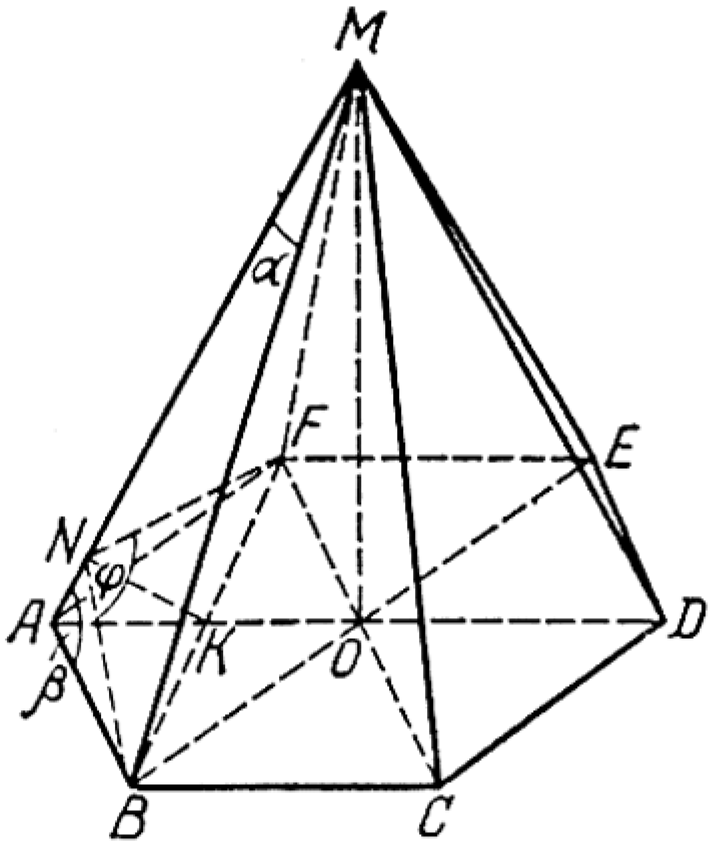


Fig. 128

650.* The isosceles triangles BMA and FMA (Fig. 128) are congruent. Therefore, their altitudes dropped from the vertices B and F pass through one and the same point N on their common side and are equal to each other: $BN=FN$. The angle BNF is equal to φ (prove it!). The angle $\beta=\angle BAM$ is expressed through the required angle $\alpha=\angle BMA$ by the formula

$$\beta=90^\circ-\frac{\alpha}{2}$$

First we find the trigonometric function of the angle β. From the right-angled triangle ABN we have $\sin\beta=\frac{BN}{a}$ (a is the side of the base). From the isosceles triangle BNF we find $BN=\frac{BK}{\sin\frac{\varphi}{2}}$. But $BK=\frac{a\sqrt{3}}{2}$ (as the altitude of the equilateral triangle ABO). Consequently,

$$\sin\beta=\frac{\sqrt{3}}{2\sin\frac{\varphi}{2}},$$

i.e.

$$\sin\left(90^\circ-\frac{\alpha}{2}\right)=\frac{\sqrt{3}}{2\sin\frac{\varphi}{2}}$$

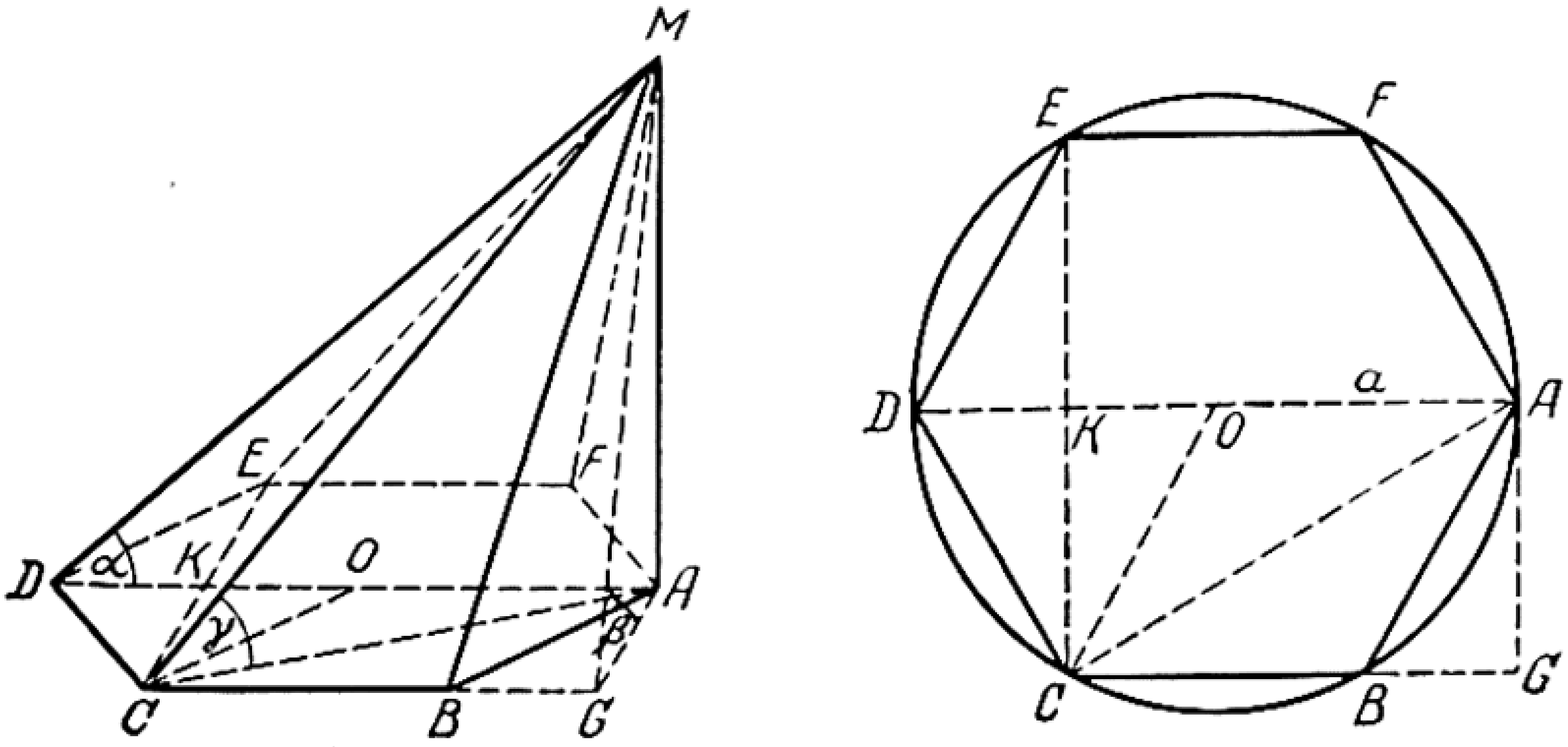
Note. The dihedral angle at the edge of a regular hexagonal pyramid always exceeds $\angle FAB$ (compare the triangles BNF and BAF), i.e. it is more than 120°. Therefore the quantity $\frac{\sqrt{3}}{2\sin\frac{\varphi}{2}}$ is always less than unity.

---

* For drawing a regular hexagon see the Note to Problem 598.

Answer: $\alpha = 2 \arccos \dfrac{\sqrt{3}}{2 \sin \frac{\varphi}{2}}$.

651. The faces AMF and AMB (Fig. 129*a*) passing through the edge AM (perpendicular to the plane $ABCDEF$) form right angles with the plane of the base. Find the total sum of the angles formed by the faces EMF and CMB with the plane of the base. Drop a perpendicular AG from A to CB (this line should be parallel to CE, see Fig. 129*b*). Then $\beta = \angle AGM$ (prove it!). We have $\tan \beta = \dfrac{H}{AG}$, where $AG = CK = \dfrac{a\sqrt{3}}{2}$ (Fig. 129*b*). But from the triangle AMD we have $\tan \alpha = \dfrac{H}{2a}$; hence,



(*a*) (*b*)

Fig. 129

$$\tan \beta = \frac{2H}{a\sqrt{3}} = \frac{4 \tan \alpha}{\sqrt{3}}$$

Since $AC \perp DC$ (prove it!), $\gamma = \angle ACM$ is the plane angle of the dihedral angle at which the face DCM (as also DEM) is inclined to the plane of the base. From the triangle ACM we have $\tan \gamma = \dfrac{H}{AC}$, where $AC = a\sqrt{3}$ (Fig. 129*b*).

Answer: $\beta = \arctan \dfrac{4 \tan \alpha}{\sqrt{3}}$; $\gamma = \arctan \dfrac{2 \tan \alpha}{\sqrt{3}}$.

652. Through a straight line we can draw a plane perpendicular to another straight line only if these lines are perpendicular to each other. Let us prove that $BC \perp AS$ (Fig. 130). Draw a plane ASO through the edge AS and altitude SO. Since A and O belong to plane ASO and at the same time to the plane of the base ABC, these planes intersect along a straight line AO, i.e. along the altitude AD of the isosceles triangle ABC. The triangles OCD and OBD are congruent (prove it!), therefore $OB = OC$, consequently the inclined lines SC and SB are also equal to each other and, hence, SD being the median of the isosceles triangle BSC also serves as its altitude. Since, as has been proved, AD and SD are perpendicular to the edge BC, then the edge BC is perpendicular to the plane ADS and, hence, to AS lying in this plane, which completes the proof.

To draw through BC a plane perpendicular to AS it is sufficient to drop a perpendicular DE to AS. The plane BEC is perpendicular to the edge AS, since two straight lines lying on it (DE and BC) are perpendicular to AS. Cutting the dihedral angle the plane ADS which is perpendicular to the edge BC yields an angle ADE (the plane angle of this dihedral angle).

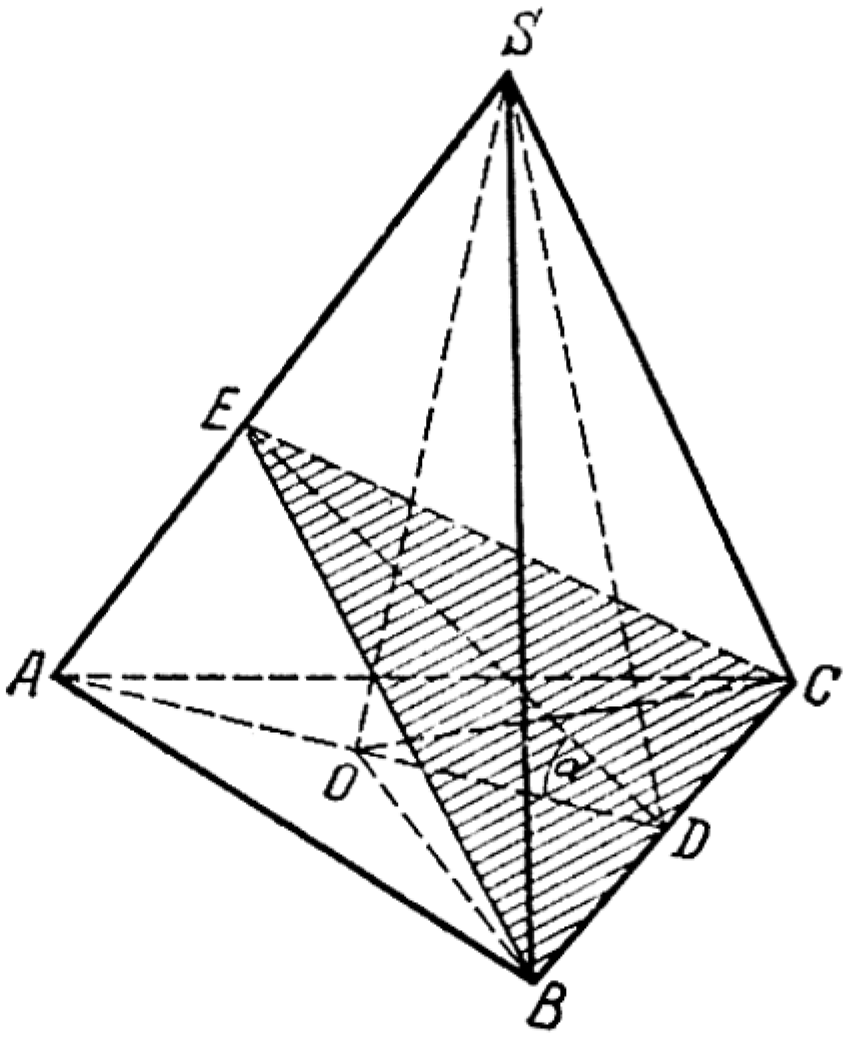


Fig. 130

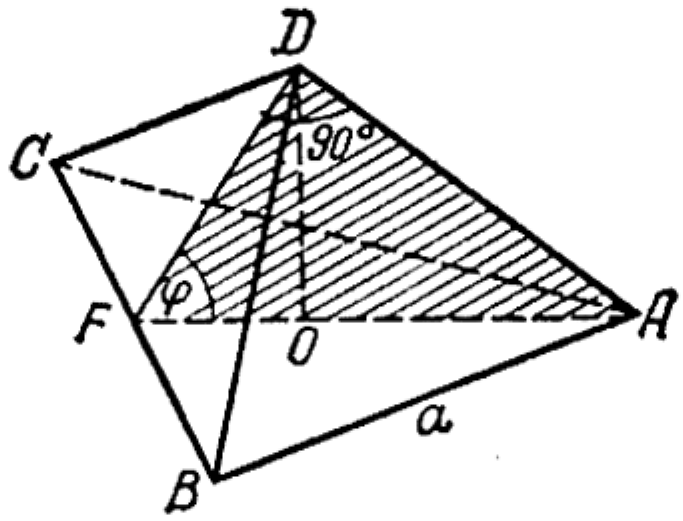


Fig. 131

The triangle ASD is an isosceles one (since the altitude SO passes through the midpoint of the base AD). Consequently,

$$\angle ASD = 2\ \angle ASO = 2\alpha$$

($\angle ASO = \angle ADE = \alpha$ as angles with perpendicular sides). The ratio of the volume V_1 of the pyramid $SBCE$ to the volume V of the pyramid $ABCE$ (these pyramids have a common base BCE) is equal to the ratio of their altitudes, i.e. $V_1 : V = SE : AE$. From the triangle DSE we have

$$SE = DE \cdot \cot \angle ESD = DE \cdot \cot 2\alpha$$

from the triangle AED we find

$$AE = DE \cdot \tan \alpha$$

Hence,

$$V_1 : V = \cot 2\alpha : \tan \alpha$$

Answer: $V_1 = V \cot \alpha \cot 2\alpha$.

653.* To draw a section bisecting the dihedral angle at the edge AD (Fig. 131) it is necessary to have the plane angle of this dihedral angle. Such is the angle BDC, since the plane BDC is perpendicular to the edge AD. Indeed, in any regular pyramid the lateral edge AD is perpendicular to the opposite side BC of the base (proved as in the preceding problem); furthermore, in the given case the edge AD is perpendicular to FD. Indeed, by hypothesis the triangle AFD is a right-angled one, and since its angles at the vertices A and F are necessarily acute, $\angle ADF$ is a right angle. Since $OF = \frac{1}{2} OA = \frac{1}{2} R$,

---

* For drawing a regular triangular pyramid see Problem **603**.

$$OD = \sqrt{OF \cdot OA} = \frac{R}{\sqrt{2}}$$

$\left(\text{where } R = \frac{a}{\sqrt{3}}\right)$. The angle $\varphi = \angle AFD$ measures the angle of inclination of the face BCD to the plane of the base. We have

$$\tan \varphi = \frac{OD}{OF} = \frac{R}{\sqrt{2}} : \frac{R}{2} = \sqrt{2}$$

Note. The lateral edge AD forms a right angle with the edge BD (and the edge CD); since the pyramid is a regular one, the edges BD and DC also form a right angle.

Answer: $V = \frac{a^3 \sqrt{2}}{24}$; $\varphi = \arctan \sqrt{2}$.

654.* The only quantity which remains unknown and is necessary to compute the total surface area of the pyramid is the slant height ND. It is determined in the following way: first find the line-segments AM and MD (Fig. 132) into which the edge AD is divided by the perpendicular NM (N is the midpoint of BC). Then from the triangle ANM, where $AN = \frac{q\sqrt{3}}{2}$, find MN, and finally, from the triangle NMD find ND.

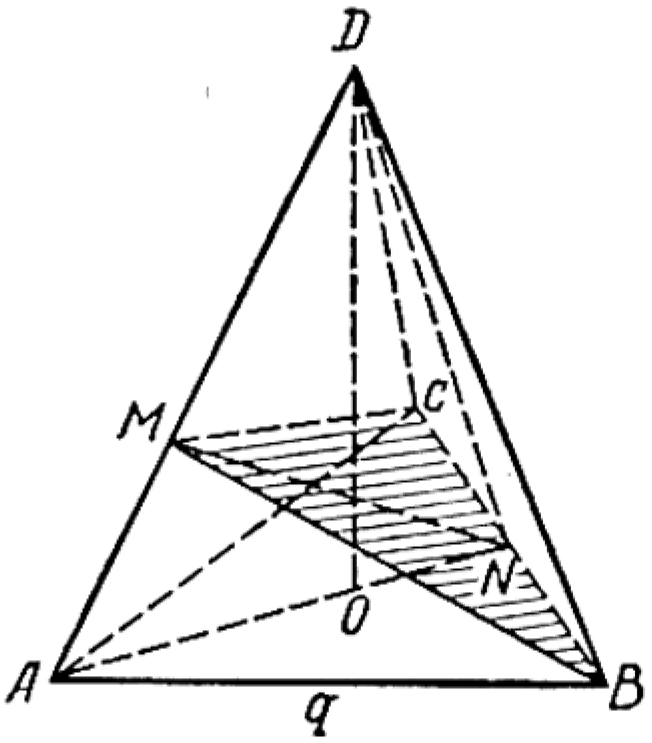


Fig. 132

From the given condition it is not clear which ratio — $AM : MD$ or $MD : AM$ — is equal to $m : n$, therefore we may put $MD = mx$, $MA = nx$, so that $AD = (m + n) x$. From similarity of the triangles AMN and ADO we have

$$\frac{AM}{AO} = \frac{AN}{AD}$$

where

$$AN = \frac{q\sqrt{3}}{2}$$

and

$$AO = \frac{2}{3} AN = \frac{q\sqrt{3}}{3}$$

We get the equation

$$nx(m+n)x = \frac{q\sqrt{3}}{2} \cdot \frac{q\sqrt{3}}{3}$$

where

$$x = \frac{q}{\sqrt{2n(m+n)}},$$

so that

$$MD=\frac{mq}{\sqrt{2n(m+n)}} \text{ and } AM=\frac{nq}{\sqrt{2n(m+n)}}.$$

Furthermore

$$MN^2=AN^2-AM^2=\frac{q^2(n+3m)}{4(m+n)}$$

and

$$ND^2=MD^2+MN^2=\frac{q^2(n+2m)}{4n}$$

Now we find

$$S_{total}=\frac{q^2\sqrt{3}}{4}+\frac{3q\cdot ND}{2}$$

Answer: $S_{total}=\frac{q^2\sqrt{3}}{4}\cdot\left[1+\sqrt{\frac{3(n+2m)}{n}}\right]$.

655. We have (Fig. 133): $\angle BD_1A=\alpha$ and $\angle BD_1C=\alpha$ (prove it!). The triangles BD_1A and BD_1C are congruent (prove it!). Consequently, the base $ABCD$ is a square with side $a=d\sin\alpha$. Then we find

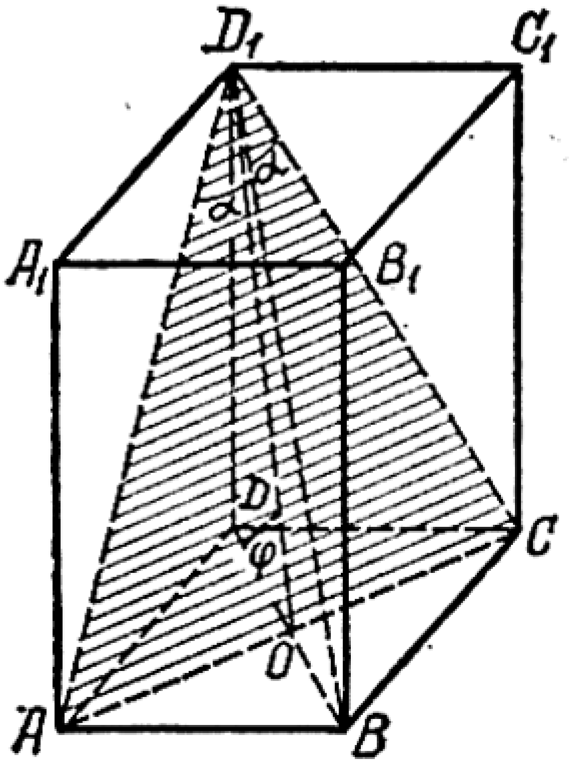


Fig. 133

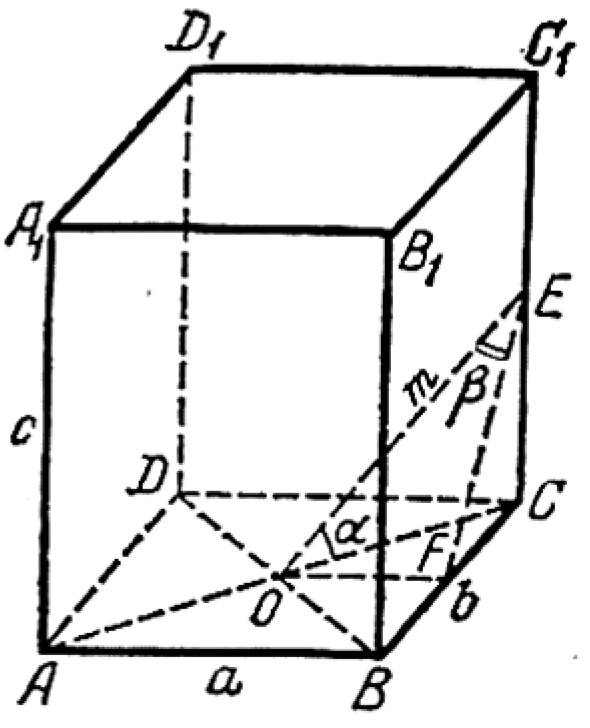


Fig. 134

$$AD_1=d\cos\alpha$$

and

$$H=\sqrt{AD_1^2-AD^2}=\sqrt{d^2\cos^2\alpha-d^2\sin^2\alpha}=d\sqrt{\cos 2\alpha}$$

The plane ACD_1 forms with the plane of the base the angle

$$\varphi=\angle DOD_1;\ \tan\varphi=\frac{DD_1}{OD}=H:\frac{a}{\sqrt{2}}.$$

Answer: $V=d^3\sin^2\alpha\sqrt{\cos 2\alpha}$; $\varphi=\arctan\left(\frac{\sqrt{2\cos 2\alpha}}{\sin\alpha}\right)$.

656. The angle $EOC=\alpha$ (Fig. 134). To construct the angle β formed by the line-segment OE with the lateral face BB_1C_1C draw $OF\perp BC$. Then FE is

the projection of OE on this face, and so $\angle OEF = \beta$. Let us introduce the following notation: $AB = a$, $BC = b$ and $CC_1 = c$, then $V = abc$ and $S_{lat} = 2(a + c)\,b$. From $\triangle OEF$ we have

$$\frac{a}{2} = OF = m \sin\beta = m \sin 2\alpha$$

$$FE = m\cos\beta = m\cos 2\alpha$$

from $\triangle\ OEC$ we have

$$\frac{c}{2} = EC = m\ \sin\ \alpha$$

from $\triangle\ FEC$ we have

$$\frac{b}{2} = FC = \sqrt{FE^2 - EC^2} = m\sqrt{\cos^2 2\alpha - \sin^3\alpha}$$

Reduce the radicand to the form convenient for taking logarithms:

$$\cos^2 2\alpha - \sin^2\alpha = \frac{1+\cos 4\alpha}{2} - \frac{1-\cos 2\alpha}{2} = \frac{\cos 4\alpha + \cos 2\alpha}{2} = \cos 3\alpha\cos\alpha$$

Hence,

$$b = 2m\sqrt{\cos 3\alpha\cos\alpha}$$

Note. The angle $\beta = \angle OEF$ is less than $\angle OEC = 90° - \alpha$ (compare their sines!). And since by hypothesis $\beta = 2\alpha$, then $2\alpha < 90° - \alpha$. Hence, it must be $\alpha < 30°$.

Answer: $V = 8m^3 \sin 2\alpha \sin\alpha\sqrt{\cos 3\alpha\cos\alpha}$

$$S_{lat} = 16\,m^2 \sin\frac{3\alpha}{2}\cos\frac{\alpha}{2}\sqrt{\cos 3\alpha\cos\alpha}$$

657. (a) *Drawing.* The semicircle is represented by a semiellipse (AB a diameter of the ellipse; Fig. 135[*]), DC is drawn parallel to AB. Straight lines perpendicular to AB, are represented by straight lines parallel to the tangent lines AM and BL.

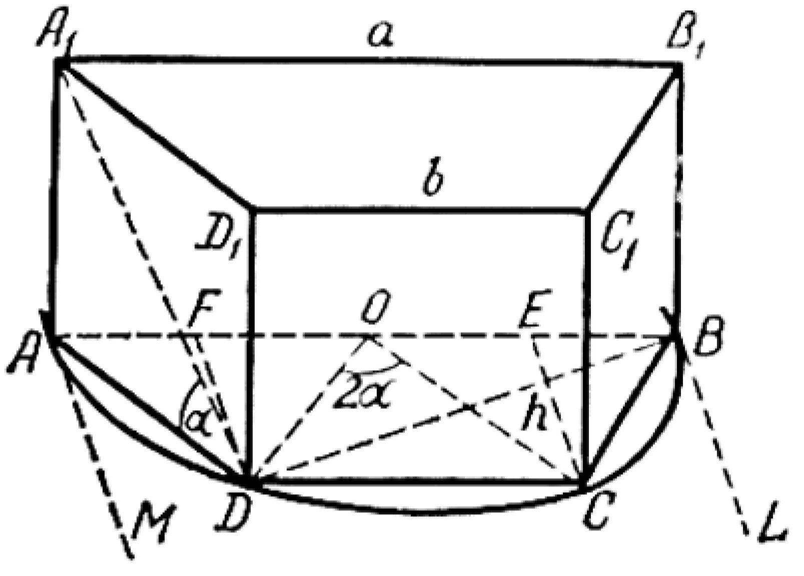


Fig. 135

(b) *Solution.* Let us introduce the following notation: $AB = a$; $DC = b$; $DF = CE = h$; then

$$V = \frac{a+b}{2}hH$$

By hypothesis $a = 2R$; the side b is found by the law of sines from the triangle BCD, in which $\angle DBC$ is measured by half the arc $\overset{\frown}{DC} = 2\alpha$; we have $b = 2R\sin\alpha$. From the triangle ODF, where $OD = R$ and $\angle AOD$ is measured by the arc $\overset{\frown}{AD} = \dfrac{180° - 2\alpha}{2} = 90° - \alpha$, we find

$$h = FD = R\sin(90° - \alpha) = R\cos\alpha$$

The altitude H is found from the triangle A_1AD, where $\angle A_1DA = \alpha$ (prove it!) and AD can be determined from the right-angled triangle ADB, where $\angle ABD$

---

[*] For constructing an ellipse see Problem 613.

subtended by the arc $\breve{AD}$ is equal to $\left(45^\circ - \frac{\alpha}{2}\right)$. We get

$$H = 2R \sin\left(45^\circ - \frac{\alpha}{2}\right) \tan\alpha$$

Consequently,

$$V = 2R^3 (1 + \sin\alpha) \sin\left(45^\circ - \frac{\alpha}{2}\right) \tan\alpha \cos\alpha$$

After a number of simplifications we get

$$1 + \sin\alpha = 2\cos^2\left(45^\circ - \frac{\alpha}{2}\right)$$

and so on.

Answer: $V = R^3 \sin 2\alpha \cos\left(45^\circ - \frac{\alpha}{2}\right)$.

658. The projection of the diagonal D_1B on the lateral face AA_1D_1D (Fig. 136) is AD_1; therefore $\angle AD_1B = \beta$. The angle α between the cutting plane DBB_1D_1

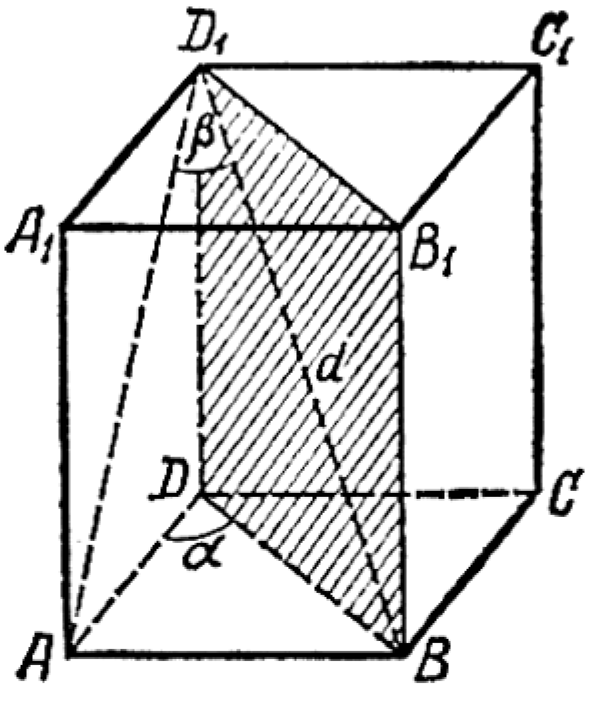


Fig. 136

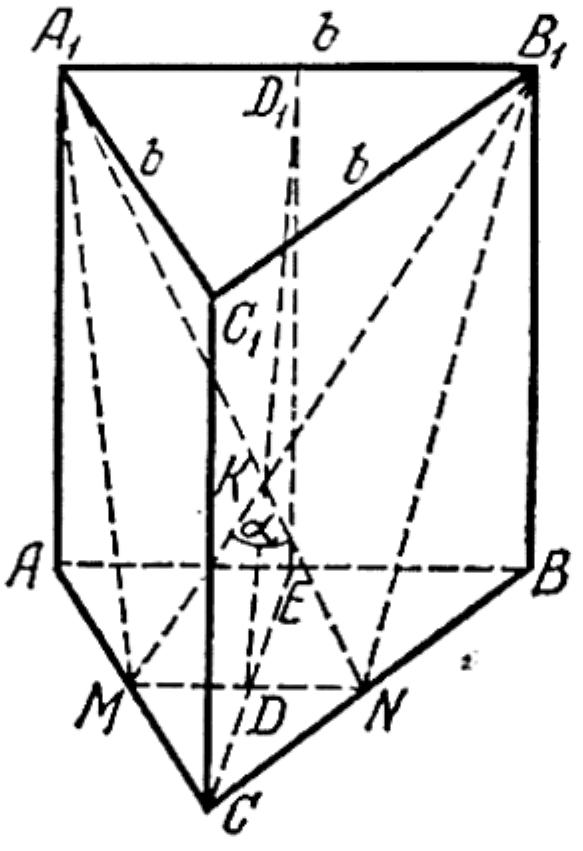


Fig. 137

and the face ADD_1A_1 is measured by the angle ADB (prove it!). From the triangle AD_1B we find AB and AD_1; from the triangle ABD we find AD. $DD_1 = H$ is determined from the triangle AD_1D

$$H = \sqrt{AD_1^2 - AD^2} = \sqrt{d^2\sin^2\alpha - d^2\cos^2\alpha\cot^2\alpha} = \frac{d}{\sin\alpha}\sqrt{\sin^4\alpha - \cos^4\alpha} =$$
$$= \frac{d}{\sin\alpha}\sqrt{-\cos 2\alpha}$$

Note. The angle β is always less than the angle α (compare their tangents!) Since by hypothesis $\beta = 90^\circ - \alpha$, we have $90^\circ - \alpha < \alpha$, hence, $\alpha > 45^\circ$. From the inequality

$$45^\circ < \alpha < 90^\circ$$

it follows that the angle 2α belongs to the second quadrant, and so $\cos 2\alpha < 0$, and $-\cos 2\alpha > 0$. For computation purposes it is convenient to substitute the expression $\cos(180° - 2\alpha)$ for $-\cos 2\alpha$, since the angle $180° - 2\alpha$ belongs to the first quadrant.

Answer: $V = d^3 \cos\alpha \cot^2\alpha \sqrt{\cos(180° - 2\alpha)}$

659. The drawn lines are A_1N and B_1M (Fig. 137). The quadrilateral A_1B_1NM is an isosceles trapezoid (prove it!). From the isosceles triangle MKN, where $\angle MKN = \alpha$ and $MN = \frac{b}{2}$, we have

$$KD = \frac{b}{4} \cot \frac{\alpha}{2}$$

From the triangle A_1KB_1 we find

$$KD_1 = \frac{b}{2} \cot \frac{\alpha}{2}$$

Adding these equalities, we obtain

$$DD_1 = \frac{3b}{4} \cot \frac{\alpha}{2}$$

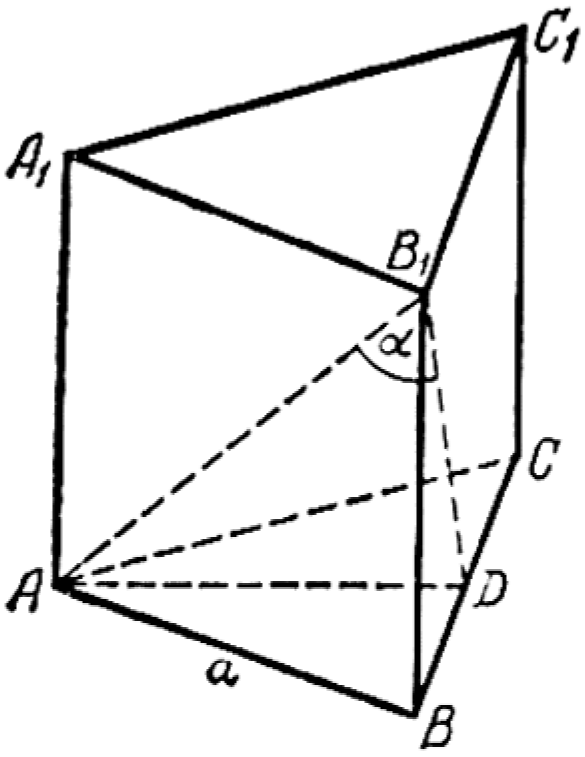


Fig. 138

From the triangle DED_1, where $DE = \frac{1}{2} CE = \frac{1}{4} b\sqrt{3}$, we find

$$H = ED_1 = \frac{3b}{4} \sqrt{\cot^2 \frac{\alpha}{2} - \frac{1}{3}} = \frac{3b}{4} \sqrt{\cot^2 \frac{\alpha}{2} - \cot^2 60°} =$$

$$= \frac{3b}{4} \sqrt{\left(\cot \frac{\alpha}{2} + \cot 60°\right)\left(\cot \frac{\alpha}{2} - \cot 60°\right)} =$$

$$= \frac{3b \sqrt{\sin\left(60° + \frac{\alpha}{2}\right) \sin\left(60° - \frac{\alpha}{2}\right)}}{4 \sin \frac{\alpha}{2} \sin 60°}.$$

Answer: $V = \frac{3b^3}{8 \sin \frac{\alpha}{2}} \sqrt{\sin\left(60° + \frac{\alpha}{2}\right) \sin\left(60° - \frac{\alpha}{2}\right)}$.

660. To construct the angle formed by the diagonal AB_1 and the lateral face BB_1C_1C we have to find the projection of AB_1 on this face (Fig. 138). The point A is projected into the midpoint D of BC (prove it!). The projection is B_1D, hence $\angle AB_1D = \alpha$. From $\triangle B_1BD$ we find

$$H = BB_1 = \sqrt{B_1D^2 - BD^2}$$

B_1D is found from the triangle AB_1D. The expression obtained for H is transformed in the same way as in the preceding problem.

Answer: $S_{lat} = \frac{3a^2 \sqrt{\sin(60° + \alpha) \sin(60° - \alpha)}}{\sin\alpha}$.

661. The projection of the diagonal AB_1 on the face AA_1C_1C is AC_1 (Fig. 139), hence, $\angle B_1AC_1=\beta$. The altitude of the prism

$$CC_1=\sqrt{AC_1^2-AC^2}$$

where AC_1 is determined from $\triangle B_1AC_1$; we have

$$CC_1=\sqrt{b^2\tan^2\alpha\cot^2\beta-b^2}=b\cot\beta\sqrt{\tan^2\alpha-\tan^2\beta}=$$

$$=\frac{b}{\cos\alpha\sin\beta}\sqrt{\sin(\alpha+\beta)\sin(\alpha-\beta)}$$

Answer: $V=\dfrac{b^3\tan\alpha}{2\cos\alpha\sin\beta}\sqrt{\sin(\alpha+\beta)\sin(\alpha-\beta)}$.

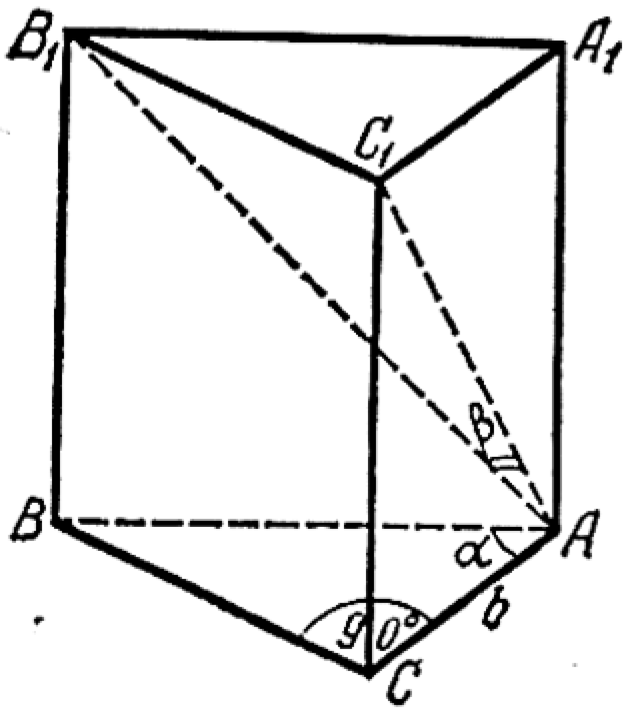


Fig. 139

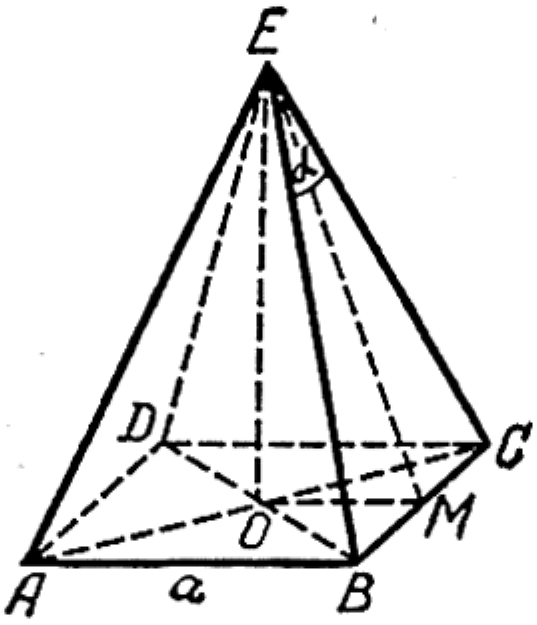


Fig. 140

662. By hypothesis $a^2+2a\cdot ME=S$ (Fig. 140). But from the triangle BME we have $ME=\frac{a}{2}\cot\frac{\alpha}{2}$; hence, $S=a^2\left(1+\cot\frac{\alpha}{2}\right)$; whence $a=$ $=\sqrt{\dfrac{S}{1+\cot\frac{\alpha}{2}}}$. From the triangle OME we now find

$$H=\sqrt{ME^2-\left(\frac{a}{2}\right)^2}=\frac{1}{2}\sqrt{a^2-\left(\cot^2\frac{\alpha}{2}-1\right)}=$$

$$=\frac{1}{2}\sqrt{\frac{S\left(\cot^2\frac{\alpha}{2}-1\right)}{\cot\frac{\alpha}{2}+1}}=\frac{1}{2}\sqrt{S\left(\cot\frac{\alpha}{2}-1\right)}$$

The expression $\cot\frac{\alpha}{2}-1$ can be transformed as follows

$$\cot\frac{\alpha}{2}-1=\cot\frac{\alpha}{2}-\cot 45^\circ=\frac{\sin\left(45^\circ-\frac{\alpha}{2}\right)}{\sin 45^\circ\sin\frac{\alpha}{2}}=\frac{\sqrt{2}\sin\left(45^\circ-\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}}$$

Answer: $H=\sqrt{\dfrac{S\sin\left(45^\circ-\dfrac{\alpha}{2}\right)}{2\sqrt{2}\sin\dfrac{\alpha}{2}}}$.

663. From the triangle AOM (Fig. 141), where

$$\angle AOM=\frac{180^\circ}{n}, \quad \text{we have} \quad OM=\frac{a}{2}\cot\frac{180^\circ}{n}$$

hence,

$$S_{base}=\frac{na^2}{4}\cot\frac{180^\circ}{n}$$

From the triangle EOM we find

$$H=\sqrt{ME^2-OM^2}=\frac{a}{2}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{180^\circ}{n}}$$

The radicand is transformed as in Problem 659.

Answer: $V=\dfrac{na^3\cot\dfrac{180^\circ}{n}\sqrt{\sin\left(\dfrac{180^\circ}{n}-\dfrac{\alpha}{2}\right)\sin\left(\dfrac{180^\circ}{n}+\dfrac{\alpha}{2}\right)}}{24\sin\dfrac{\alpha}{2}\sin\dfrac{180^\circ}{n}}$.

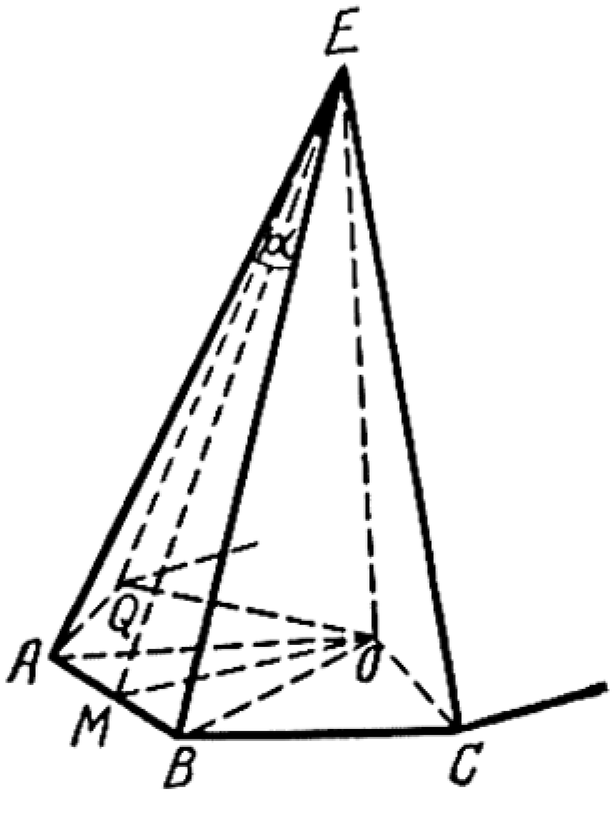


Fig. 141

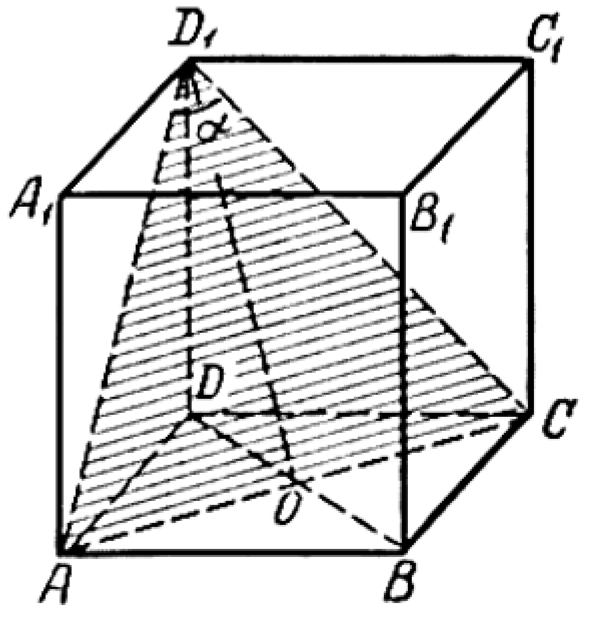


Fig. 142

664. Denoting (Fig. 142) $OD=OA$ by x, we get

$$OD_1=AO\cdot\cot\frac{\alpha}{2}=x\cot\frac{\alpha}{2}$$

and

$$H=DD_1=\sqrt{OD_1^2-OD^2}=x\sqrt{\cot^2\frac{\alpha}{2}-1}$$

The total area S of the pyramid D_1ADC is equal to

$$S=DO\cdot AO+AD\cdot H+AO\cdot OD_1=x^2+x\sqrt{2}x\sqrt{\cot^2\frac{\alpha}{2}-1}+x\cdot x\cot\frac{\alpha}{2},$$

whence

$$x^2=\frac{S\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}+\sqrt{2\cos\alpha}+\cos\frac{\alpha}{2}}$$

The total surface area of the prism

$$S_{total}=4x^2+4\cdot x\sqrt{2}\cdot H=4x^2\left(1+\frac{\sqrt{2\cos\alpha}}{\sin\frac{\alpha}{2}}\right)$$

Answer: $S_{total}=\dfrac{4S\left(\sin\frac{\alpha}{2}+\sqrt{2\cos\alpha}\right)}{\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}+\sqrt{2\cos\alpha}}$.

665. The altitude DO passes through the centre O (Fig. 143*a*) of the circle circumscribed about the triangle ABC, where $AB=AC=2l\sin\frac{\alpha}{2}$ and $BC=2l\sin\frac{\beta}{2}$ *. The point O lies on the perpendicular KO to the side AB drawn through the midpoint of AB. Therefore, from similarity of the triangles AOK and ABL we get proportion $AO:\frac{1}{2}AB=AB:AL$, whence

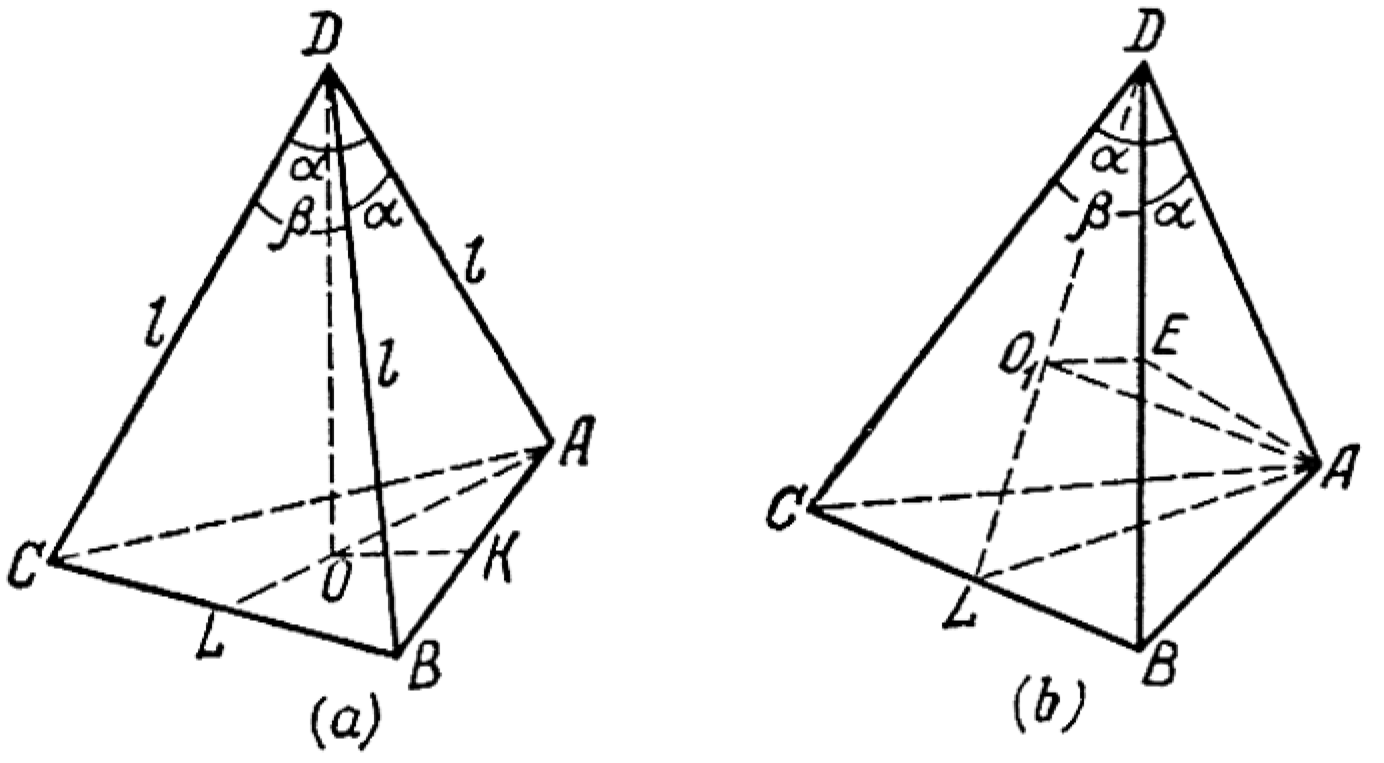


Fig. 143

$$AO=\frac{\frac{1}{2}AB^2}{AL}=\frac{2l^2\sin^2\frac{\alpha}{2}}{\sqrt{4l^2\sin^2\frac{\alpha}{2}-l^2\sin^2\frac{\beta}{2}}}$$

---

* See the Preliminary Notes to Problem 611.

Then from the triangle AOD we find

$$H=\sqrt{l^2-AO^2}=l\sqrt{\frac{\sin^2\alpha-\sin^2\frac{\beta}{2}}{4\sin^2\frac{\alpha}{2}-\sin^2\frac{\beta}{2}}}$$

and

$$V=\frac{1}{3\cdot 2}\,BC\cdot AL\cdot H=\frac{1}{3}\,l^3\sin\frac{\beta}{2}\sqrt{\sin^2\alpha-\sin^2\frac{\beta}{2}}$$

The radicand may be transformed in the same way as in Problem 656.

Alternate method. Let the face BDC (Fig. 143*b*) be the base of the pyramid. Its area is $S_{base}=\frac{1}{2}\,l^2\sin\beta$. The face BDC is perpendicular to the plane ADL (prove it!) and, consequently, the altitude of the pyramid AO_1 lies in this plane. Draw O_1E perpendicular to BD. From similarity of the triangles O_1DE and BDL we have $\frac{O_1D}{ED}=\frac{BD}{DL}$, where from the triangle ADE

$$ED=l\cos\alpha,\quad BD=l\quad\text{and}\quad DL=l\cos\frac{\beta}{2}$$

hence

$$O_1D=\frac{l\cos\alpha}{\cos\frac{\beta}{2}}$$

From the triangle ADO_1, we find

$$H=AO_1=\sqrt{AD^2-DO_1^2}=$$

$$=\frac{l}{\cos\frac{\beta}{2}}\sqrt{\cos^2\frac{\beta}{2}-\cos^2\alpha}$$

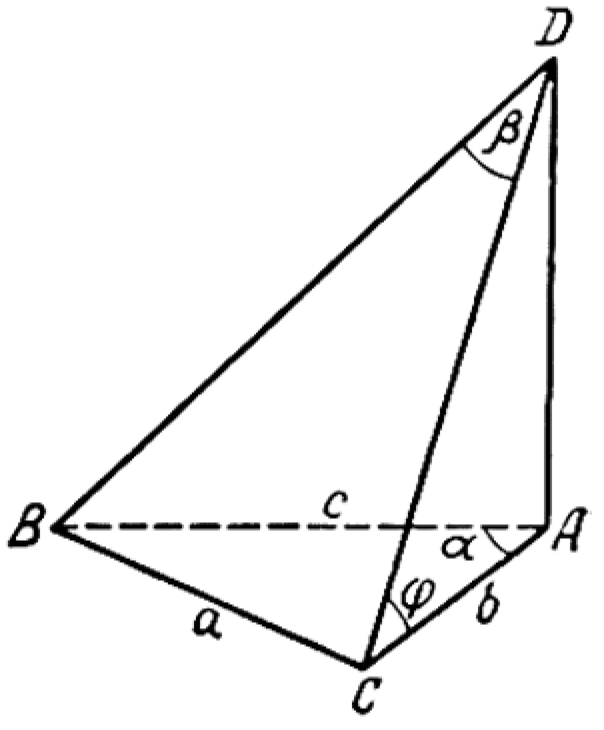


Fig. 144

Answer: $V=\frac{1}{3}\,l^3\sin\frac{\beta}{2}\sqrt{\sin\left(\alpha+\frac{\beta}{2}\right)\sin\left(\alpha-\frac{\beta}{2}\right)}$.

666. Since the triangle ABC (Fig. 144) is the projection of the triangle DBC, DA is perpendicular to the base. The area of the triangle ABC is

$$S_1=\frac{1}{2}\,ab=\frac{1}{2}\,a^2\cot\alpha$$

The area of the triangle BCD is

$$S_2=\frac{1}{2}\,a^2\cot\beta$$

By hypothesis

$$\frac{1}{2}\,a^2(\cot\beta-\cot\alpha)=S$$

whence $a=\sqrt{\frac{2S}{\cot\beta-\cot\alpha}}$.

The area of the face DAC is $S_3=\frac{1}{2}bH$, and that of the face DAB, $S_4=\frac{1}{2}cH$. Consequently,

$$S_4-S_3=\frac{1}{2}H(c-b)=\frac{1}{2}aH(\csc\alpha-\cot\alpha)$$

The altitude H is determined from the triangle ACD:

$$H=\sqrt{DC^2-AC^2}=\sqrt{a^2\cot^2\beta-a^2\cot^2\alpha}$$

Hence,

$$S_4-S_3=\frac{1}{2}a^2\sqrt{\cot^2\beta-\cot^2\alpha}\,(\csc\alpha-\cot\alpha)=$$

$$=\frac{1}{2}\frac{2S}{\cot\beta-\cot\alpha}\sqrt{\cot^2\beta-\cot^2\alpha}\,(\csc\alpha-\cot\alpha)=$$

$$=\frac{S(1-\cos\alpha)}{\sin\alpha}\sqrt{\frac{\cot^2\beta-\cot^2\alpha}{(\cot\beta-\cot\alpha)^2}}=S\tan\frac{\alpha}{2}\sqrt{\frac{\cot\beta+\cot\alpha}{\cot\beta-\cot\alpha}}$$

The lateral faces ADC and ADB form right angles with the base. The face BDC forms with the base an angle which is measured by the plane angle $DCA=\varphi$

$$\cos\varphi=\frac{AC}{DC}=\frac{\cot\alpha}{\cot\beta}.$$

Answer: $S_4-S_3=S\tan\frac{\alpha}{2}\sqrt{\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}}$; $\varphi=\arccos\left(\frac{\cot\alpha}{\cot\beta}\right)$.

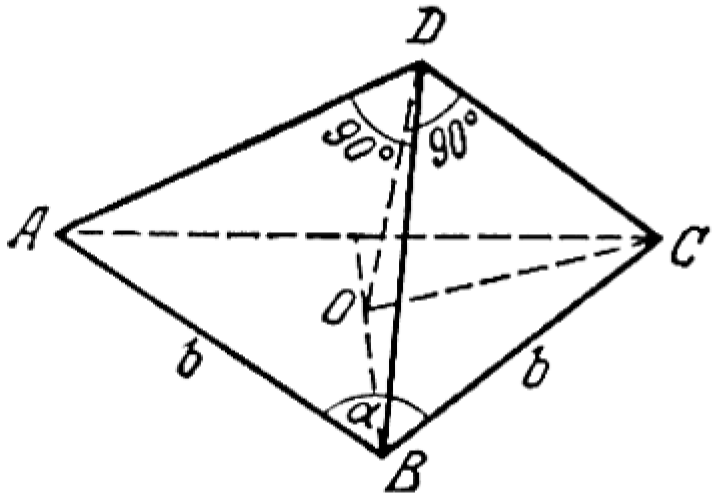


Fig. 145

667. All the lateral edges of the pyramid are equal as sides of isosceles right-angled triangles (Fig. 145), therefore the altitude DO of the pyramid passes through the centre O of the circle circumscribed about the base;

$$S_{base}=\frac{1}{2}b^2\sin\alpha$$

From the triangle DOC we find

$$H=\sqrt{DC^2-OC^2}$$

where $DC=\frac{b}{\sqrt{2}}$ and $OC=R$ is the radius of the circle circumscribed about the triangle ABC. Since the triangle ABC is an isosceles one, $\angle BAC=90°-\frac{\alpha}{2}$ and, hence, by the law of sines

$$BC=2R\sin\left(90°-\frac{\alpha}{2}\right)$$

whence

$$OC=R=\frac{b}{2\cos\frac{\alpha}{2}}$$

Answer: $V=\frac{1}{6}b^3\sin\frac{\alpha}{2}\sqrt{\cos\alpha}$.

668. The altitude passes through the centre of the circle circumscribed about the base * (Fig. 146). The bisectors of the angles AED and BEC are also medians of the isosceles triangles AED and BEC. The area of the section MEN is equal to $\frac{MN}{2}\cdot OE$ and $\frac{MN}{2}=AK=l\sin\frac{\alpha}{2}$. From the triangle EOK we find

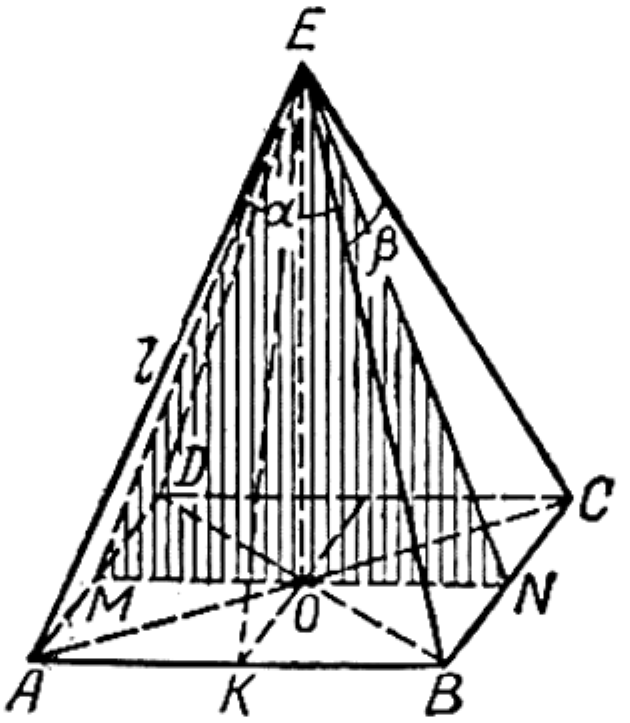


Fig. 146

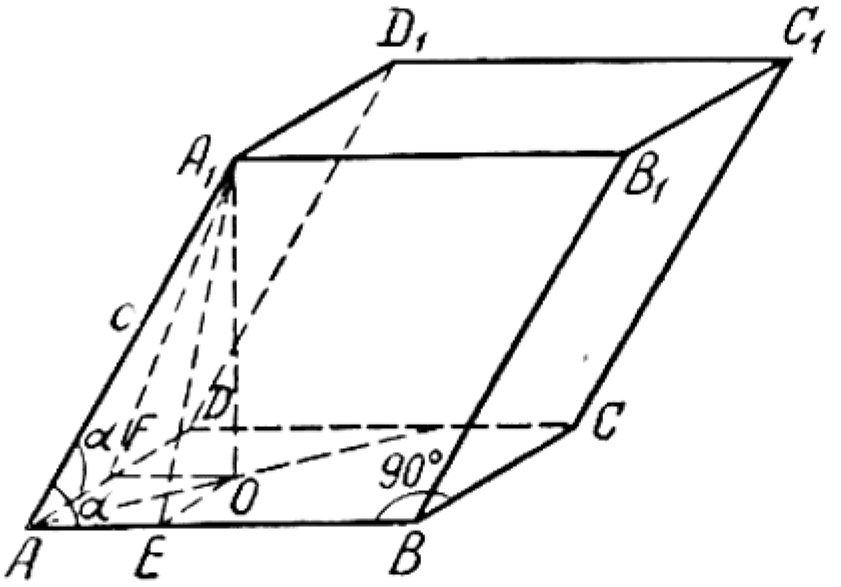


Fig. 147

$$OE=\sqrt{EK^2-OK^2}$$

where $EK=l\cos\frac{\alpha}{2}$ and $OK=BN=l\sin\frac{\beta}{2}$, thus

$$OE=l\sqrt{\cos^2\frac{\alpha}{2}-\sin^2\frac{\beta}{2}}$$

Answer: $S_{sec}=l^2\sin\frac{\alpha}{2}\sqrt{\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}$.

669. Through the vertex A_1 (Fig. 147) draw planes A_1EO perpendicular to AB and A_1FO perpendicular to AD. These planes are perpendicular to the base (prove it!), and the line A_1O along which they intersect is the altitude of the parallelepiped. The right-angled triangles A_1AE and A_1AF thus formed are congruent (since they have a common hypotenuse $AA_1=c$ and equal angles $\angle A_1AE=\angle A_1AF=\alpha$). Consequently, $A_1E=A_1F$ and therefore, the triangles A_1OE and A_1OF are congruent; and hence, $OE=OF$ and AO is the bisector

---

* See the Preliminary Notes to Problem 611.

of the angle BAD. We have $H = \sqrt{A_1E^2 - OE^2}$. Since $AEOF$ is a square, $OE = AE$. AE and A_1E are found from the triangle AA_1E; we get $H = c\sqrt{\sin^2\alpha - \cos^2\alpha} = c\sqrt{-\cos 2\alpha}$.

Note. In the trihedral angle at the vertex A either of the two equal face angles is equal to α, the third being a right one; consequently, the sum of two face angles 2α must be more than the third one (90°), i.e. $2\alpha > 90°$ or $\alpha > 45°$. At this condition $-\cos 2\alpha > 0$, and, hence, H has a real value. The lateral edge AA_1 forms an angle $\angle A_1AO = \varphi$ with the base, since AO is the projection of the edge on the base

$$\cos\varphi = \frac{AO}{AA_1} = \sqrt{2}\cos\alpha$$

Answer: $V = abc\sqrt{\cos(180° - 2\alpha)}$; $S_{lat} = 2c(a+b)\sin\alpha$

$\varphi = \arccos(\sqrt{2}\cos\alpha)$

670. The construction here is the same as in the preceding problem. The bisector of the angle BAD is the diagonal AC of the rhombus (Fig. 148)

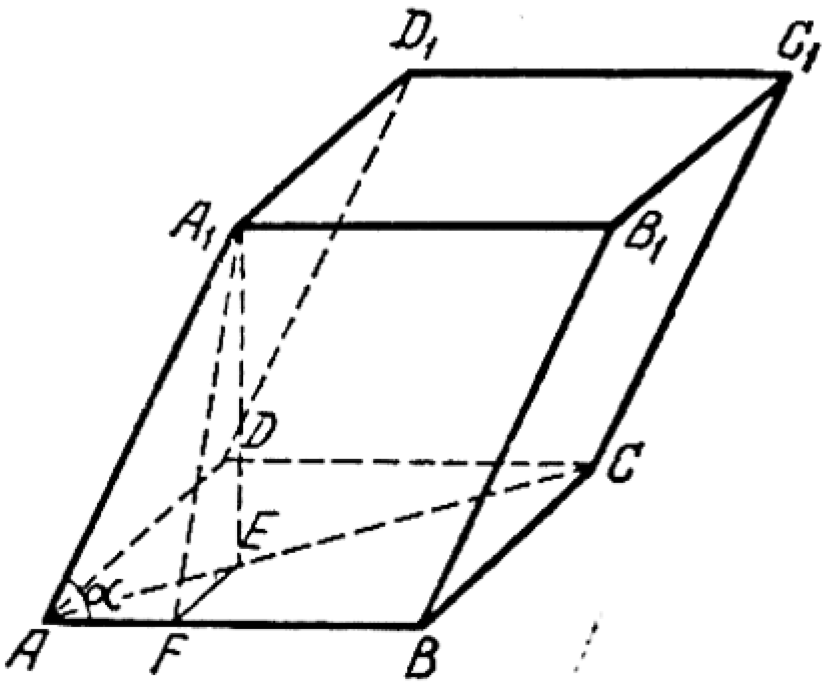


Fig. 148

$$S_{base} = a^2\sin\alpha$$

From the triangle AA_1E we find

$$H = \sqrt{AA_1^2 - AE^2}$$

where $AA_1 = a$; to determine AE first find AF from AA_1F, and then AE from the right-angled triangle AEF. We get

$$AE = \frac{a\cos\alpha}{\cos\frac{\alpha}{2}}$$

whence

$$H = \frac{a}{\cos\frac{\alpha}{2}}\sqrt{\cos^2\frac{\alpha}{2} - \cos^2\alpha}$$

Answer: $V = 2a^3\sin\frac{\alpha}{2}\sqrt{\sin\frac{3\alpha}{2}\sin\frac{\alpha}{2}}$.

671. The problem is solved analogously to the preceding one. We can use the same figure (148), introducing the notation: $\angle BAD = \alpha$ and $\angle A_1AD = \varphi$ instead of $\alpha = \angle A_1AB$.

Answer: $V = 2a^2b\sin\frac{\alpha}{2}\sqrt{\sin\left(\varphi - \frac{\alpha}{2}\right)\sin\left(\varphi + \frac{\alpha}{2}\right)}$.

672. The base $ABCD$ is a rectangle (Fig. 149). To construct a plane angle of the dihedral angle D_1ACD draw a plane through the edge DD_1 and perpendicular to AC. The lines along which this plane intersects the faces of the dihedral angle D_1ACD form the plane angle $D_1ED = \varphi$. We have

$$\cos\varphi = \frac{DE}{D_1E} = \frac{h_1}{h}$$

Let us introduce the following notation:

$$AB = DC = a$$

$$BC = AD = b\ (a > b),\ DD_1 = H$$

$$D_1E = h,\ DE = h_1$$

In the isosceles triangle AOB the sum of interior angles at the base AB is equal to the exterior angle 2α, hence, $\angle BAC = \alpha$. From the triangle ABC we find

$$a = 2R\cos\alpha;\quad b = 2R\sin\alpha$$

From $\triangle DEC$, where $\angle ACD = \alpha$, we find

$$h_1 = a\sin\alpha = 2R\cos\alpha\sin\alpha \text{ and } EC = a\cos\alpha = 2R\cos^2\alpha$$

From $\triangle D_1EC$ we find

$$h = EC\cdot\tan\beta = 2R\cos^2\alpha\tan\beta$$

From $\triangle D_1DE$ we find

$$H = \sqrt{D_1E^2 - DE^2} = \sqrt{h^2 - h_1^2} = \sqrt{4R^2\cos^4\alpha\tan^2\beta - 4R^2\sin^2\alpha\cos^2\alpha} =$$
$$= 2R\cos^2\alpha\sqrt{\tan^2\beta - \tan^2\alpha}.$$

Transform the expression $\tan^2\beta - \tan^2\alpha$ as in Problem 659.

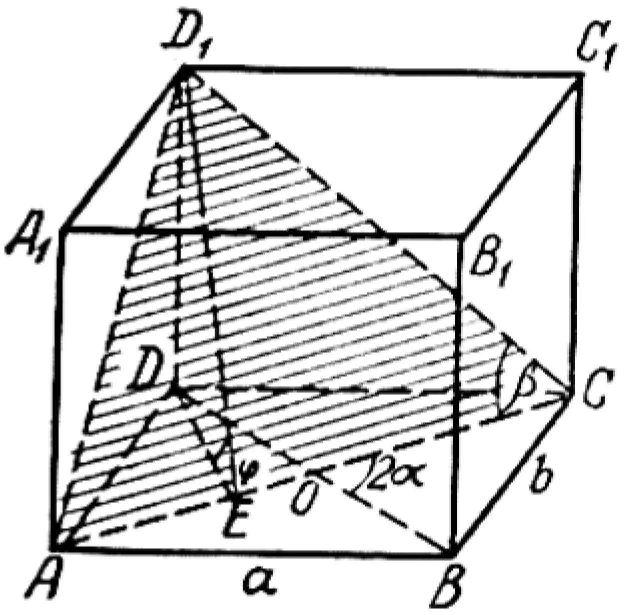


Fig. 149

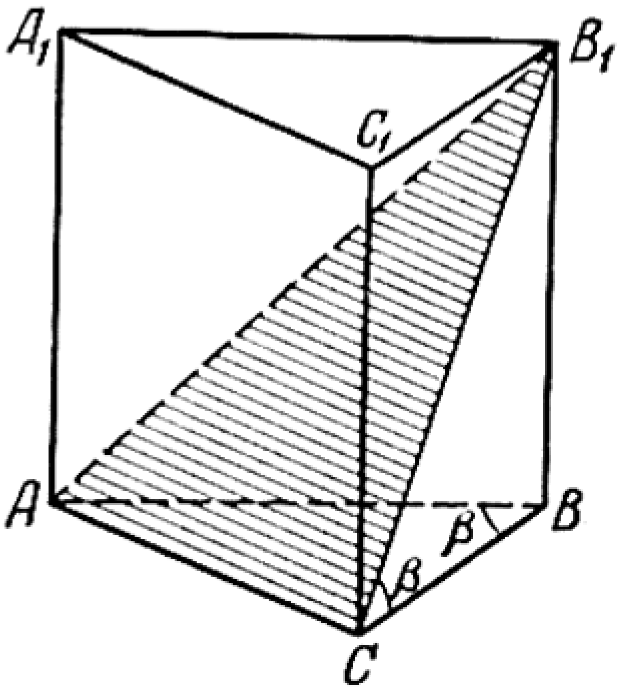


Fig. 150

Answer: $S_{lat} = 8R^2\cos\alpha\cos(45^\circ - \alpha)\sec\beta\cdot\sqrt{2\sin(\beta+\alpha)\sin(\beta-\alpha)}$

$$S_{sec} = 2R^2\cos^2\alpha\tan\beta;\quad \varphi = \arccos\left(\frac{\tan\alpha}{\tan\beta}\right)$$

673. If the leg AC (Fig. 150) subtends the arc equal to 2β, then $\angle ABC$ is equal to β as an inscribed angle having the same arc. The plane passing through the diagonal B_1C perpendicular to the face BB_1C_1C must pass through AC, since AC is perpendicular to this face; the plane angle of the dihedral angle B_1ACB is $\angle B_1CB = \beta$. The hypotenuse AB is the diameter of the circumscribed circle and, hence, $AB = 2R$. Let us denote: $BC = a$, $AC = b$ and $AB = c$. A quadrangular pyramid $B_1AA_1C_1C$ is cut off the prism by the plane ACB_1. Since the volume of the pyramid B_1ABC is equal to one third of the volume of the prism, the volume of the remaining portion, i.e. of the quadrangular pyramid $B_1AA_1C_1C$ is equal to $\frac{2}{3}$ of the volume of the prism. If we denote the volume of the pyramid $B_1AA_1C_1C$ by V_1, and the volume of the prism by V, then

$$V_1 = \frac{2}{3}V = \frac{2}{3}\cdot\frac{ab}{2}\cdot H = \frac{abH}{3}$$

From the triangle ABC we find a and b, and from $\triangle B_1BC$, H. For the lateral surface area we get the following expression:

$$S_{lat} = (2R\cos\beta + 2R\sin\beta + 2R)\cdot 2R\cos\beta\tan\beta = 4R^2\sin\beta\,(\cos\beta + \sin\beta + 1)$$

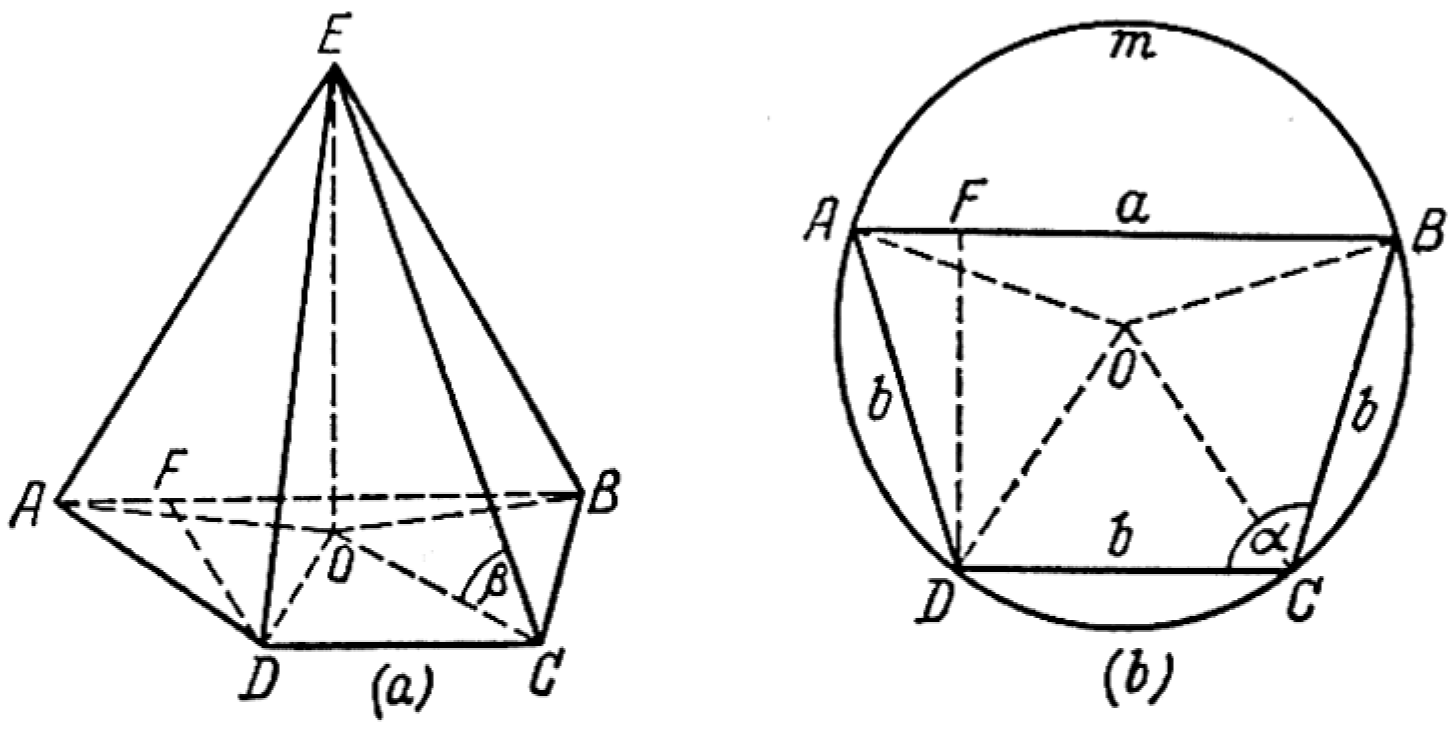
The expression in parentheses can be reduced to the form convenient for taking logarithms:

$$\cos\beta + \sin\beta + 1 = (1+\cos\beta) + \sin\beta =$$
$$= 2\cos^2\frac{\beta}{2} + 2\sin\frac{\beta}{2}\cos\frac{\beta}{2} = 2\cos\frac{\beta}{2}\left(\cos\frac{\beta}{2} + \sin\frac{\beta}{2}\right) =$$
$$= 2\cos\frac{\beta}{2}\left[\sin\left(90^\circ - \frac{\beta}{2}\right) + \sin\frac{\beta}{2}\right] =$$
$$= 2\cos\frac{\beta}{2}\cdot 2\sin 45^\circ\cos\left(45^\circ - \frac{\beta}{2}\right) = 2\sqrt{2}\cos\frac{\beta}{2}\cos\left(45^\circ - \frac{\beta}{2}\right)$$

Answer: $S_{lat} = 8\sqrt{2}\,R^2\sin\beta\cos\frac{\beta}{2}\cos\left(45^\circ - \frac{\beta}{2}\right)$

$V_1 = \frac{4}{3}R^3\sin\beta\sin 2\beta.$

674. The altitude EO (Fig. 151a) passes through the centre O of the circle circumscribed about the trapezoid $ABCD$*. The arcs $\overset{\frown}{AD}$, $\overset{\frown}{DC}$ and $\overset{\frown}{CB}$ (Fig. 151b)



(a) (b)

Fig. 151

are equal (since by hypothesis the sides AD, DC and CB are equal), and $\angle B = 180^\circ - \alpha$ is measured by half the arc $\overset{\frown}{ADC}$. Hence each of the arcs

---

* See the Preliminary Notes to Problem 611.

$\breve{AD}$, $\breve{DC}$ and $\breve{CB}$ contains $180^\circ - \alpha$; consequently, the arc $\overset{\frown}{AmB}$ is equal to $360^\circ - 3(180^\circ - \alpha) = 3\alpha - 180^\circ$. From the triangle AOB, where $AB = a$, we find:

$$AO = R = \frac{a}{2\sin\frac{3\alpha - 180^\circ}{2}} = -\frac{a}{2\cos\frac{3\alpha}{2}}$$

$\left(\text{the quantity } \cos\frac{3\alpha}{2}\right.$ is negative, since α is an obtuse angle and so $135^\circ < \left.\frac{3\alpha}{2} < 270^\circ\right)$. From the triangle ODC we find

$$DC = b = 2R\sin\frac{180^\circ - \alpha}{2} = -\frac{a\cos\frac{\alpha}{2}}{\cos\frac{3\alpha}{2}}$$

From the triangle ADF, where $AD = b$ and $\angle A = 180^\circ - \alpha$, we find the altitude of the trapezoid

$$DF = h = b\sin\alpha = -\frac{a\sin\alpha\cos\frac{\alpha}{2}}{\cos\frac{3\alpha}{2}}$$

From the triangle BOE (see Fig. 151*a*) where $OB = R$ and $\angle OBE = \beta$, we find $H = R\tan\beta$. The area of the base

$$S = \frac{1}{2}(a+b)h = -\frac{a^2\left(\cos\frac{3\alpha}{2} - \cos\frac{\alpha}{2}\right)\cos\frac{\alpha}{2}\sin\alpha}{2\cos^2\frac{3\alpha}{2}} = \frac{a^2\sin^3\alpha}{2\cos^2\frac{3\alpha}{2}}$$

Answer: $V = -\dfrac{a^3\sin^3\alpha\tan\beta}{12\cos^3\frac{3\alpha}{2}} = \dfrac{a^3\sin^3\alpha\tan\beta}{12\cos^3\left(180^\circ - \frac{3\alpha}{2}\right)}$

675. The altitude EO passes through the centre O of the circle circumscribed about the trapezoid $ABCD$* (Fig. 152). The angle $ACB = 90^\circ$ must be subtended by the diameter as one inscribed in this circle. In other words, the centre O lies on the side AB. The trapezoid $ABCD$, being inscribed in the circle, is an isosceles one, and thus $\angle DAB = \angle CBA$.

Let us introduce the following notation: $AB = a$; $DC = b$; $\angle AEB = \varphi = 2\alpha$. By hypothesis, $\frac{1}{2}aH = S$ and from the isosceles triangle AEB we have $a = 2H\tan\frac{\varphi}{2} = 2H\tan\alpha$.

From the two equations we find

$$H = \sqrt{S\cot\alpha} \quad \text{and} \quad a = 2\sqrt{S\tan\alpha}$$

---

* See the Preliminary Notes to Problem 611.

The side $b = DC$ is determined from the triangle ADC inscribed in the circle of the diameter a. In this triangle

$$\angle DAC = \angle DAB - \angle CAB = \angle CBA - \angle CAB$$

Since the triangle ACB is a right-angled one, $\angle CBA = 90^\circ - \angle CAB$. Hence,

$$\angle DAC = 90^\circ - 2\,\angle CAB = 90^\circ - 2\alpha$$

and we have

$$b = a\sin(90^\circ - 2\alpha) = a\cos 2\alpha$$

Finally,

$$CN = h = AC\cdot\sin\alpha = a\cos\alpha\sin\alpha$$

Now we get

$$V = \frac{1}{3}\cdot\frac{a+b}{2}hH = \frac{1}{6}a^2(1+\cos 2\alpha)\cos\alpha\sin\alpha H =$$

$$= \frac{1}{6}\cdot 4S\tan\alpha 2\cos^2\alpha\cos\alpha\sin\alpha\sqrt{S\cot\alpha} \equiv \frac{\sin^2 2\alpha}{3}\sqrt{S^3\cot\alpha}$$

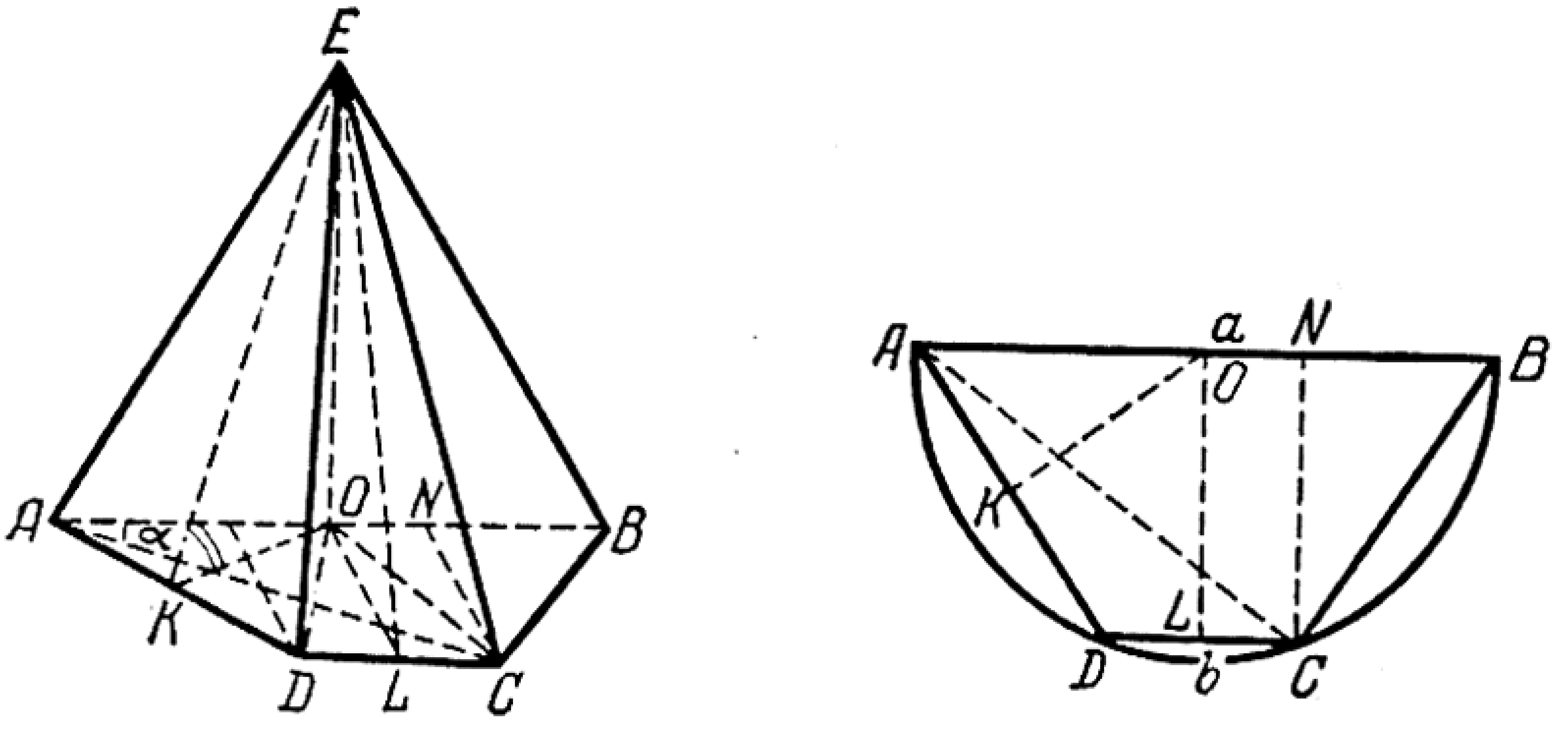


Fig. 152

The face ABE forms a right angle with the plane $ABCD$. To determine the angle φ_1 formed by the face ADE and the plane $ABCD$ drop a perpendicular from O onto AD (it is depicted by a straight line OK parallel to the diagonal BD so that the latter is perpendicular to AD; the diagonal BD is not shown in the drawing; $\angle EKO = \varphi_1$). In the triangle AOK the angle OAK is equal to $\angle ABC = 90^\circ - \angle CAB = 90^\circ - \alpha$. Therefore

$$OK = AO\cdot\sin(90^\circ - \alpha) = \frac{a}{2}\cos\alpha$$

and

$$\tan\varphi_1 = \frac{H}{OK} = \frac{2H}{a\cos\alpha} = \frac{2H}{2H\tan\alpha\cos\alpha} = \frac{1}{\sin\alpha}$$

To determine the angle φ_2 formed by the face DCE and the plane $ABCD$, draw $OL \perp DC$; $\angle ELO = \varphi_2$. Since $OL = NC = h$, we have

$$\tan\varphi_2 = \frac{H}{h} = \frac{H}{a\cos\alpha\sin\alpha} = \frac{1}{2\sin^2\alpha}$$

Answer: $V=\dfrac{\sin^2 2\alpha}{3}\sqrt{S^3\cot\alpha}$

$$\left\{\begin{aligned}\varphi_1&=\arctan(\csc\alpha)\\ \varphi_2&=\arctan\left(\frac{1}{2}\csc^2\alpha\right)\end{aligned}\right.$$

676. It is required to determine (Fig. 153) the sum of the areas of the triangles ABC, ABD and ACD. The area of the triangle ABC is equal to

$$S_1=\frac{1}{2}AB\cdot CE=\frac{1}{4}a^2\sqrt{3}$$

The area of the triangle ABD is equal to

$$S_2=\frac{1}{2}AB\cdot DE=AB\cdot\frac{1}{2}\frac{CE}{\cos\varphi}=\frac{S_1}{\cos\varphi}$$

the area of the triangle ACD is equal to

$$S_3=\frac{1}{2}AC\cdot CD=\frac{1}{2}AB\cdot CD=\frac{1}{2}AB\cdot CE\cdot\tan\varphi=S_1\tan\varphi$$

Consequently,

$$S_{lat}=S_1+S_2+S_3=\frac{a^2\sqrt{3}}{4\cos\varphi}(1+\cos\varphi+\sin\varphi)$$

The expression in parentheses is transformed as in Problem **673** to be equal to $2\sqrt{2}\cos\frac{\varphi}{2}\cos\left(45^\circ-\frac{\varphi}{2}\right)$. If in the denominator of the formula for S_{lat} we substitute $\sin(90^\circ-\varphi)$ for $\cos\varphi$, then the expression for S_{lat} can be reduced by

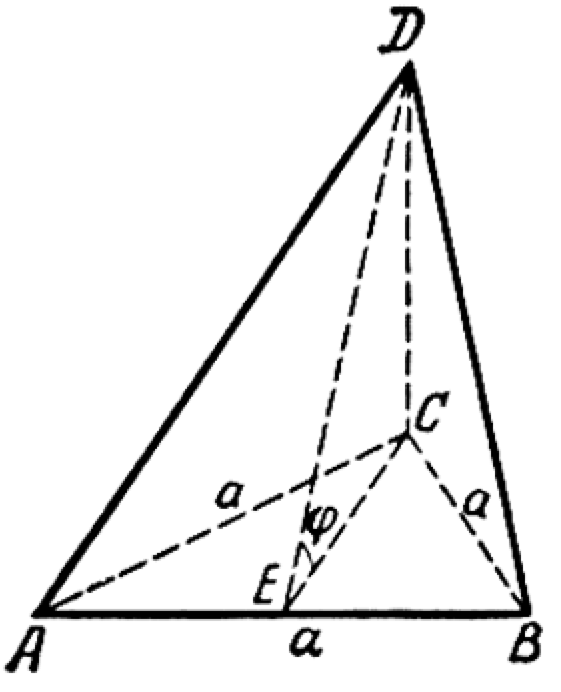


Fig. 153

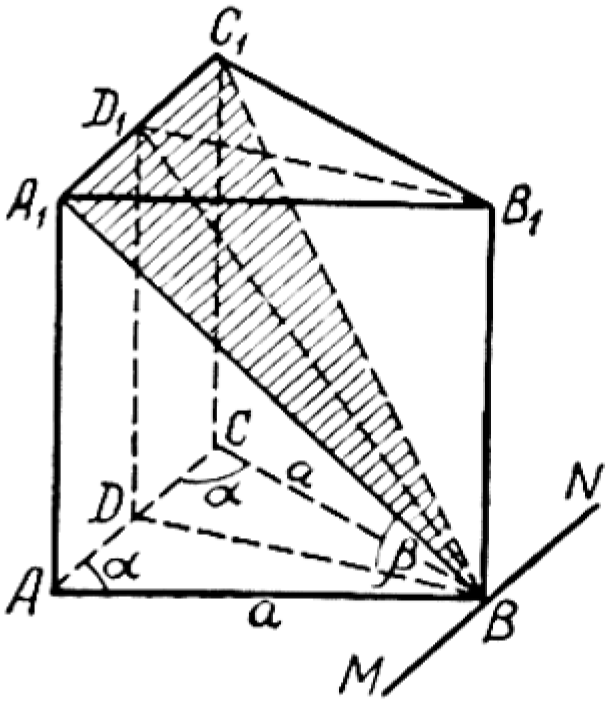


Fig. 154

$$\cos\left(45^\circ-\frac{\varphi}{2}\right)$$

Answer: $$S_{lat}=\frac{a^2\sqrt{6}\cos\frac{\varphi}{2}}{4\sin\left(45^\circ-\frac{\varphi}{2}\right)}.$$

677. Since the plane of the base ABC (Fig. 154) passes through AC, and the cutting plane A_1BC_1 through A_1C_1 parallel to AC, the edge MN of the dihedral angle β is parallel to AC and A_1C_1. Therefore to construct the plane angle draw $BD \perp AC$ and $BD_1 \perp A_1C_1$ (D and D_1 are the midpoints of AC and A_1C_1). We have

$$S_{lat} = (2AB + AC)\cdot DD_1 = (2AB + AC)\cdot BD \cdot \tan\beta =$$
$$= 2a^2(1 + \cos\alpha)\sin\alpha\tan\beta$$

The volume V_1 of the quadrangular pyramid $BACC_1A_1$ is equal to $\frac{2}{3}$ of the volume V of the prism (see Problem 673) and, hence,

$$V_1 = \frac{2}{3} S \cdot DD_1$$

where

$$S = \frac{1}{2} a^2 \sin(180° - 2\alpha) = \frac{1}{2} a^2 \sin 2\alpha$$

Answer: $S_{lat} = 4a^2 \cos^2\frac{\alpha}{2}\sin\alpha\tan\beta$;

$$V_1 = \frac{a^3}{3}\sin 2\alpha\sin\alpha\tan\beta.$$

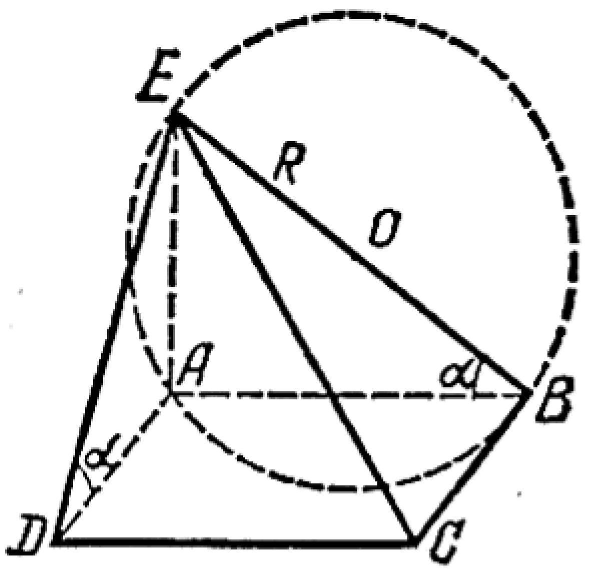


Fig. 155

678. As in Problem 630 let us prove that the face DCE (Fig. 155) is inclined to the base $ABCD$ at an angle $\alpha = \angle ADE$ and the face BCE at an equal angle $\alpha = \angle ABE$; both faces are right-angled triangles ($\angle CDE = \angle CBE = 90°$).

The area of the triangle ADE (as also the area of the triangle ABE) is equal to $S_1 = \frac{1}{2} AB \cdot AE$. From the triangle ABE, where $BE = 2R$, we find

$$AB = 2R\cos\alpha; \quad AE = 2R\sin\alpha$$

and thus $S_1 = 2R^2\sin\alpha\cos\alpha$.

The area of the triangle CDE (as also of the triangle CBE) is equal to

$$S_2 = \frac{1}{2} BC \cdot BE = \frac{1}{2} AB \cdot BE = 2R^2\cos\alpha$$

We have

$$S_{total} = S + 2S_1 + 2S_2 = 4R^2(\cos^2\alpha + \cos\alpha\sin\alpha + \cos\alpha) =$$
$$= 4R^2\cos\alpha(\cos\alpha + \sin\alpha + 1)$$

The expression in parentheses is transformed as in Problem 673.

Answer: $S_{total} = 8\sqrt{2}R^2\cos\alpha\cos\frac{\alpha}{2}\cos\left(45° - \frac{\alpha}{2}\right)$.

679. The cutting plane ECD (Fig. 156) parallel to the hypotenuse AB intersects the face ABB_1A_1 along a straight line ED which is parallel to AB. Drop perpendiculars CM and CF to AB and ED to get a right-angled triangle CMF in which $\angle CFM = \beta$ (prove it!). Consequently,

$$\triangle CMF = \triangle CMB$$

(they have a common leg MC and $\angle CBM = 90° - \alpha$ and by hypothesis $\beta = 90° - \alpha$).

It is required to find the volume V of the pyramid $CABDE$, whose base $ABDE$ is a rectangle, and the altitude is equal to $CM = a \sin\beta = a\cos\alpha$. We have

$$V=\frac{1}{3}\cdot AB\cdot MF\cdot CM=\frac{1}{3}\cdot AB\cdot MB\cdot CM=\frac{1}{3}BC^2\cdot CM=\frac{1}{3}a^3\cos\alpha$$

(the leg BC is a mean proportional between AB and MB)

Then we have

$$S_{lat}=(BC+AB+AC)\,H=aH\left(1+\frac{1}{\sin\alpha}+\cot\alpha\right)$$

here aH is the area of the face CBB_1C_1, which by hypothesis is equal to the area S_{sec} of the triangle CDE. Consequently,

$$aH=S_{sec}=\frac{1}{2}AB\cdot CF=\frac{1}{2}AB\cdot CB=\frac{a^2}{2\sin\alpha}$$

Hence,

$$S_{lat}=\frac{a^2}{2\sin\alpha}\left(1+\frac{1}{\sin\alpha}+\cot\alpha\right)=\frac{a^2}{2\sin^2\alpha}(\sin\alpha+1+\cos\alpha$$

The expression in parentheses is transformed as in Problem 673.

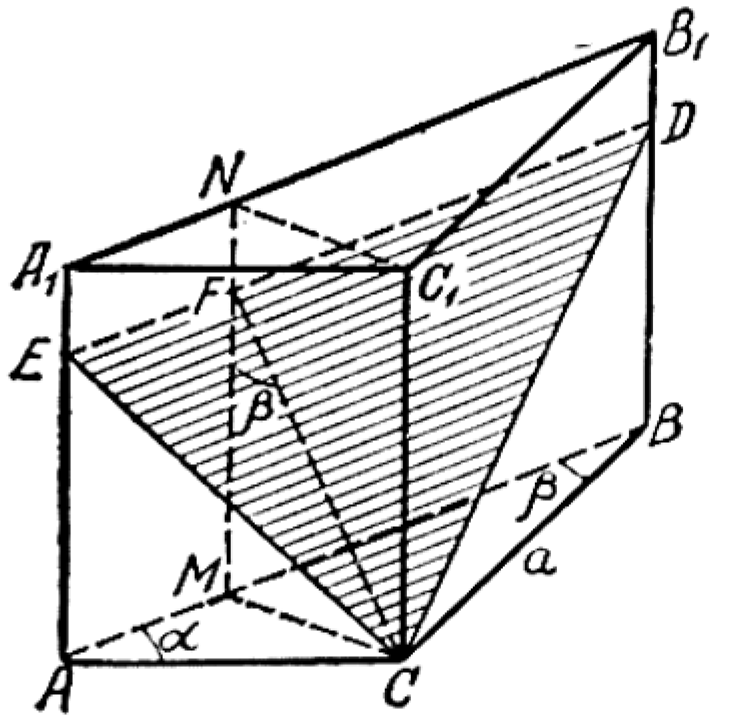


Fig. 156

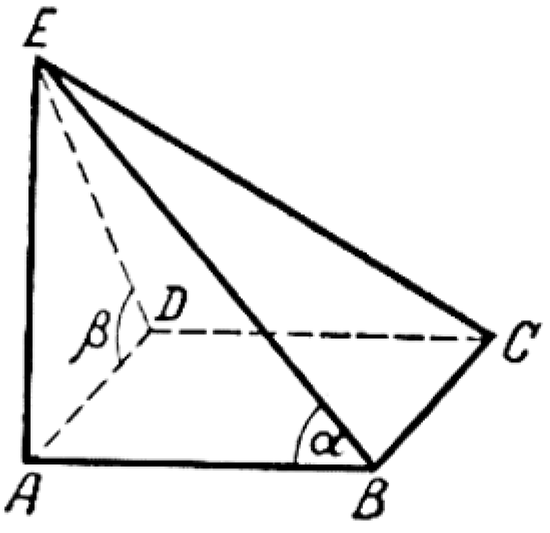


Fig. 157

For the plane CDE to intersect the face ABB_1A_1 it is necessary 'that 'the line-segment $MF=MB=a\sin\alpha$ be less than the line-segment $MN=H=$ $=\frac{a^2}{2\sin\alpha}:a=\frac{a}{2\sin\alpha}$. From the ;inequality $a\sin\alpha<\frac{a}{2\sin\alpha}$ we find $\sin^2\alpha<\frac{1}{2}$, i.e. $\sin\alpha<\frac{\sqrt{2}}{2}$. Hence the angle α must be less than 45°.

Answer: $V=\frac{a^3\cos\alpha}{3}$; $S_{lat}=\frac{\sqrt{2}\,a^2\cos\frac{\alpha}{2}\cos\left(45^\circ-\frac{\alpha}{2}\right)}{\sin^2\alpha}$; $\alpha<45^\circ$.

680. (Fig. 157). The lateral surface of the pyramid is

$$S_{lat}=\frac{H^2\cot\alpha}{2}+\frac{H^2\cot\beta}{2}+\frac{H^2\cot\beta}{2\sin\alpha}+\frac{H^2\cot\alpha}{2\sin\beta}$$

Hence,

$$S_{lat}=\frac{H^2}{2\sin\alpha\sin\beta}(\cos\alpha\sin\beta+\sin\alpha\cos\beta+\cos\beta+\cos\alpha)$$

The expression in parentheses can be reduced to the form convenient for taking logarithms, taking into account that $\cos\alpha\sin\beta+\sin\alpha\cos\beta=$ $=\sin(\alpha+\beta)$ and

$$\cos\beta+\cos\alpha=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

We get

$$\sin(\alpha+\beta)+2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}+$$
$$+2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}=2\cos\frac{\alpha+\beta}{2}\left(\sin\frac{\alpha+\beta}{2}+\cos\frac{\alpha-\beta}{2}\right)$$

Substituting $\sin\left(90^\circ-\frac{\alpha-\beta}{2}\right)$ for $\cos\frac{\alpha-\beta}{2}$ and transforming the expression in parentheses, we get

$$4\cos\frac{\alpha+\beta}{2}\cos\left(45^\circ-\frac{\alpha}{2}\right)\cos\left(45^\circ-\frac{\beta}{2}\right)$$

Answer: $S_{lat}=\dfrac{2H^2\cos\frac{\alpha+\beta}{2}\cos\left(45^\circ-\frac{\alpha}{2}\right)\cos\left(45^\circ-\frac{\beta}{2}\right)}{\sin\alpha\sin\beta}$.

681. Let $r = ON$ be the radius of the circle inscribed in the base of the pyramid*. From the triangle DON (Fig. 158) we have $DO = H = r\tan\alpha$. Since the centre O of the inscribed circle lies at the point of intersection of the bisectors of the angles A and B, $\angle OAM=\frac{\alpha}{2}$ and

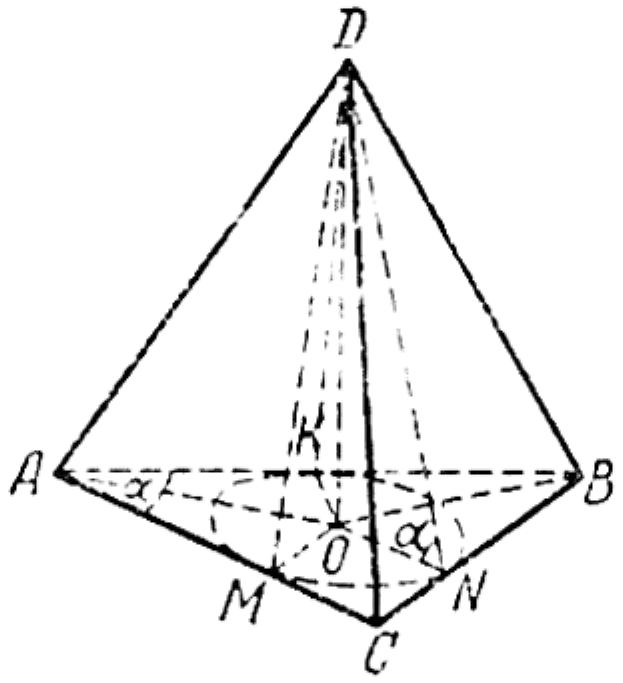


Fig. 158

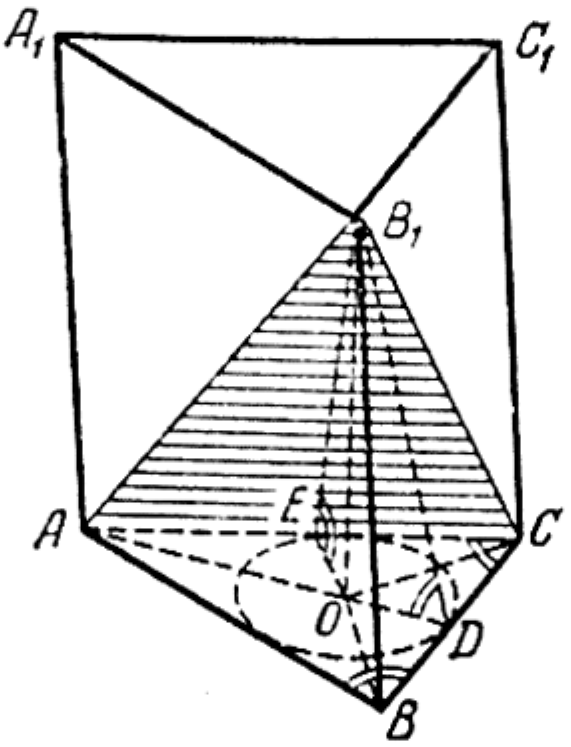


Fig. 159

$$\angle OBN=\frac{90^\circ-\alpha}{2}=45^\circ-\frac{\alpha}{2}$$

---

* See the Preliminary Notes to Problem 617.

Since the angle C is a right one, the quadrilateral $MCNO$ is a square and $MC = CN = r$. Hence,

$$AC = b = AM + MC = r\left(\cot\frac{\alpha}{2} + 1\right)$$

and

$$CB = a = r\left[\cot\left(45^\circ - \frac{\alpha}{2}\right) + 1\right]$$

The bracketed expression is transformed as in Problem 662 and we get

$$S_{base} = \frac{1}{2}\,ab = \frac{1}{2}\,\frac{\sqrt{2}\,r\cos\frac{\alpha}{2}}{\sin\left(45^\circ - \frac{\alpha}{2}\right)}\;\frac{\sqrt{2}\,r\sin\left(45^\circ + \frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}} =$$

$$= r^2\cot\frac{\alpha}{2}\cot\left(45^\circ - \frac{\alpha}{2}\right)$$

Consequently

$$V = \frac{1}{3}\,S_{base}\cdot H = \frac{1}{3}\,r^3\tan\alpha\cot\frac{\alpha}{2}\cot\left(45^\circ - \frac{\alpha}{2}\right)$$

This expression can be simplified if we take into consideration that

$$\tan\alpha = \frac{\sin\alpha'}{\cos\alpha} = \frac{\sin\alpha}{\sin(90^\circ - \alpha)} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\sin\left(45^\circ - \frac{\alpha}{2}\right)\cos\left(45^\circ - \frac{\alpha}{2}\right)}$$

The lateral and total surface areas can be found by the formulas

$$S_{lat} = \frac{S_{base}}{\cos\alpha}; \qquad S_{total} = \frac{2S_{base}\cos^2\frac{\alpha}{2}}{\cos\alpha}\,^*$$

Answer: $V = \dfrac{r^3\cos^2\frac{\alpha}{2}}{3\sin^2\left(45^\circ - \frac{\alpha}{2}\right)}$; $S_{lat} = \dfrac{r^2\cot\frac{\alpha}{2}}{2\sin^2\left(45^\circ - \frac{\alpha}{2}\right)}$;

$$S_{total} = \frac{r^2\cot\frac{\alpha}{2}\cos^2\frac{\alpha}{2}}{\sin^2\left(45^\circ - \frac{\alpha}{2}\right)}.$$

682. The plane cuts from the prism a pyramid B_1ABC (Fig. 159), whose altitude passes through the centre O of the circle inscribed in the base of the pyramid; therefore all the lateral faces are inclined to the base at one and the same angle α, consequently,

---

* See the Notes to Problems 617 and 618.

$$S_{total}=\frac{2S_{base}\cos^2\frac{\alpha}{2}}{\cos\alpha}.^{*}$$

We find

$$S_{base}=\frac{BC\cdot AD}{2}=DC\cdot AD.$$

From $\triangle OCD$, where $OD=r$, and $\angle OCD=\frac{\alpha}{2}$, find $DC=r\cot\frac{\alpha}{2}$. From $\triangle ADC$ find $AD=DC\cdot\tan\alpha=r\cot\frac{\alpha}{2}\tan\alpha$. Hence

$$S_{base}=r^2\cot^2\frac{\alpha}{2}\tan\alpha \quad\text{and}\quad S_{total}=\frac{2r^2\cot^2\frac{\alpha}{2}\tan\alpha\cos^2\frac{\alpha}{2}}{\cos\alpha}.$$

The obtained expressions may be simplified by representing $\tan\alpha$ in the form

$$\frac{\sin\alpha}{\cos\alpha}=\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos\alpha}$$

The volume of the prism

$$V=S_{base}\cdot H$$

where

$$H=r\tan\alpha$$

(from $\triangle B_1OD$).

Answer: $S_{total}=\dfrac{4r^2\cos^4\frac{\alpha}{2}\cot\frac{\alpha}{2}}{\cot^2\alpha}$;

$$V=r^3\cot^2\frac{\alpha}{2}\tan^2\alpha.$$

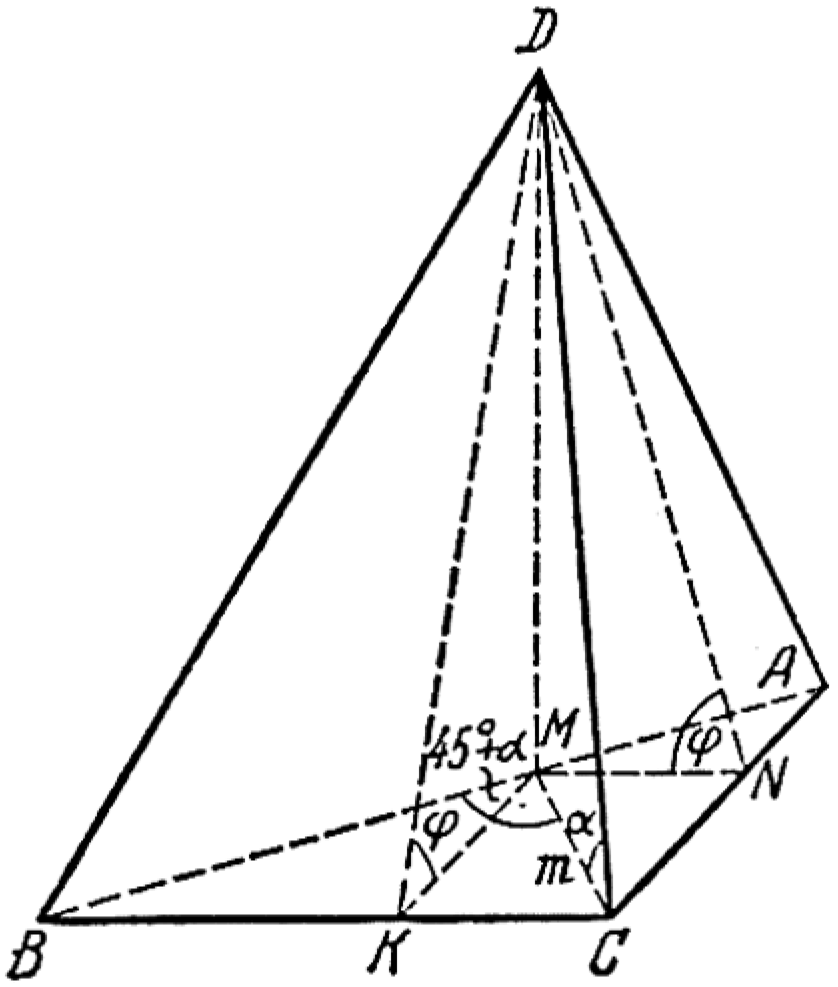


Fig. 160

683. From $\triangle BMC$ (Fig. 160), where $\angle MCB=45°$, and $\angle MBC=180°-(45°+\alpha)-45°=90°-\alpha$, according to the law of sines, we have

$$\frac{BC}{\sin(45°+\alpha)}=\frac{m}{\sin(90°-\alpha)}$$

Hence,

$$BC=a=\frac{m\sin(45°+\alpha)}{\cos\alpha}$$

---

* See the Notes to Problems 617 and 618.

From $\triangle ABC$ we find

$$AC = b = a \cot \alpha = \frac{m \sin (45^\circ + \alpha)}{\sin \alpha}$$

From $\triangle DCM$ we find

$$H = m \tan \alpha$$

The angles DNM and DKM are plane angles of the dihedral angles $DACB$ and $DBCA$; they are equal to each other, since the following triangles are congruent pairwise: MKC and MNC (by hypotenuse and an acute angle), DMK and DNM (by hypotenuse and a leg). Let us denote them by φ; then $\tan \varphi = \frac{H}{MN}$, where $MN = \frac{m}{\sqrt{2}}$.

Answer: $V = \frac{1}{6} m^3 \frac{\sin^2 (45^\circ + \alpha)}{\cos^2 \alpha}$; $\varphi = \arctan (\sqrt{2} \tan \alpha)$.

684. Let ABE (Fig. 161a) be the first, and ADE the second lateral face. By hypothesis they are inclined to the base at one and the same angle α. Consequently, the point O, through which the altitude passes, lies on the diagonal AC. Indeed, if we drop perpendiculars OM and ON* from O (Fig. 161b) to the sides AB and AD, then $\angle OME = \alpha$ and $\angle ONE = \alpha$ (prove it!); hence,

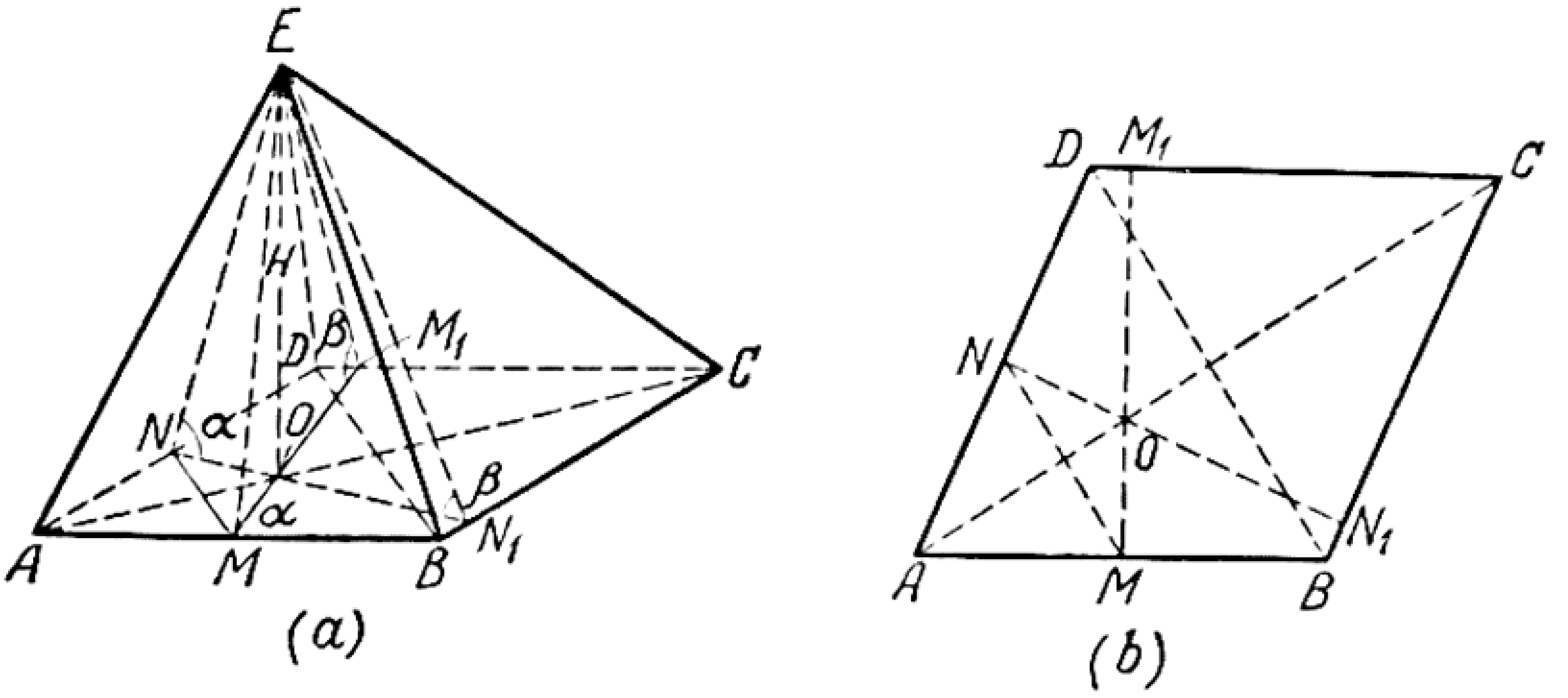


Fig. 161

$$OM = H \cot \alpha$$

and

$$ON = H \cot \alpha$$

i.e. $OM = ON$. Hence, the point O lies on the bisector of the angle BAD, i.e. on the diagonal AC of the rhombus $ABCD$.

But then we also have $OM_1 = ON_1$ (OM_1 and ON_1 are extensions of OM and ON), whence it follows that the triangles OM_1E and ON_1E are congruent and, consequently, $\angle ON_1E = \angle OM_1E$, which completes the proof.

---

* On the drawing (Fig. 161a) one of these perpendiculars, say OM, may be shown by an arbitrary straight line, but the second one is then constructed in a quite definite manner, since MN must be parallel to the diagonal BD. It is easily proved in Fig. 161b.

From the triangle OME we find $OM = H \cot \alpha$ and from the triangle OM_1E we have $OM_1 = H \cot \beta$. Consequently, the altitude of the rhombus is equal to $h = MM_1 = H (\cot \alpha + \cot \beta)$.

Hence,

$$V = \frac{1}{3} S_{base} H = \frac{1}{3} ahH = \frac{1}{3} aH^2 (\cot \alpha + \cot \beta)$$

$$S_{total} = S_{base} + 2S_{ABE} + 2S_{BEC} = a (h + ME + N_1E)$$

where

$$ME = \frac{H}{\sin \alpha}, \quad N_1E = \frac{H}{\sin \beta}.$$

Then

$$S_{total} = aH \left(\cot \alpha + \frac{1}{\sin \alpha} + \cot \beta + \frac{1}{\sin \beta}\right) = aH \left(\frac{1 + \cos \alpha}{\sin \alpha} + \frac{1 + \cos \beta}{\sin \beta}\right)$$

Expressing the numerators and denominators through $\frac{\alpha}{2}$ and $\frac{\beta}{2}$ and reducing the fractions, we get

$$S_{total} = aH \left(\cot \frac{\alpha}{2} + \cot \frac{\beta}{2}\right)$$

Answer: $V = \frac{1}{3} aH^2 (\cot \alpha + \cot \beta) = \frac{1}{3} aH^2 \frac{\sin (\alpha + \beta)}{\sin \alpha \sin \beta}$;

$$S_{total} = aH \left(\cot \frac{\alpha}{2} + \cot \frac{\beta}{2}\right) = \frac{aH \sin \frac{\alpha + \beta}{2}}{\sin \frac{\alpha}{2} \sin \frac{\beta}{2}}.$$

685. Let $\angle A$ (Fig. 162) be the acute angle of the rhombus, so that AC is the greater diagonal and $\angle OAD = \frac{\alpha}{2}$. Draw $MK \perp AC$ and $MN \perp BD$[*]. Let φ be the angle at which the plane EAC is inclined to the base. Then $\angle MKE = \varphi$ and $\angle MNE = \psi$. To determine H express MK and MN through H; we obtain $MK = H \cot \varphi$ and $MN = H \cot \psi$; substitute these expressions into the relationship

$$a = AD = AM + MD = \frac{MK}{\sin \frac{\alpha}{2}} + \frac{MN}{\cos \frac{\alpha}{2}}$$

We get

$$a = H \left(\frac{\cot \varphi}{\sin \frac{\alpha}{2}} + \frac{\cot \psi}{\cos \frac{\alpha}{2}}\right)$$

---

[*] In Fig. 162 MK should be drawn parallel to BD, and MN parallel to AC, since the diagonals of a rhombus are mutually perpendicular (see the footnote overleaf).

Answer: $$V=\frac{a^3\sin\alpha}{3\left(\dfrac{\cot\varphi}{\sin\dfrac{\alpha}{2}}+\dfrac{\cot\psi}{\cos\dfrac{\alpha}{2}}\right)}=\frac{a^3\sin^2\alpha}{6\left(\cos\dfrac{\alpha}{2}\cot\varphi+\sin\dfrac{\alpha}{2}\cot\psi\right)},$$

where the larger diagonal of the rhombus serves as the edge of the dihedral angle φ and the smaller one, of the dihedral angle ψ.

686. The line-segment AB (see Fig. 163) depicts the hypotenuse of the base. To construct the plane angle α we have to intersect the edge BB_1 by a plane perpendicular to this edge. In this case such a plane can be drawn through the leg AC. To prove this, we have to prove that $AC \perp BB_1$.

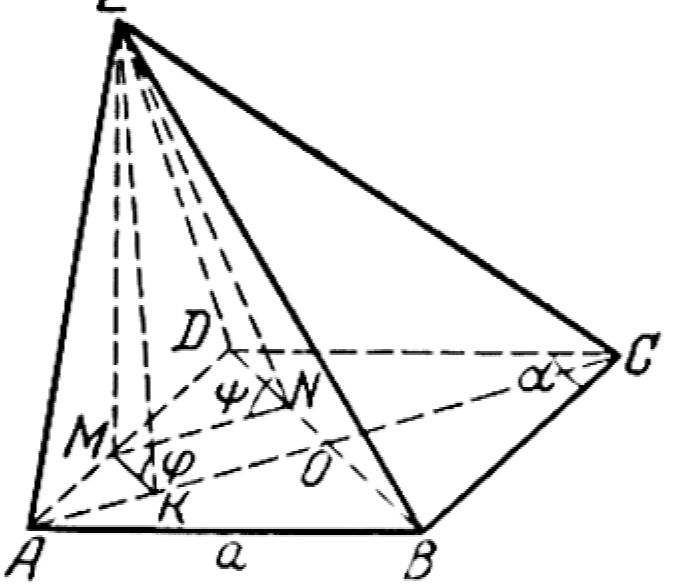


Fig. 162

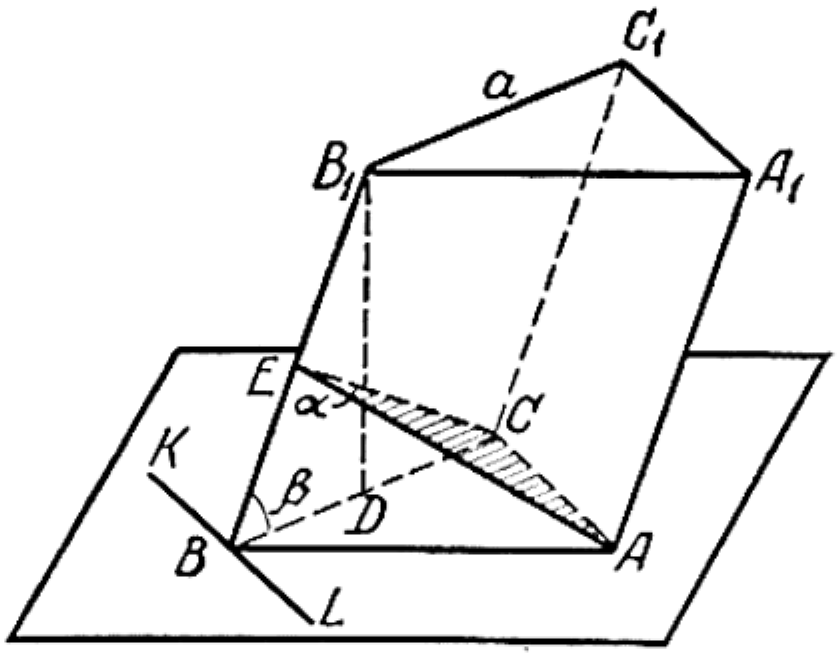


Fig. 163

By hypothesis the vertex B_1 is projected into the point D (the midpoint of BC) which lies on the leg BC. Consequently, if a straight line KL is drawn through B and perpendicular to BC, then KL is also perpendicular to BB_1 (the theorem on three perpendiculars). And hence $AC \parallel KL$, $AC \perp BB_1$, which completes the proof.

Through AC draw a plane AEC perpendicular to BB_1. The lateral surface area of the prism is equal to the perimeter $CE + AC + AE$ of the perpendicular section figure multiplied by the edge BB_1. From the right-angled triangle BCE, where $\angle CBE = \beta$ (prove it!) and $BC = a$, we find $CE = a\sin\beta$. The straight line KL, and hence, the line AC which is parallel to it are perpendicular to the face BB_1C_1C. Therefore the triangle ACE is a right one at the vertex C. Hence, $AC = CE\tan\alpha$ and $AE=\dfrac{CE}{\cos\alpha}$, thus

$$CE+AC+AE=a\sin\beta\left(1+\tan\alpha+\frac{1}{\cos\alpha}\right)$$

From the triangle BDB_1, where $BD=\dfrac{a}{2}$ we find the edge BB_1. We get $BB_1=\dfrac{a}{2\cos\beta}$, hence,

$$S_{lat}=(CE+AC+AE)\cdot BB_1=\frac{a^2\tan\beta}{2}\left(1+\tan\alpha+\frac{1}{\cos\alpha}\right)$$

Transform the expression in parentheses in the same way as in Problem **673**, and $\cos\alpha$ in the same way as in Problem **681**.

Answer: $$S_{lat}=\frac{a^2\tan\beta\cos\frac{\alpha}{2}}{\sqrt{2}\sin\left(45^\circ-\frac{\alpha}{2}\right)}.$$

687. As in the preceding problem, let us prove that the edge $AA_1\perp BC$ (Fig. 164), and hence, $BB_1\perp BC$ and the face BB_1C_1C is a rectangle. $\angle A_1AC = \angle A_1AB = 2\alpha$ (for the proof see Problem 669) and, consequently, the face $AA_1C_1C = AA_1B_1B$. Point E is the midpoint of the side AB and $EO\perp AB$ (O is the centre of the circle circumscribed about the triangle ABC); then $A_1E\perp AB$ (by the theorem on three perpendiculars). According to the law of sines we have

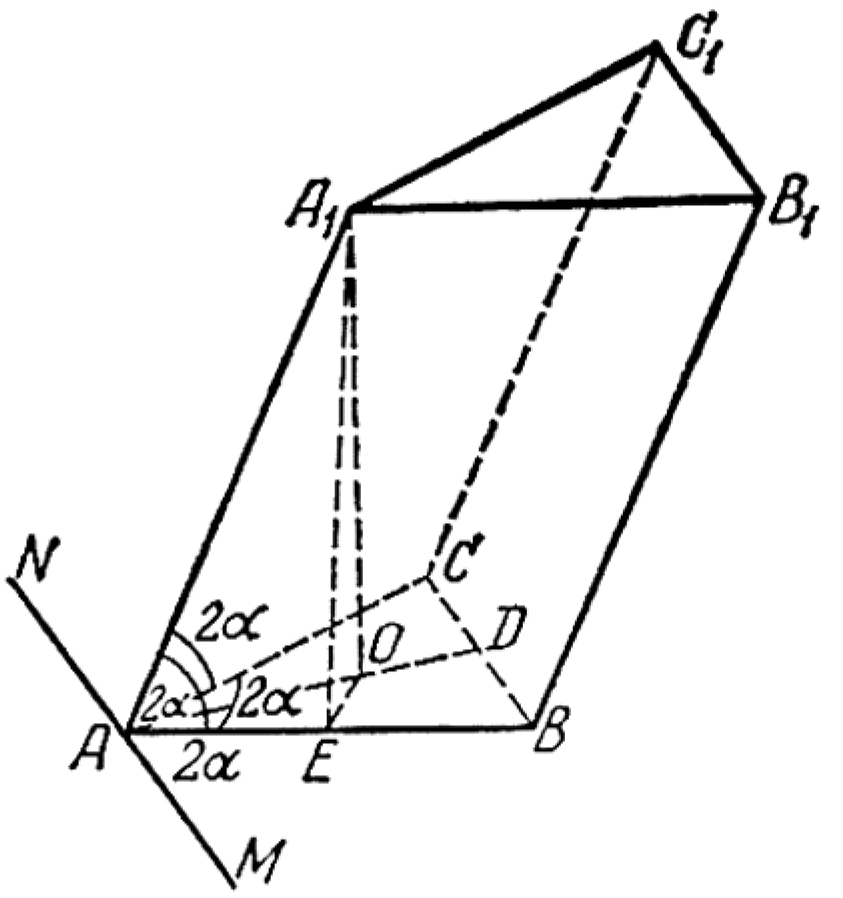


Fig. 164

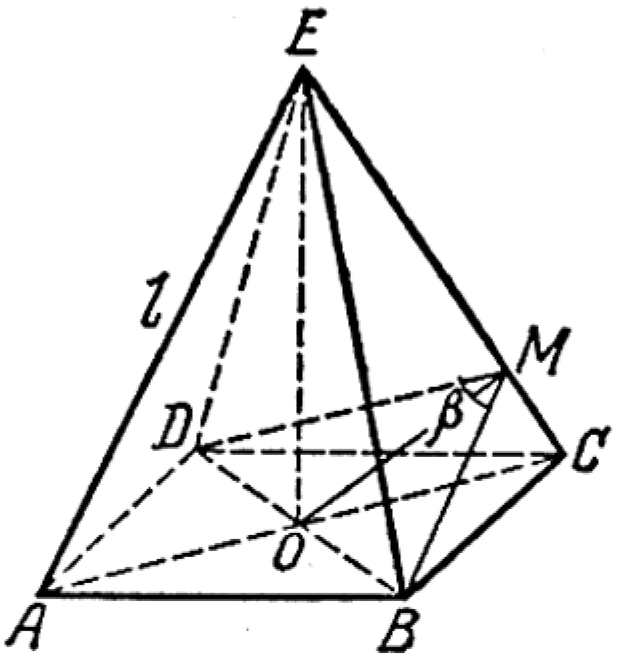


Fig. 165

$$AB = 2R\sin(90^\circ-\alpha) = 2R\cos\alpha;$$

then

$$S_{base}=\frac{1}{2}AB^2\cdot\sin 2\alpha=2R^2\cos^2\alpha\sin 2\alpha.$$

From the triangle AA_1E we have

$$AA_1=l=\frac{AE}{\cos 2\alpha}=\frac{AB}{2\cos 2\alpha}=\frac{R\cos\alpha}{\cos 2\alpha}$$

From $\triangle AA_1O$ we find

$$H=\sqrt{l^2-R^2}=\frac{R}{\cos 2\alpha}\sqrt{\cos^2\alpha-\cos^2 2\alpha}$$

(Transform the radicand in the same way as in Problem 656). The side $BC=2BD=2\cdot AB\cdot\sin\alpha$. Hence

$$=S_{base}\cdot H=2R^2\cos^2\alpha\sin 2\alpha\,\frac{R}{\cos 2\alpha}\sqrt{\cos^2\alpha-\cos^2 2\alpha}=$$

$$=2R^3\cos^2\alpha\tan 2\alpha\sqrt{\cos^2\alpha-\cos^2 2\alpha}$$

and

$$S_{lat}=2S_{AA_1B_1B}+S_{BB_1C_1C}=2l\cdot AB\cdot\sin 2\alpha+2l\cdot AB\cdot\sin\alpha=$$
$$=2l\cdot AB\,(\sin 2\alpha+\sin\alpha)$$

Answer: $V=2R^3\cos^2\alpha\tan 2\alpha\sqrt{\sin 3\alpha\sin\alpha}$;

$$S_{lat}=\frac{8R^2\cos^2\alpha\sin\frac{3\alpha}{2}\cos\frac{\alpha}{2}}{\cos 2\alpha}.$$

688. Draw the altitude OM in the triangle OCE (Fig. 165); then $\angle BMD=\beta$ (prove it!). Denote $OC=OB$ by x and find x from the formula $OC^2=CE\cdot CM$, where $CE=l$ and $CM=\sqrt{x^2-OM^2}$. From the triangle OMB we find

$$OM=OB\cot\frac{\beta}{2}=x\cot\frac{\beta}{2}$$

hence

$$CM=x\sqrt{1-\cot^2\frac{\beta}{2}}$$

Substituting into the formula $OC^2=CE\cdot CM$, we get the equation

$$x^2=lx\sqrt{1-\cot^2\frac{\beta}{2}}$$

The root $x=0$ does not obviously meet the given condition and we have

$$x=OC=l\sqrt{1-\cot^2\frac{\beta}{2}}$$

Consequently,

$$H=\sqrt{CE^2-OC^2}=\sqrt{l^2-x^2}=l\cot\frac{\beta}{2}$$

Now we find

$$V=\frac{1}{3}2x^2H$$

Note. The quantity of $\cos\beta$ is negative, since $\frac{\beta}{2}>45°$ $\left(\text{as }\tan\frac{\beta}{2}=\frac{OB}{OM}=\frac{OC}{OM}\right.$, but the inclined line OC is longer than the perpendicular OM, hence $\left.\tan\frac{\beta}{2}>1\right)$.

Answer: $V=\frac{2}{3}l^3\cot\frac{\beta}{2}\left(1-\cot^2\frac{\beta}{2}\right)=-\frac{2}{3}l^3\frac{\cot\frac{\beta}{2}\cos\beta}{\sin^2\frac{\beta}{2}}$.

689. From the triangle A_1FE (Fig. 166), where $\angle A_1FE=\alpha$, we find $FE=H\cot\alpha$ and from the triangle A_1CE, where $A_1C=d$, we find $EC=$

$=\sqrt{d^2-H^2}$ and, consequently,

$$EK=\frac{EC}{\sqrt{2}}=\sqrt{\frac{d^2-H^2}{2}}$$

Now we find the sides of the bases

$$AB=a=EK+EF$$

and

$$A_1B_1=EG=b=EK-GK=EK-EF$$

So that for the quantity

$$a^2+ab+b^2$$

entering the formula for the volume of a truncated pyramid we get the following expression:

$$(EK+EF)^2+(EK+EF)(EK-EF)+(EK-EF)^2=3EK^2+EF^2$$

Answer: $V=\frac{H}{3}(3\cdot EK^2+EF^2)=\frac{H}{6}[3(d^2-H^2)+2H^2\cot^2\alpha]$.

690. We can use the same drawing (Fig. 166) as in the preceding problem, introducing the following notation: $AA_1=l$ and $\angle A_1AC=\beta$. From the right-angled triangle AA_1C we find $AC=\frac{l}{\cos\beta}$, hence $a=FK=\frac{l}{\sqrt{2}\cos\beta}$. From the triangle AA_1E we find $H=l\sin\beta$ and $AE=l\cos\beta$, hence $FE=\frac{l\cos\beta}{\sqrt{2}}$; consequently,

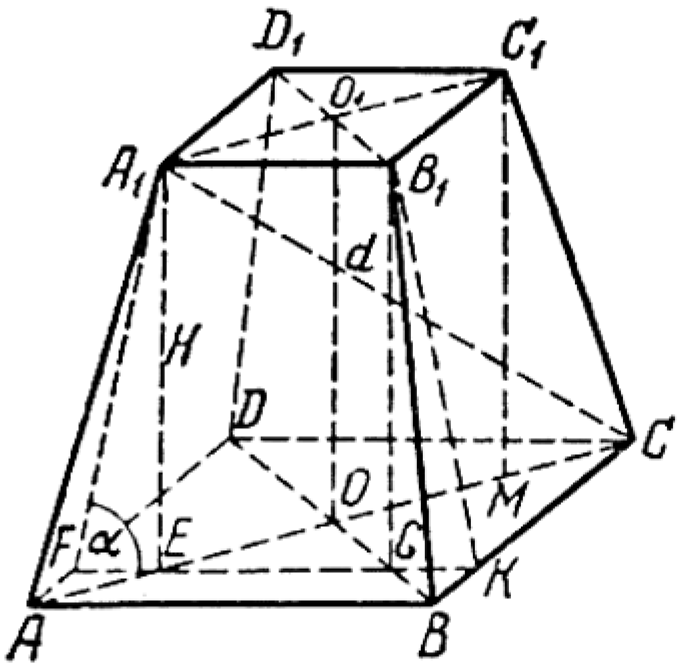


Fig. 166

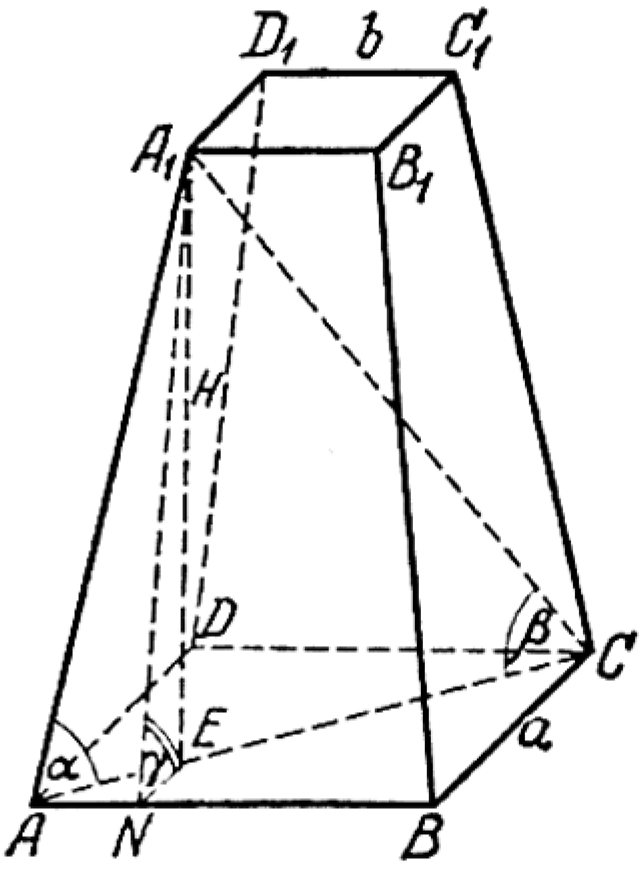


Fig. 167

$$b=EG=FK-2FE=\frac{l}{\sqrt{2}\cos\beta}(1-2\cos^2\beta)=-\frac{l\cos 2\beta}{\sqrt{2}\cos\beta}$$

Now we get

$$V=\frac{H}{3}(a^2+ab+b^2)=\frac{l^3\sin\beta}{6\cos^2\beta}(1-\cos 2\beta+\cos^2 2\beta)$$

If both the numerator and denominator are multiplied by $(1+\cos 2\beta)$ (applying the formula for a sum of cubes), we get a somewhat simpler expression.

Note. The angle β must exceed 45°, since $FK > 2\cdot FE$. Therefore, $\cos 2\beta < 0$.

Answer: $V=\frac{l^3\sin\beta}{6\cos^2\beta}(1-\cos 2\beta+\cos^2 2\beta)=\frac{l^3\sin\beta\,(1+\cos^3 2\beta)}{12\cos^4\beta}$.

691. From the triangles AA_1E and EA_1C (Fig. 167) * we have

$$AE=H\cot\alpha \quad\text{and}\quad EC=H\cot\beta$$

The lateral surface area is equal to

$$S_{lat}=4\cdot\frac{a+b}{2}\cdot A_1N=2(a+b)\cdot A_1N$$

The slant height A_1N is found from the triangle A_1EN, where

$$EN=\frac{AE}{\sqrt{2}}=\frac{H}{\sqrt{2}}\cot\alpha$$

We get

$$A_1N=H\sqrt{1+\frac{1}{2}\cot^2\alpha}$$

The sum

$$a+b=AB+A_1B_1=2A_1B_1+2AN=2\cdot NB=EC\cdot\sqrt{2}=H\cdot\sqrt{2}\cot\beta$$

Consequently,

$$S_{lat}=2H\sqrt{2}\cot\beta H\sqrt{1+\frac{1}{2}\cot^2\alpha}$$

Answer: $S_{lat}=2H^2\cot\beta\sqrt{2+\cot^2\alpha}$.

692. In the triangle A_1EN (Fig. 167), where

$$EN=AN=\frac{AB-A_1B_1}{2}=\frac{a}{2}(\sqrt{3}-1)$$

we find

$$H=A_1E=\frac{a}{2}(\sqrt{3}-1)\tan\gamma$$

and

$$A_1N=\frac{a(\sqrt{3}-1)}{2\cos\gamma}$$

We obtain now

$$V=\frac{H}{3}(3a^2+a^2+c^2\sqrt{3})=\frac{a^3}{6}(\sqrt{3}-1)(4+\sqrt{3})\tan\gamma$$

---

* For depicting a truncated pyramid see page 261.

and

$$S_{lat}=2\,(AB+A_1B_1)\,A_1N=2a\,(\sqrt{3}+1)\cdot\frac{a\,(\sqrt{3}-1)}{2\cos\gamma}=\frac{2a^2}{\cos\gamma}$$

Consequently,

$$S_{total}=S_{lat}+3a^2+a^2=\frac{2a^2\,(1+2\cos\gamma)}{\cos\gamma}$$

The expression in parentheses can be reduced to a form convenient for taking logarithms

Answer: $V=\dfrac{a^3\,(3\sqrt{3}-1)\tan\gamma}{6}\approx 0.7a^3\tan\gamma$;

$$S_{total}=\frac{2a^2\,(1+2\cos\gamma)}{\cos\gamma}=\frac{8a^2\cos\left(\frac{\gamma}{2}+30^\circ\right)\cos\left(\frac{\gamma}{2}-30^\circ\right)}{\cos\gamma}.$$

693. Let us denote the side of the cube by x (Fig. 168). From the similarity of the triangles EO_1K_1 and EOC we have

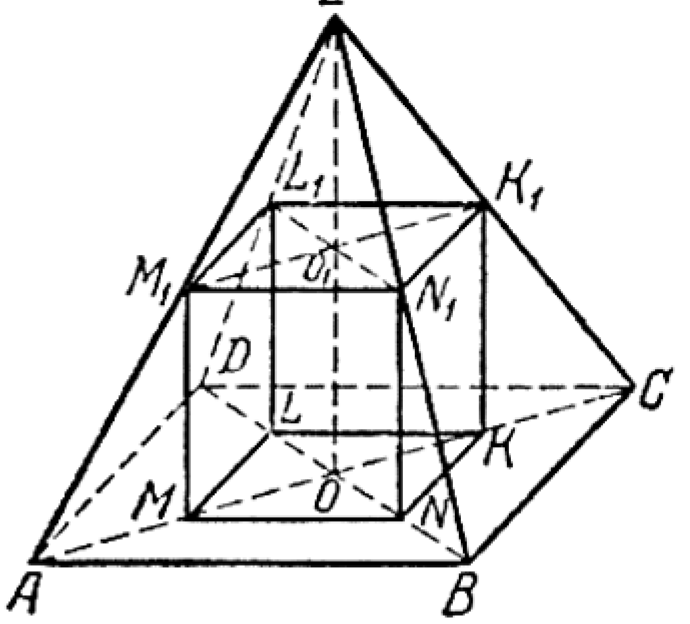


Fig. 168

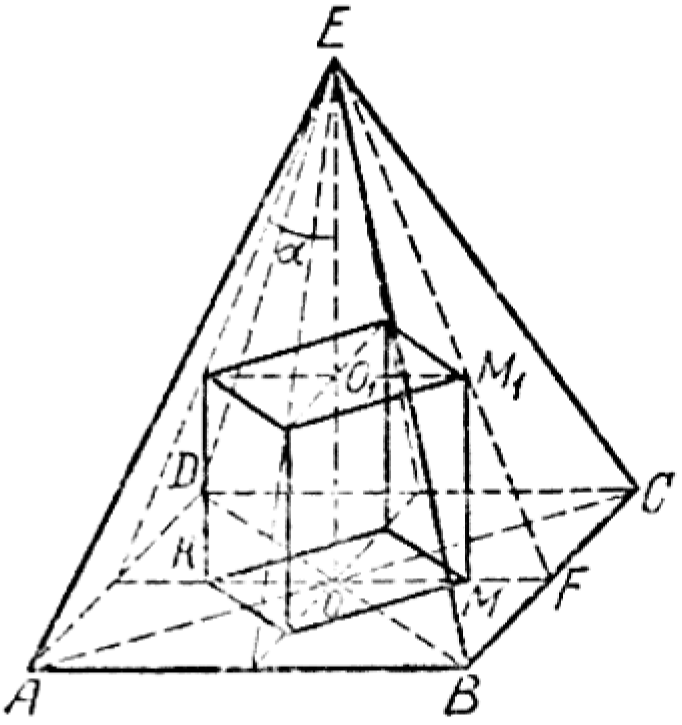


Fig. 169

$$\frac{EO_1}{EO}=\frac{O_1K_1}{OC}$$

Here

$$EO_1=EO-OO_1=H-x,\ EO=H,\ O_1K_1=\frac{x}{\sqrt{2}},\quad OC=\sqrt{l^2-H^2}$$

Consequently,

$$\frac{H-x}{H}=\frac{x}{\sqrt{2}\,\sqrt{l^2-H^2}}$$

Answer: $x=\dfrac{H\sqrt{2\,(l^2-H^2)}}{H+\sqrt{2\,(l^2-H^2)}}$.

694. From the triangle EOF (Fig. 169), where $OF=\frac{a}{2}$ and $\angle\, OEF=\alpha$, we have $H=\frac{a}{2}\cot\alpha$. Consequently, the volume of the pyramid

$$V=\frac{1}{3}a^2H=\frac{1}{6}a^3\cot\alpha$$

Let us express the side a through the edge of the cube $x=MM_1$. We have

$$a=2OF=2OM+2MF=KM+2MM_1\cdot\tan\alpha=x\sqrt{2}+2x\tan\alpha$$

Consequently,

$$V=\frac{x^3(\sqrt{2}+2\tan\alpha)^3\cot\alpha}{6}$$

Here $x^3=V_1$ is the volume of the cube.

Answer: $\frac{V}{V_1}=\frac{(\sqrt{2}+2\tan\alpha)^3\cot\alpha}{6}$.

695. (a) *Drawing*. Let us first depict the section $A_1M_1B_1$ (Fig. 170) containing the "upper" face of the cube $K_1L_1M_1N_1$ (this is a right-angled triangle with the right angle at the vertex M_1). Since the vertices K_1, L_1, M_1, N_1 lie on the lateral faces, they are found on the sides of the triangle $A_1M_1B_1$ (M_1 coincides with the vertex of the right angle; M_1K_1 represents the bisector of the right angle, since $M_1N_1=M_1L_1$). Now construct the cube $KLMNK_1L_1M_1N_1$. Inside the quadrilateral $K_1L_1M_1N_1$ take an arbitrary point O_1 depicting the point of intersection of the altitude DO and the face $K_1L_1M_1N_1$ and join it with the point O situated likewise in the quadrilateral $KLMN$. Draw O_1A_1, O_1B_1, O_1M_1 and then OA, OB, OM parallel to them, respectively. The points A, B, C of intersection of DA_1, DB_1, DM_1 and OA, OB, OM (respectively) are the vertices of the base of the pyramid.

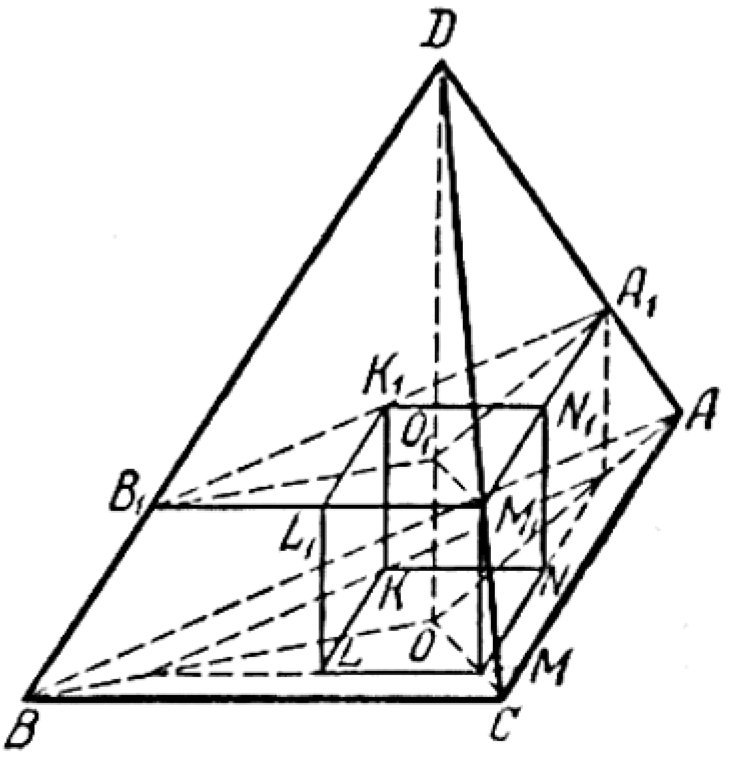


Fig. 170

(b) *Solution*. By hypothesis, $AC=6$; $BC=8$; $DO=24$*. Denote the edge of the cube by x. Then $OO_1=x$ and $DO_1=24-x$. By the property of sections parallel to the base of the pyramid we have $B_1M_1:BC=DO:DO$, i.e. $B_1M_1:8\doteq(24-x):24$, whence

$$B_1M_1=\frac{8(24-x)}{24}=\frac{24-x}{3}$$

Since the triangles $K_1B_1L_1$ and ABC are similar, we have

$$K_1L_1:B_1L_1=6:8$$

---

* The figure is drawn not to scale.

here

$$K_1L_1 = x \quad \text{and} \quad B_1L_1 = B_1M_1 - M_1L_1 = \frac{24-x}{3} - x = \frac{24-4x}{3}$$

Hence, $x : \dfrac{24-4x}{3} = 6:8$, whence $x = 3$.

Answer: 3.

696. The section BCC_1B_1 (Fig. 171) is a trapezoid (prove it!). Draw the plane MNE (M and N are the midpoints of the sides AD and BC) to intersect the plane BCC_1B_1 along NK (K is the midpoint of B_1C_1). We have $\angle NME = \angle MNE = \alpha$ and $\angle MNK = \beta$ (prove it!). The altitude KN of the trapezoid BCC_1B_1 is found from the triangle KNM, where $MN = a$ and $\angle MKN = 180° - (\alpha + \beta)$. By the law of sines $\dfrac{KN}{\sin\alpha} = \dfrac{a}{\sin(\alpha+\beta)}$, i.e. $KN = \dfrac{a\sin\alpha}{\sin(\alpha+\beta)}$. Now we find the upper base of the trapezoid (B_1C_1); since the triangle ADE is similar to B_1C_1E, we have

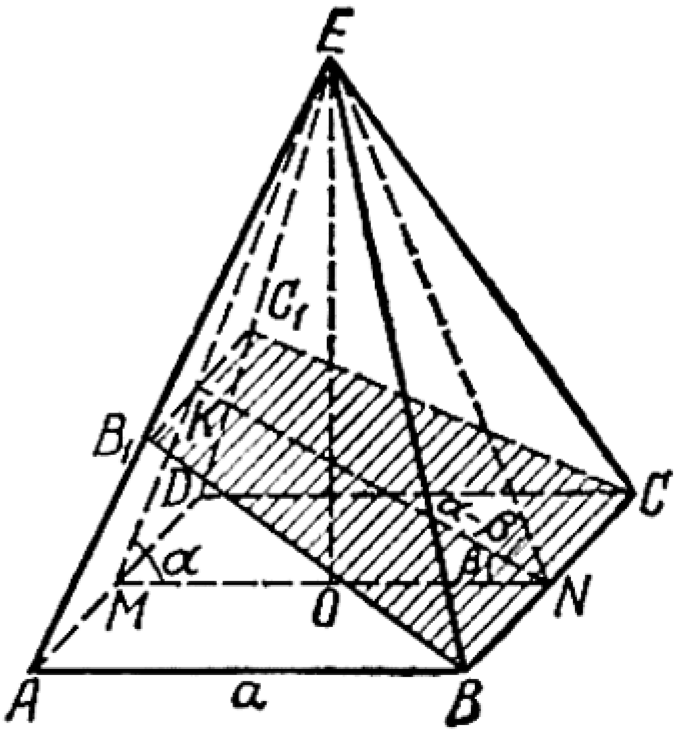


Fig. 171

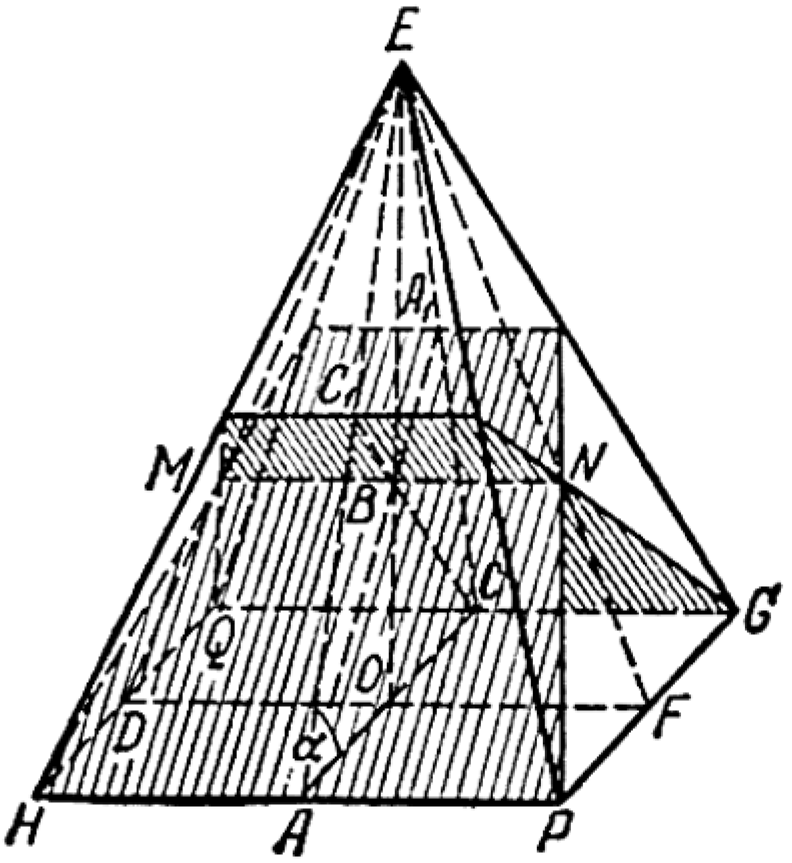


Fig. 172

$$B_1C_1 = \frac{a \cdot KE}{ME} = \frac{a \cdot KE}{NE}$$

The ratio $\dfrac{KE}{NE}$ is found from the triangle KNE, where $\angle KNE = \alpha - \beta$ and $\angle NKE = \alpha + \beta$ (as an exterior one for $\triangle KNM$). We get

$$\frac{KE}{NE} = \frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}$$

hence,

$$B_1C_1 = \frac{a\sin(\alpha-\beta)}{\sin(\alpha+\beta)}$$

The section area

$$S_{sec} = \frac{a + \dfrac{a\sin(\alpha-\beta)}{\sin(\alpha+\beta)}}{2} \cdot \frac{a\sin\alpha}{\sin(\alpha+\beta)}$$

Answer: $S_{sec}=\dfrac{a^2\sin^2\alpha\cos\beta}{\sin^2(\alpha+\beta)}$.

697. (a). *Drawing.* On constructing the pyramid $EHPGQ$ (Fig. 172) draw the line MN of intersection of the planes. It is parallel to the side HP and intersects the axis OE at the point B. The end points M and N of the line-segment MN lie on the slant heights EF and ED. Draw PN and GN, HM and QM to depict the planes intersecting along MN. Mark the points A_1 and C_1 of intersection of AB and CB with the slant heights EA and EC, respectively (A and C are the midpoints of HP and QG). The angle ABC is the plane angle of the obtained dihedral angle. By hypothesis, $\angle ABC = 90°$, i.e. the triangle ABC is an isosceles right-angled one and

$$BO=AO=\frac{a}{2}$$

(b) *Solution.* From the similarity of the triangles EMN and EDF, wherein $DF = a$, we have $MN = a\cdot\dfrac{EB}{EO}$. The angle OAE is the plane angle of the dihedral angle α, hence $EO = AO\tan\alpha=\dfrac{a}{2}\tan\alpha$. Furthermore, $EB = EO - BO = \dfrac{a}{2}(\tan\alpha - 1)$. Consequently,

$$MN=a\cdot\frac{\tan\alpha-1}{\tan\alpha}=a\,(1-\cot\alpha)$$

Answer: $MN=a\,(1-\cot\alpha)=\dfrac{\sqrt{2}a\sin(\alpha-45°)}{\sin\alpha}$.

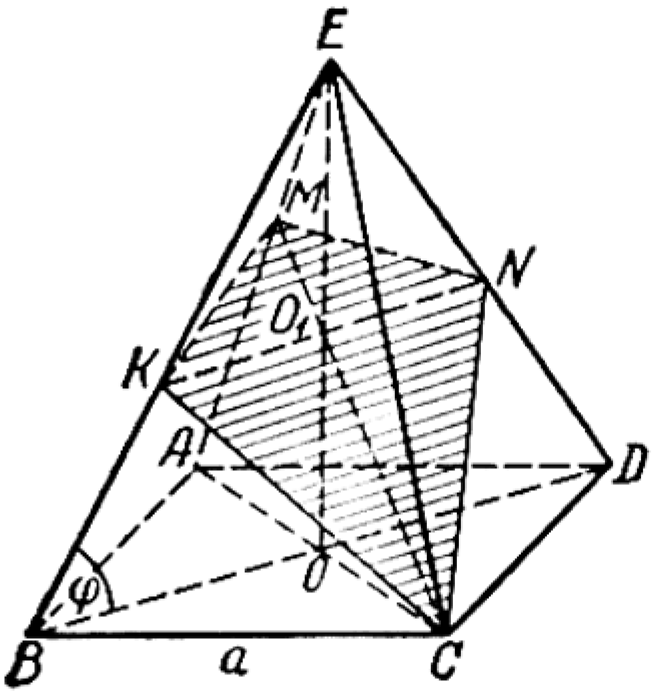


Fig. 173

698. (a) *Drawing.* Draw the straight line CM (Fig. 173) depicting the perpendicular dropped from C to AE. Through the point O_1 of intersection of CM and EO draw KN parallel to BD. The quadrilateral $KCNM$ represents the section. The proof follows from the solution below.

(b) *Solution.* Since the plane $KCNM$ is perpendicular to the edge AE, the sides MK and MN, as well as the diagonal CM of the section $KCNM$, are perpendicular to AE. Since the diagonal CM lies in the plane of the isosceles triangle AEC, it intersects EO which is the altitude of this triangle. On the other hand, the diagonal KN contained in the plane of the triangle BED (and, as we are just going to prove, is parallel to the base BD of this triangle) also intersects EO which is the altitude of the triangle BED. And since the plane $KCNM$ and the line OE have only one common point O_1, the diagonals KN and MC intersect at this point.

The plane $KCNM$ is perpendicular to the edge AE; therefore the angles EMK and EMN are the right ones. The right-angled triangles EMK and EMN are congruent (prove it!); consequently, $MK = MN$ and $EK = EN$. It follows from the last equality that $KN \parallel BD$ and that $KO_1 = O_1N$. Hence, the diagonals MC and KN are mutually perpendicular and $S_{sec} = \dfrac{1}{2}\,MC\cdot KN$.

The diagonal MC is found from the right-angled triangle AMC, wherein $\angle CAM = \varphi$ and $AC = a\sqrt{2}$. We get $MC = a\sqrt{2}\sin\varphi$.

The diagonal KN is found from the isosceles triangle KEN, wherein $\angle EKN = \varphi$. We have $KN = 2\cdot O_1E\cot\varphi$, where $O_1E = OE - OO_1$. The line-segment OE is determined from the triangle AOE (or BOE); we find $OE = \frac{a\sqrt{2}}{2}\tan\varphi$. The line-segment OO_1 is determined from the triangle OCO_1, wherein $\angle OCO_1 = 90° - \angle MAC = 90° - \varphi$. We find

$$OO_1 = OC\cdot\tan(90° - \varphi) = \frac{a\sqrt{2}}{2}\cot\varphi$$

Now we get

$$KN = 2\cdot O_1E\cot\varphi = 2\left(\frac{a\sqrt{2}}{2}\tan\varphi - \frac{a\sqrt{2}}{2}\cot\varphi\right)\cot\varphi = a\sqrt{2}\,(1-\cot^2\varphi).$$

Hence,

$$S_{sec} = \frac{1}{2}MC\cdot KN = a^2(1-\cot^2\varphi)\sin\varphi = -\frac{a^2\cos 2\varphi}{\sin\varphi}$$

Note. For the plane $KCNM$, which is perpendicular to AE, to yield a section of the pyramid it is necessary that the point M of its intersection with AE lie on the line-segment AE itself (but not on its extension), for which purpose the angle AEC must be acute, i.e. $\angle AEC = 180° - 2\varphi < 90°$. Consequently, $\varphi > 45°$, and therefore $\cos 2\varphi$ is a negative quantity.

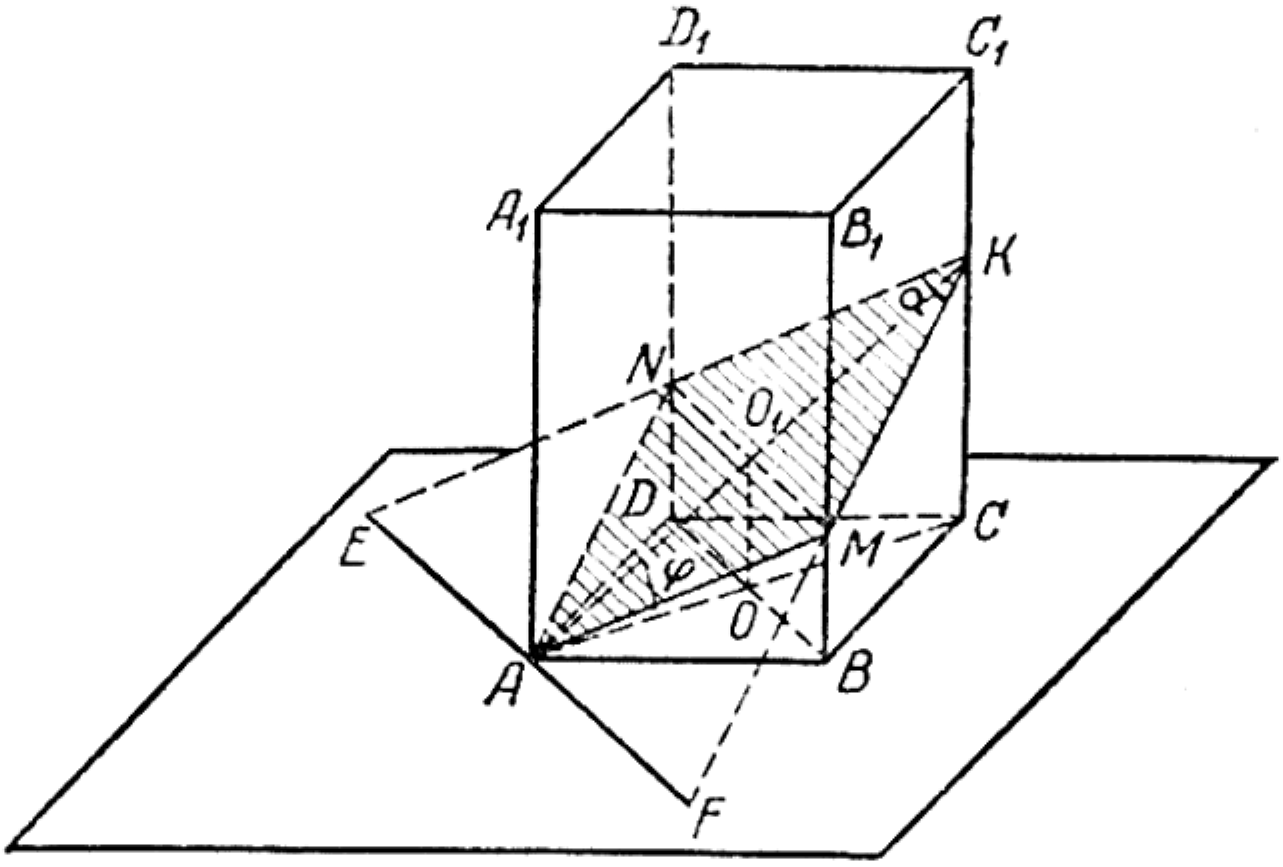


Fig. 174

Answer: $S_{sec} = -\frac{a^2\cos 2\varphi}{\sin\varphi} = \frac{a^2\cos(180° - 2\varphi)}{\sin\varphi}$.

699. The quadrilateral $AMKN$ (Fig. 174), yielded by the section of the lateral surface of the prism, is always a parallelogram (prove it!). For the section figure to be a rhombus it is necessary that $AM = AN$. Since the triangles ADN and ABM are congruent (prove it!), $DN = BM$. Hence, MN is parallel to BD and to the plane $ABCD$ as well. Consequently, the line EF of intersection of

the planes $AMKN$ and $ABCD$ is parallel to the diagonal MN (as well as to the diagonal BD) and, hence, perpendicular to the other diagonal AK of the rhombus (and also to the diagonal AC). Therefrom it follows that $\varphi = \angle CAK$ is the plane angle of the required dihedral angle. The line OO_1, joining the centre of the rhombus O_1 with the centre of the base of the prism is perpendicular to the base (prove it!).

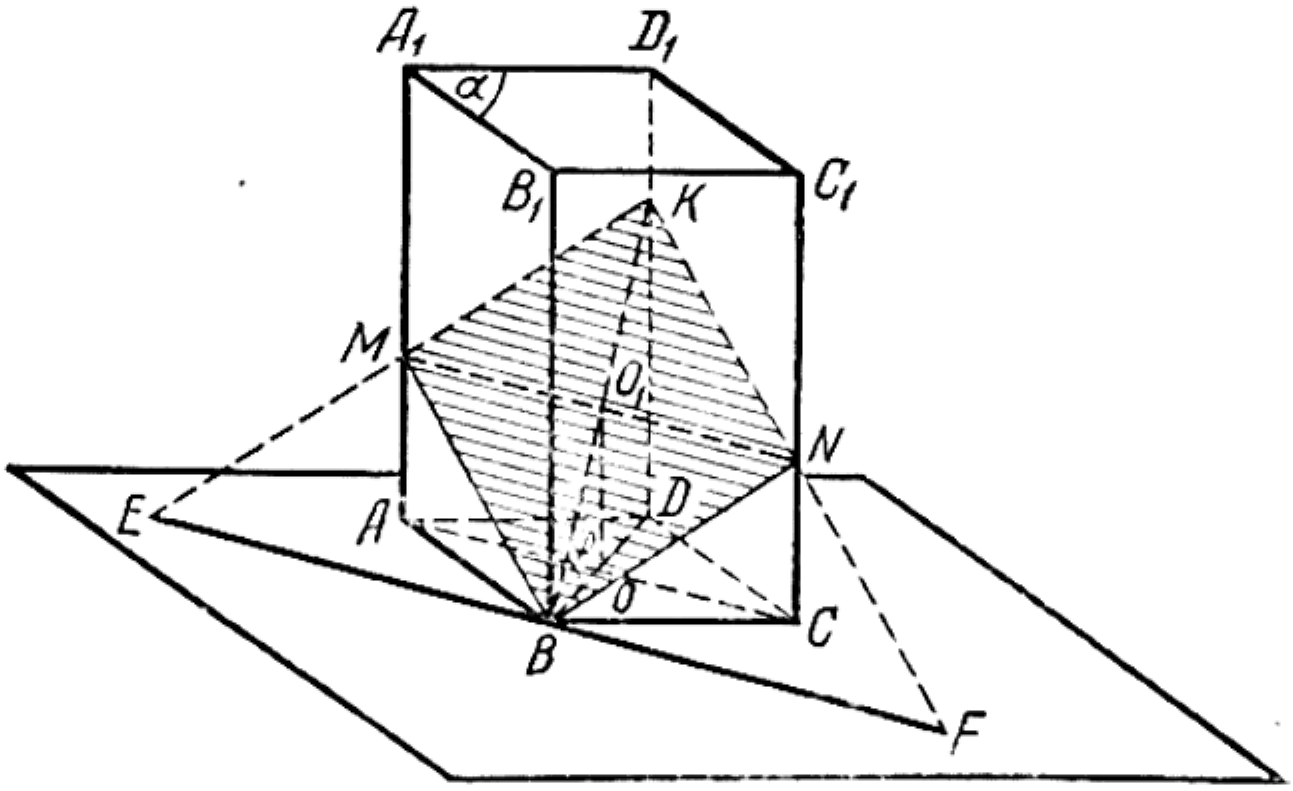


Fig. 175

From the triangle AOO_1 we find

$$\cos\varphi = \frac{AO}{AO_1} = \frac{OB}{AO_1} = \frac{O_1M}{AO_1} = \tan\frac{\alpha}{2}$$

Note. The plane drawn through the straight lines AM and AN intersects the edge CC_1 only if $CC_1 \geqslant CK$, i.e. if the altitude of the prism is not less than

$$a\sqrt{2}\tan\varphi = \frac{a\sqrt{2}\sqrt{1-\cos^2\varphi}}{\cos\varphi} = \frac{a\sqrt{2}\sqrt{1-\tan^2\dfrac{\alpha}{2}}}{\tan\dfrac{\alpha}{2}} = \frac{a\sqrt{2\cos\alpha}}{\sin\dfrac{\alpha}{2}}$$

Otherwise the required section can be drawn neither through the point A, nor through any other point on the edge AA_1.

Answer: $\varphi = \arccos\tan\frac{\alpha}{2}$. The problem is solvable only if $H \geqslant \dfrac{a\sqrt{2\cos\alpha}}{\sin\dfrac{\alpha}{2}}$.

700*. (See the solution of the preceding problem.) Since $MN = AC$ (Fig. 175) and $BK > BD$, and by hypothesis, $BK = MN$, we have $AC > BD$, i.e. AC is the greater diagonal of the rhombus, hence, $\angle ABC$ is an obtuse angle, and $\angle BAD$, an acute one.

The angle $\varphi = \angle OBO_1$ is the plane angle of the required dihedral angle. From the triangle OO_1B we have $\cos\varphi = \dfrac{OB}{O_1B}$, where $OB = OA\cdot\tan\dfrac{\alpha}{2}$.

---

* For drawing a right prism see Fig. 83.

And since $OA = O_1M = O_1B$, then $\cos\varphi = \frac{O_1B \cdot \tan\frac{\alpha}{2}}{O_1B} = \tan\frac{\alpha}{2}$. Here $\tan\frac{\alpha}{2} < 1$, since α is an acute angle.

Answer: $\varphi = \arccos\tan\frac{\alpha}{2}$; the problem is solvable only if

$$DD_1 \geqslant \frac{BD\sqrt{\cos\alpha}}{\sin\frac{\alpha}{2}}$$

701. Cf. the preceding problem. The area S_{sec} of the rhombus $BNKM$ (Fig. 176) is

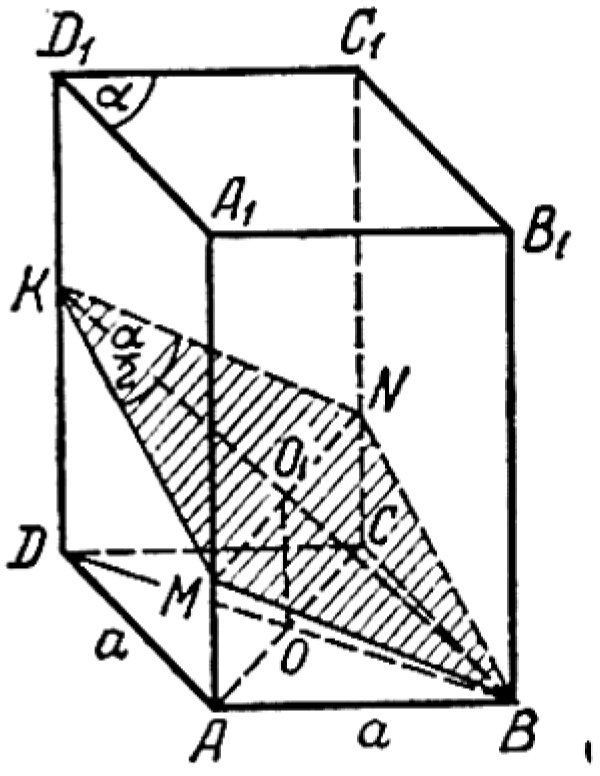


Fig. 176

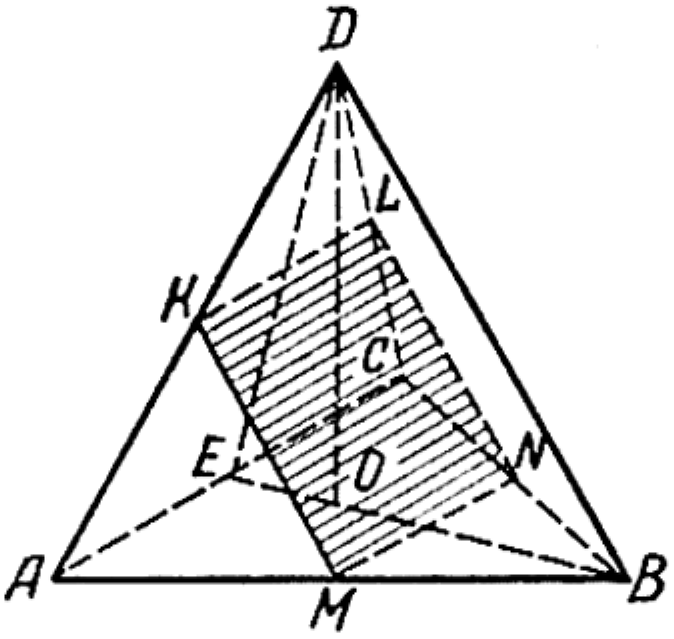


Fig. 177

$$S_{sec} = \frac{1}{2} \cdot MN \cdot BK = 2MO_1 \cdot BO_1$$

From the triangle MO_1B, wherein $\angle MBO_1 = \frac{\alpha}{4}$, we find:

$$BO_1 = MO_1 \cdot \cot\frac{\alpha}{4}$$

Hence,

$$S_{sec} = 2 \cdot MO_1^2 \cot\frac{\alpha}{4} = 2AO^2 \cot\frac{\alpha}{4}$$

AO is found from the triangle AOB, wherein $AB = a$ and $\angle ABO = \frac{\alpha}{2}$. We get $AO = a\sin\frac{\alpha}{2}$.

Answer: $S_{sec} = 2a^2 \sin^2\frac{\alpha}{2} \cot\frac{\alpha}{4}$.

702.* Let the cutting plane be drawn through the midpoint M (Fig. 177) of the edge AB and parallel to the edges AC and BD. The edge AC is contained in the plane ABC. Therefore, the plane drawn through M and parallel to AC intersects the face ABC along MN parallel to AC. Hence, MN is a midline of the triangle ABC $\left(MN=\frac{1}{2}AC=\frac{b}{2}\right)$, i.e. N is the midpoint of the edge BC. The edge BD lies in the plane BCD, and the cutting plane is parallel to this edge. Therefore, $NL \parallel BD$ $\left(NL=\frac{1}{2}BD=\frac{b}{2}\right)$ and L is the midpoint of the edge CD. Similarly, we prove that $MK=\frac{b}{2}$, and that K is the midpoint of the edge AD.

---

* For drawing a regular triangular pyramid see Fig. 82.

Consequently,

$$KL \parallel AC \text{ and } KL=\frac{b}{2}$$

Hence, the section $MNLK$ is a rhombus. Furthermore, the angle NMK is a right one. Indeed, the edge BD is contained in the plane BDE (E is the midpoint of AC), which is perpendicular to the edge AC. Consequently, $BD \perp AC$. But, as has been proved, $MK \parallel BD$ and $MN \parallel AC$, hence, $MK \perp MN$, wherefrom it follows that $MNLK$ is a square with the side $\frac{b}{2}$.

Answer: $S_{sec}=\frac{b^2}{4}$.

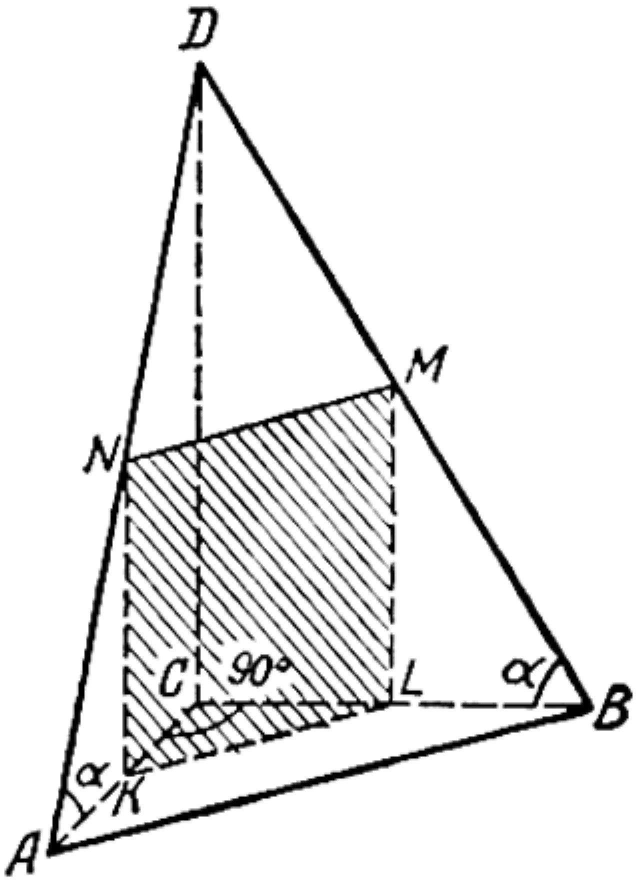


Fig. 178

703. Let CD (Fig. 178) be the lateral edge perpendicular to the base. Since, by hypothesis, $\angle DAC=\angle DBC=\alpha$, we have $AC=CB$, i.e. the triangle ABC is an isosceles one at the vertex C of the pyramid and, hence, by hypothesis, $\angle C=90°$.

Any section of the pyramid perpendicular to the base ABC is a quadrilateral $NKLM$ with two right angles ($\angle NKL$ and $\angle KLM$). For this quadrilateral to become a square the following condition should be satisfied: $KN=KL=LM=x$. From the congruence of the triangles AKN and BLM (prove it!) it follows that $AK=BL$, hence, $KC=CL$, and $KC=\frac{KL}{\sqrt{2}}=\frac{x}{\sqrt{2}}$. From the triangle AKN we find $AK=KN\cot\alpha=x\cot\alpha$. Since $KC+AK=AC=a$, we get the equation

$$\frac{x}{\sqrt{2}}+x\cot\alpha=a$$

whence

$$x=\frac{a\sqrt{2}}{1+\sqrt{2}\cot\alpha}$$

Answer: $S_{sec}=x^2=\frac{2a^2}{(1+\sqrt{2}\cot\alpha)^2}$.

704. The section yields the trapezoid MA_1B_1N (Fig. 179) equal to the lateral face DD_1C_1C (prove it!). In the cut-off portion $A_1B_1C_1D_1MNCD$ we have $A_1D_1=$

$= B_1C_1 = NC = MD$ (as the segments of parallel lines contained between parallel planes). The obtained solid is an oblique prism with the base CC_1D_1D. Draw the plane $FGQQ_1$ through the slant height FG of the frustum of a pyramid and the apothem OG of the base; we get $\angle FGL = \alpha$ (prove it!). The perpendicular LK dropped from L to GF is the altitude of the prism (prove it!). From the triangle LKG, wherein $LG = Q_1F = a$, we have $LK = a \sin\alpha$. From the triangle FLG we find $FG = \frac{LG}{\cos\alpha} = \frac{a}{\cos\alpha}$. The volume of the prism is computed by the formula

$$V = \frac{D_1C_1 + DC}{2} \cdot FG \cdot LK$$

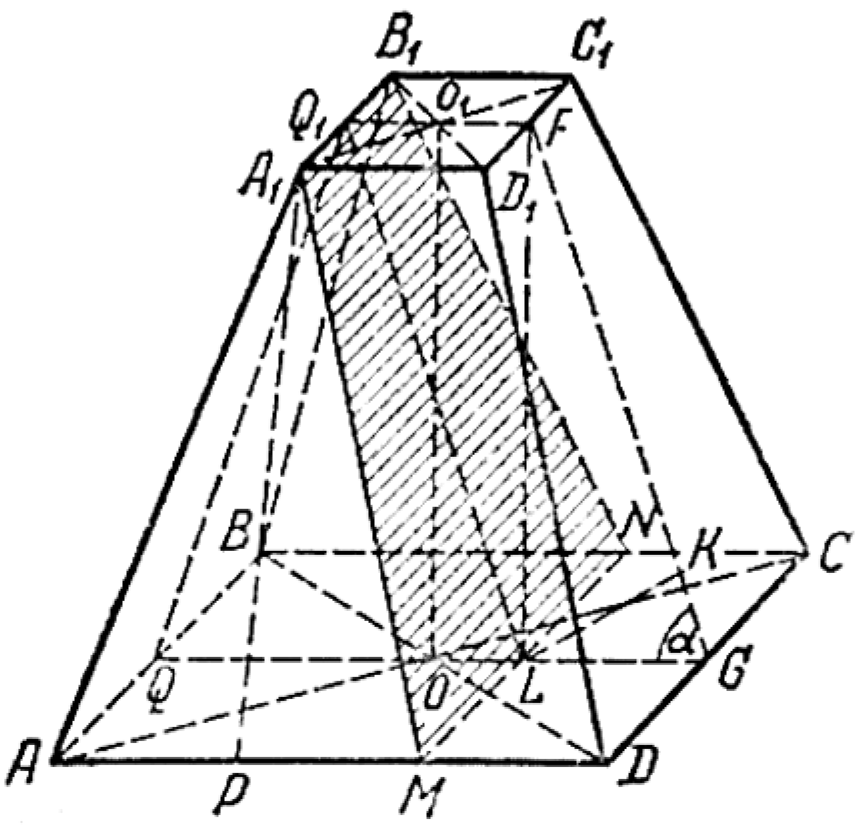


Fig. 179

Now we find the total surface area S of the solid AMA_1B_1NB cut off by the plane A_1B_1NM. The face AA_1B_1B is equal to the section MA_1B_1N (prove it!). Each of these faces has an area $S_1 = \frac{a+3a}{2} \cdot QQ_1$, where $QQ_1 = FG = \frac{a}{\cos\alpha}$. Either of the faces AA_1M and BNB_1 has an area $S_2 = \frac{AM \cdot A_1P}{2}$, where $AM = AD - MD = 3a - a = 2a$ and $A_1P = FG = \frac{a}{\cos\alpha}$. The area of the face $ABNM$ is $S_3 = AM \cdot AB = 2a \cdot 3a$. We have

$$S = 2S_1 + 2S_2 + S_3.$$

Answer: $V = 2a^3 \tan\alpha$; $S = \dfrac{12a^2 \cos^2 \frac{\alpha}{2}}{\cos\alpha}$.

Preliminary Notes to Problems **705** *to* **708**

When solving Problems 705 to 708 use should be made of the following theorem.

If a polygon $ABCDE\ldots$ contained in a plane P is orthogonally projected on a plane P_1 as a polygon $A_1B_1C_1D_1E_1\ldots$, then the area S of the polygon $ABCDE\ldots$ and the area S_1 of the polygon $A_1B_1C_1D_1E_1\ldots$.. are related in the following way

$$S_1 = S\cos\alpha,$$

where α is the angle between the planes P and P_1.

Proof. First consider the case when the projected figure is the triangle ABC (Fig. 180*a*), whose side AB is parallel to the projection plane P_1. Draw the plane Q through AB and parallel to the plane P_1 (E is the point of intersection with the projecting line CC_1). We get the triangle ABE congruent to the triangle $A_1B_1C_1$. Draw the altitude CD of the triangle ABC; ED is then the altitude of the triangle AEB, and the angle $\alpha = \angle EDC$ is the plane angle of the dihedral angle $CABE$ equal to the angle between the planes P and P_1. From the triangle DCE we find $DE = CD\cos\alpha$. Consequently,

$$S_1 = \frac{1}{2} AB \cdot DE = \frac{1}{2} AB \cdot DC \cos\alpha = S\cos\alpha$$

Then consider the case when the projected figure is the triangle LMN (Fig. 180*b*), whose sides are not parallel to the plane P_1. Such a triangle can be divided into two triangles of the type considered above. For this purpose it is sufficient to draw the plane Q parallel to P_1 through one of its vertices M which should be neither the closest to, nor the remotest from the plane P_1. This plane intersects the triangle LMN along the straight line KM parallel to P_1. If S' and S'' are the respective areas of the triangles KMN and LMK, and S_1' and S_1'', the areas of their projections (i.e. of the triangles $K_1M_1N_1$ and $L_1M_1K_1$), then, as has been proved,

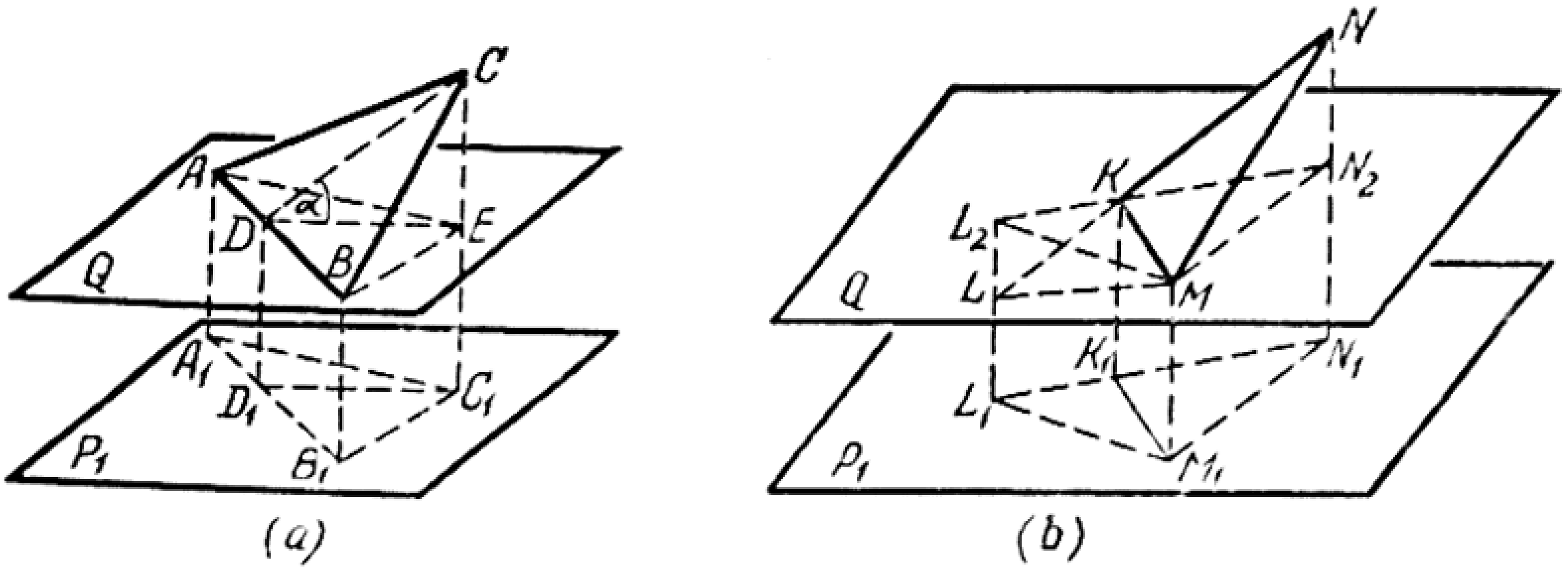


Fig. 180

$$S_1' = S' \cos\alpha \text{ and } S_1'' = S'' \cos\alpha$$

And since $S = S' + S''$ and $S_1 = S_1' + S_1''$, we have

$$S_1 = S_1' + S_1'' = S' \cos\alpha + S'' \cos\alpha = (S' + S'') \cos\alpha = S \cos\alpha$$

If the polygon has more than three sides, then we divide it into triangles and, reasoning in the same way as in the above case, prove the general theorem.

Let us draw our attention to the fact that this theorem holds true for the areas of curvilinear figures as well. To prove it we have to inscribe a polygon in the given curvilinear figure and pass to the limit.

705. We have (Fig. 181) $S_{base} = \frac{a^2\sqrt{3}}{4}$ and $H = BB_1 = BD + DB_1$. From the triangles BED and B_1E_1D (E and E_1 are the midpoints of AC and A_1C_1) we have

$$BD = BE \tan\alpha = \frac{a\sqrt{3}}{2}\tan\alpha$$

and

$$B_1D = \frac{a\sqrt{3}}{2}\tan\beta$$

Hence,

$$V = S_{base}\cdot H = \frac{3a^3}{8}(\tan\alpha + \tan\beta) = \frac{3a^3 \sin(\alpha+\beta)}{8\cos\alpha\cos\beta}$$

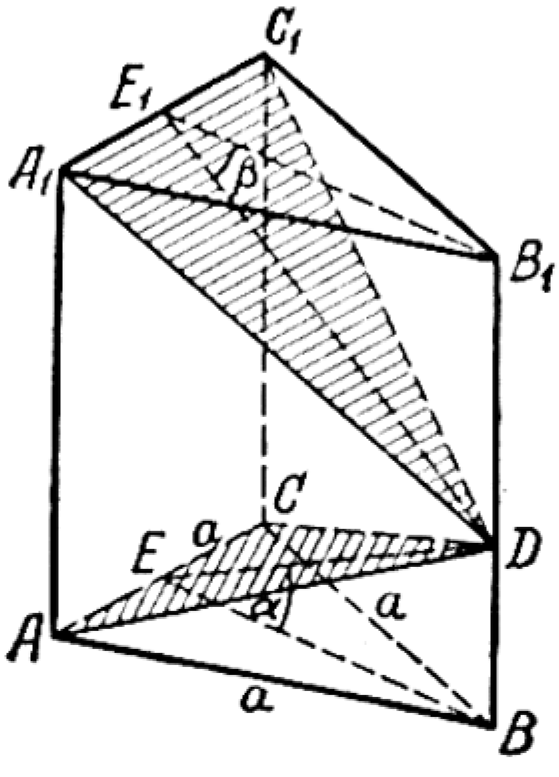


Fig. 181

The section ADC is projected on the plane of the lower base as the triangle ABC. As has been proved (see the Preliminary Notes) the area S of the section ADC

is related to the area of the triangle ABC, i.e. to the S_{base}, by the formula $S_{base}=S\cos\alpha$, hence, $S=\frac{S_{base}}{\cos\alpha}$. Proceeding in the same way (i.e. projecting the section A_1DC_1 on the upper base), we find that the area of the section A_1DC_1 is $S'=\frac{S_{base}}{\cos\beta}$. Consequently,

$$S+S'=S_{base}\left(\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}\right)$$

Answer: $V=\frac{3a^3\sin(\alpha+\beta)}{8\cos\alpha\cos\beta}$

$$S+S'=\frac{a^2\sqrt{3}}{4}\cdot\frac{\cos\alpha+\cos\beta}{\cos\alpha\cos\beta}=\frac{a^2\sqrt{3}\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\cos\alpha\cos\beta}$$

706. (a) *Drawing*. Join the midpoints K and L (Fig. 182) of the sides AB and AD. Through the point E of intersection of KL and AC draw the straight line EN (the angle NEC depicts the plane angle of the dihedral angle α). Through the point O_2 of intersection of EN and the axis OO_1 draw PM parallel to BD. The pentagon $KLMNP$ represents the section. The proof is obvious from the solution below.

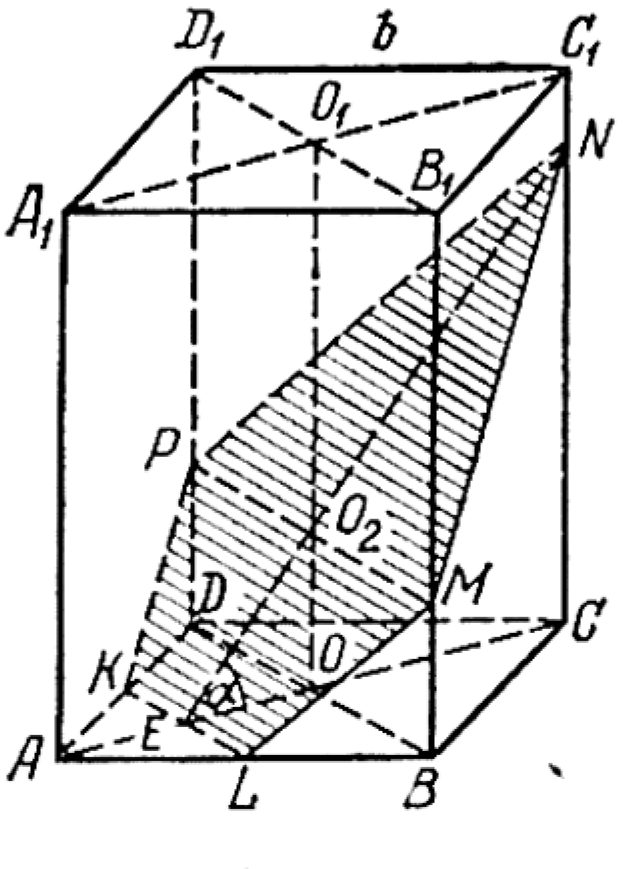


Fig. 182

(b) *Solution*. Since $KL \parallel BD$, the plane $KLMNP$ (passing through KL) intersects the diagonal plane DBB_1D_1 (passing through the diagonal BD) along the straight line PM parallel to KL and BD. The axis OO_1 of the prism lies in the diagonal plane DBB_1D_1 and, hence, intersects PM. The plane $KLMNP$ intersects the diagonal plane ACC_1A_1 (passing through the diagonal AC) along the straight line NE (E is the midpoint of KL), which also intersects the axis OO_1. But since the plane $KLMNP$, containing PM and EN, intersects the axis OO_1 only at the point O_2, both EN and MP pass through this point, i.e. the point of intersection of PM and EN lies on the axis OO_1. The lines EC and EN are perpendicular to KL (the theorem on three perpendiculars); hence, $\angle CEN=\alpha$.

The area S of the pentagon $KLBCD$ is equal to the area of the square $ABCD$ less the area of the triangle AKL, thus, $S=b^2-\frac{b^2}{8}=\frac{7}{8}b^2$. The area S_{sec} of the pentagon $KLMNP$ is determined according to the theorem proved in the Preliminary Notes to Problem 705. We have $\frac{7}{8}b^2=S_{sec}\cos\alpha$, i.e.

$$S_{sec}=\frac{7b^2}{8\cos\alpha}$$

Comparing the triangles MO_2N and BOC ($BO=MO_2$ and $MN>BC$), we make sure that $\angle MNO_2<\angle BCO$; and since $\angle BCO=45°$, $\angle MNO_2<$ $<45°$ and, consequently, the angle $\varphi=\angle MNP$ is acute. All the rest of the angles of the pentagon are obtuse (the acute angle $\angle NMO_2=90°-\angle MNO_2$

exceeds 45°; the angle MLK is equal to $180° - \angle LMO_2 = 180° - \angle NMO_2$). From the triangle MO_2N we have

$$\tan\frac{\varphi}{2} = \frac{MO_2}{NO_2}$$

but

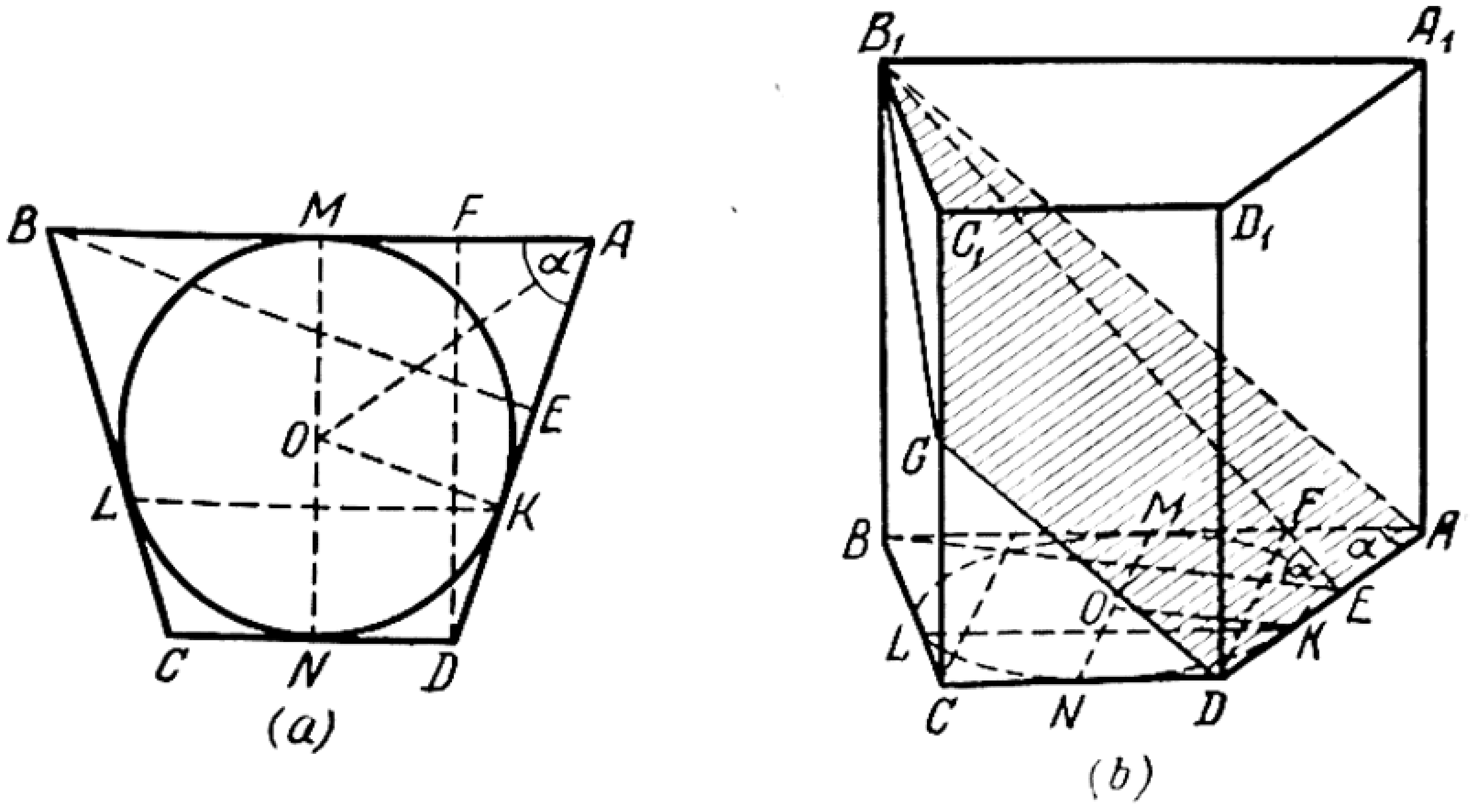
$$NO_2 = \frac{OC}{\cos\alpha} = \frac{OB}{\cos\alpha} = \frac{MO_2}{\cos\alpha}$$

Hence,

$$\tan\frac{\varphi}{2} = \cos\alpha$$

Answer: $S_{sec} = \frac{7b^2}{8\cos\alpha}$; $\varphi = 2\arctan(\cos\alpha)$.

707. (a) *Drawing*. First draw separately the base of the prism (Fig. 183*a*). Then construct an ellipse (Fig. 183*b*) depicting the circle about which the base is circumscribed*. Draw a diameter (MN) of the ellipse and through its ends draw tangent lines CD and AB representing the straight lines on which the bases of the isosceles trapezoid lie. Draw some line KL parallel to CD and AB to intersect the ellipse at points K and L. Through these points draw tangent lines DA and BC to the ellipse. The quadrilateral $ABCD$ depicts the isosceles trapezoid circumscribed about the circle. Then complete the drawing of the right prism $ABCDA_1B_1C_1D_1$. The cutting plane passing through the side AD and vertex B_1 intersects the face AA_1B_1B along the straight line AB_1 and the face DD_1C_1C (which is parallel to AA_1B_1B) along DG parallel to AB_1. The section yields the quadrilateral AB_1GD. From the point B draw the straight line BE parallel to the radius OK joining the centre O with the point of tangency K. The line represents the perpendicular dropped from B to AD. Consequently, the angle BEB_1 depicts the plane angle α.



(a) (b)

Fig. 183

---

* For constructing an ellipse see Fig. 92.

(b) *Solution.* From the triangle DFA (Fig. 183*b*), wherein $DF = MN = 2r$ and $\angle DAF = \alpha$, we find $BC = AD = \frac{2r}{\sin\alpha}$. Denote AB by a; CD, by b; $AD = BC$, by c. By the property of the circumscribed quadrilateral

$$a+b=AB+CD=AD+BC=2c=\frac{4r}{\sin\alpha}$$

We have

$$S_{base}=\frac{a+b}{2}h=\frac{2r}{\sin\alpha}\cdot 2r=\frac{4r^2}{\sin\alpha}$$

Consequently (see the Preliminary Notes to Problem 705),

$$S_{sec}=\frac{S_{base}}{\cos\alpha}=\frac{4r^2}{\sin\alpha\cos\alpha}=\frac{8r^2}{\sin 2\alpha}$$

The altitude $H = BB_1$ is found from the triangle BB_1E, BE being determined beforehand from the triangle BEA, wherein

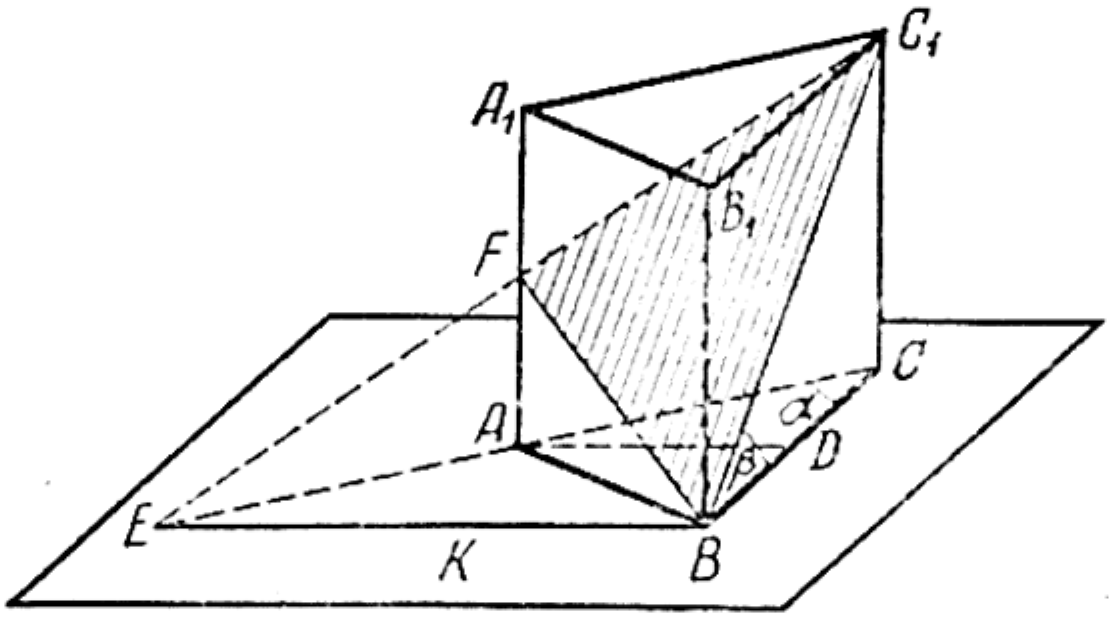


Fig. 184

$$AB=a=2AM=2OM\cdot\cot\angle OAM=2r\cot\frac{\alpha}{2}$$

We have

$$BE=a\sin\alpha \text{ and } H=BE\tan\alpha$$

Hence,

$$H=2r\cot\frac{\alpha}{2}\sin\alpha\tan\alpha$$

Now we find

$$S_{lat}=H(a+b+2c)=4Hc.$$

Answer: $S_{lat}=16r^2\tan\alpha\cot\frac{\alpha}{2}$; $S_{sec}=\frac{8r^2}{\sin 2\alpha}$.

708. (a) *Drawing.* The cutting plane P may be drawn through either of the two diagonals of the face BCC_1B_1 (Fig. 184). Let us draw it through the diagonal BC_1. By hypothesis, $P \parallel AD$. Consequently, the plane P intersects the plane containing the base ABC along the straight line BK parallel to AD (the entire line-segment BK lies outside the triangle ABC). Since the face BCC_1B_1 is perpendicular to AD, it is also perpendicular to BK; hence, $\angle CBC_1$ is the plane angle of the dihedral angle β at the edge BK.

Depict now the triangle yielded by the cutting plane P. One side of this triangle (BC_1) is known, we have only to find the opposite vertex, i.e. the point of intersection of the plane P and the edge AA_1. For this purpose it is sufficient to join the point E, at which BK intersects the extension of the edge AC, with the point C_1. The point F at which C_1F intersects the edge AA_1 is the required vertex.

Let us prove it. Since the point E lies on the line BE of intersection of the planes P and ABC, this point belongs to the plane P. On the other hand, the point E lies on the line AC of intersection of the planes ACC_1A_1 and ABC, hence, it belongs to the plane ACC_1A_1 (it is actually situated on the extension of the face ACC_1A_1). Consequently, the point E must belong to the line of intersection of the planes P and ACC_1A_1. By hypothesis, the point C_1 also belongs to the line of intersection of these planes. Consequently, the planes P and ACC_1A_1 intersect along the straight line C_1E, i.e. on this line is positioned the side (C_1F) of the section found on the face CC_1A_1A. Hence, the point F of intersection of C_1E and the edge AA_1 is the required vertex.

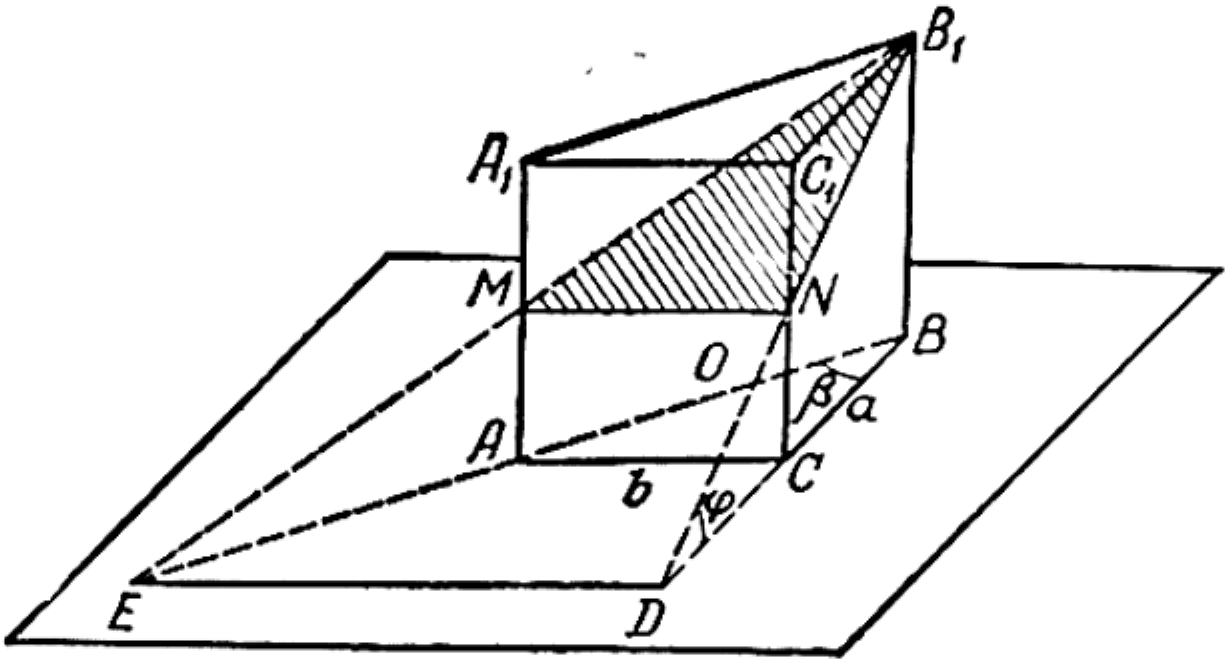


Fig. 185

(b) *Solution.* Since the triangle ABC is the projection of the triangle FBC_1, contained in the plane P, on the plane of the base, then

$$S_{sec}=\frac{S_{base}}{\cos\beta}=\frac{\frac{1}{2}a^2\sin 2\alpha}{\cos\beta}$$

where $a = AC$ is the side of the isosceles triangle ABC. Express a^2 through the lateral surface area S. We have

$$S=(2AC+BC)\cdot CC_1$$

where $AC = a$, $BC = 2a\cos\alpha$ and $CC_1 = BC\cdot\tan\beta = 2a\cos\alpha\tan\beta$. Hence, $S = 4a^2\cos\alpha\,(1+\cos\alpha)\tan\beta = 8a^2\cos\alpha\cos^2\frac{\alpha}{2}\tan\beta$.

Answer: $S_{sec}=\frac{S}{16}\,\frac{\sin 2\alpha\cot\beta}{\cos\beta\cos\alpha\cos^2\frac{\alpha}{2}}=\frac{S\tan\frac{\alpha}{2}}{4\sin\beta}$.

709. (a) *Drawing.* Extending the line-segment BC (Fig. 185) depicting a leg of the base by a length $CD = BC$, we get the point D, which is actually symmetrical to B about the leg AC. Let us take the point M in the middle of the edge AA_1 and draw the section of the prism by the plane P passing through the points B_1, M and D. To this end join B_1 with D by a line to intersect the edge CC_1 at the point N. The triangle B_1NM is the required section. Indeed, the point D lies on the line BC and, hence, belongs to the plane CBB_1C_1 (D is situated on the extension of the face CBB_1C_1). But the point D also lies in the plane P, therefore it is positioned on the line of intersection of the planes P and CBB_1C_1. The point B_1 is also found on this line. Hence, the planes P and BCC_1B_1 intersect along the straight line B_1D. The point N at which B_1D intersects the edge CC_1 is one of the vertices of the section, thus, the section of the prism is the triangle B_1NM.

Since $BC = CD$ and $CN \parallel BB_1$, CN is the midline of the triangle BB_1D, i.e. N is the midpoint of the edge CC_1. Consequently, MN is parallel to AC contained in the plane of the base. Therefore the line DE, along which the plane P intersects the plane of the base, is parallel to AC and, hence, perpendicular to the face BCC_1B_1. Therefore $\angle BDB_1$ is the plane angle of the dihedral angle φ at the edge DE.

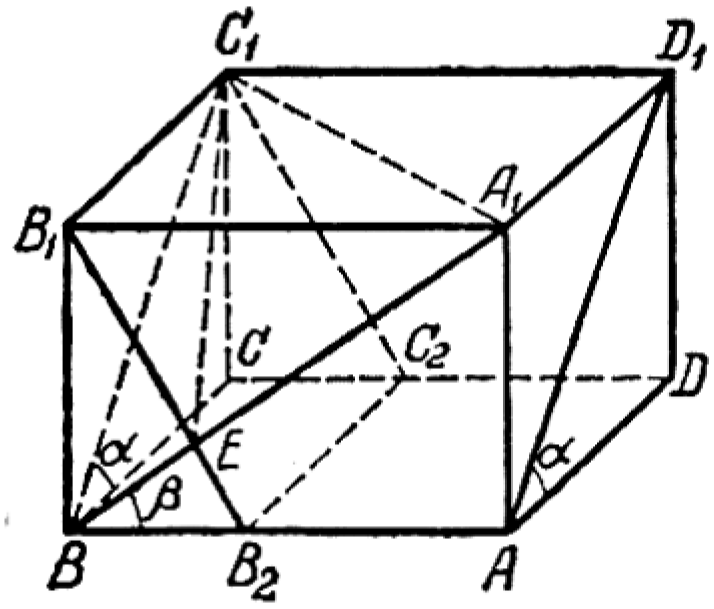


Fig. 186

(b) *Solution*. We have (see solution of the preceding problem)

$$S_{sec} = \frac{S_{base}}{\cos\varphi} = \frac{ab}{2\cos\varphi}$$

(where $a = BC$, $b = AC$), and since $b = a \tan\beta$, we get

$$S_{sec} = \frac{a^2 \tan\beta}{2\cos\varphi}$$

Find a^2. By hypothesis, β is the smallest one of the acute angles of the triangle ABC, thus, $b < a$ and the area bH of the face ACC_1A_1 is smaller than the area aH of the face BCC_1B_1. Therefore, the difference S of these areas (we assume that it is positive) is equal to $(a-b)H$. From the triangle DBB_1, wherein $BD = 2BC = 2a$, we find $H = 2a \tan\varphi$. Consequently,

$$S = 2a^2(1 - \tan\beta)\tan\varphi$$

whence we find a^2.

Answer:

$$S_{sec} = \frac{S}{4} \cdot \frac{\tan\beta}{(1-\tan\beta)\sin\varphi} = \frac{S}{4\sqrt{2}} \cdot \frac{\sin\beta}{\sin(45^\circ - \beta)\sin\varphi}$$

710. The angle between the non-intersecting diagonals BA_1 and AD_1 (Fig. 186) is equal to the angle $\varphi = \angle A_1BC_1$ between BA_1 and the line BC_1 parallel to AD_1. We have $\angle CBC_1 = \angle DAD_1 = \alpha$ and $\angle ABA_1 = \beta$. To determine the angle φ find $A_1C_1^2$ first from the triangle A_1BC_1 (by the law of cosines) and then from the right-angled triangle $A_1B_1C_1$, and equate the obtained expressions. We get

$$BA_1^2 + BC_1^2 - 2 \cdot BA_1 \cdot BC_1 \cdot \cos\varphi = B_1A_1^2 + B_1C_1^2$$

Hence

$$2 \cdot BA_1 \cdot BC_1 \cdot \cos\varphi = (BA_1^2 - B_1A_1^2) + (BC_1^2 - B_1C_1^2) = 2 \cdot BB_1^2$$

Substituting

$$BA_1 = \frac{AA_1}{\sin\beta} = \frac{BB_1}{\sin\beta}$$

(from the triangle BAA_1) and $BC_1 = \dfrac{BB_1}{\sin\alpha}$ into the last equality, we get

$$\cos\varphi = \sin\alpha\sin\beta$$

Alternate method. Through the edge B_1C_1 draw the plane $B_1C_1C_2B_2$ perpendicular to BA_1 (it is possible, since $B_1C_1 \perp BA_1$). Let E be the point of intersection of BA_1 and B_1B_2. From the right-angled triangle BC_1E we find $BE = BC_1 \cos\varphi$ and from the right-angled triangle BB_1E, wherein $\angle B_1BE = 90^\circ - \beta$, we have

$$BE = BB_1 \cos(90^\circ - \beta) = BB_1 \sin\beta$$

Now we express the line-segment BB_1 through BC_1 from the triangle BB_1C_1, wherein $\angle B_1BC_1 = 90^\circ - \alpha$. We get $BB_1 = BC_1 \sin\alpha$ and, hence,

$$BE = BC_1 \cdot \sin\alpha\sin\beta.$$

Equating the two expressions for BE, we obtain

$$BC_1 \cdot \cos\varphi = BC_1 \cdot \sin\alpha\sin\beta$$

Answer: $\cos\varphi = \sin\alpha\sin\beta$.

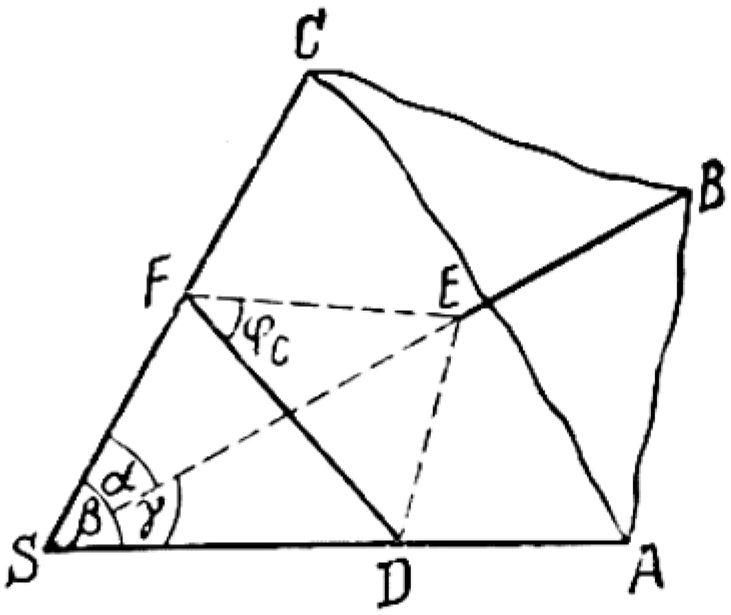


Fig. 187

711. Let us denote the dihedral angles at the edges SA, SB, SC (Fig. 187) by φ_A, φ_B, φ_C. Through a point (F) on the edge SC draw a plane (DFE) perpendicular to SF. Then $\angle DFE = \varphi_C$. Determine ED^2 first from the triangle EFD and then from the triangle ESD, and then equate the obtained expressions. We wind

$$FE^2 + FD^2 - 2FE \cdot FD \cdot \cos\varphi_C = SE^2 + SD^2 - 2 \cdot SE \cdot SD \cos\gamma$$

Hence

$$2 \cdot FE \cdot FD \cdot \cos\varphi_C = 2 \cdot SE \cdot SD \cdot \cos\gamma - (SE^2 - FE^2) - (SD^2 - FD)^2,$$

i. e.

$$2 \cdot FE \cdot FD \cdot \cos\varphi_C = 2 \cdot SE \cdot SD \cdot \cos\gamma - 2SF^2$$

Substituting into this equality

$$FE = SF \tan\alpha$$

$$FD = SF \tan\beta$$

$$SE = \frac{SF}{\cos\alpha}$$

and

$$SD = \frac{SF}{\cos\beta}$$

we obtain

$$\tan\alpha\tan\beta\cos\varphi_C = \frac{\cos\gamma}{\cos\alpha\cos\beta} - 1$$

whence

$$\cos \varphi_C = \frac{\cos \gamma - \cos \alpha \cos \beta}{\sin \alpha \sin \beta}$$

Similarly, we find $\cos \varphi_A$ and $\cos \varphi_B$.

Answer: $\cos \varphi_A = \frac{\cos \alpha - \cos \beta \cos \gamma}{\sin \beta \sin \gamma}$

$$\cos \varphi_B = \frac{\cos \beta - \cos \gamma \cos \alpha}{\sin \gamma \sin \alpha}$$

$$\cos \varphi_C = \frac{\cos \gamma - \cos \alpha \cos \beta}{\sin \alpha \sin \beta}$$

712. Solved in the same way as the preceding problem.

Answer: $\cos \gamma = \cos \alpha \cos \beta + \sin \alpha \sin \beta \cos A$.

713. See Problem 711.

Answer: the required angle contains 90°.

714. Let the point M lie on the face Q (Fig. 188). By hypothesis, AM forms an angle α with AB, and MB is perpendicular to AB. Through BM draw the plane MBN perpendicular to the edge, and drop the perpendicular MN from M to BN. The line MN is also perpendicular to NA, and $\angle MAN = \beta$ (prove it!). We also have $\varphi = \angle NBM$. The angle φ is found from the triangle NBM, wherein $MN = AM \cdot \sin \alpha$ (found from the triangle ANM) and $BM = AM \sin \alpha$ (found from the triangle AMB). We get

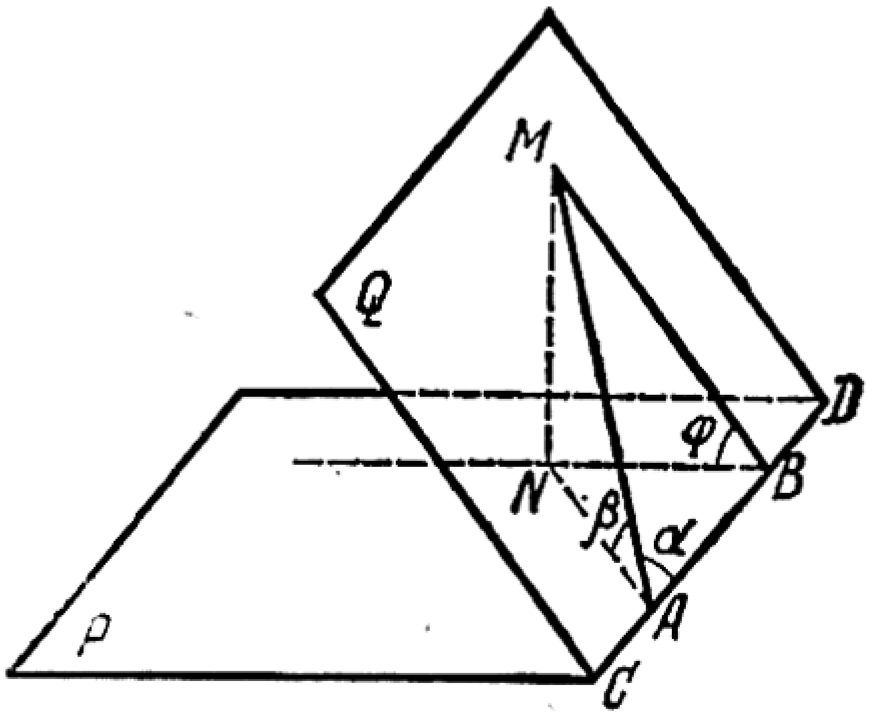


Fig. 188

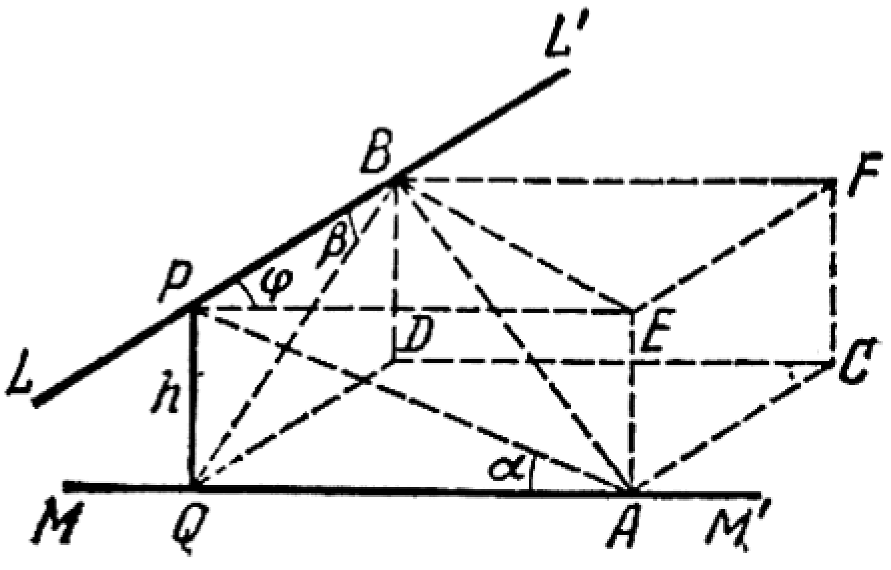


Fig. 189

$$\sin \varphi = \frac{MN}{BM} = \frac{AM \sin \beta}{AM \sin \alpha} = \frac{\sin \beta}{\sin \alpha}$$

Answer: $\sin \varphi = \frac{\sin \beta}{\sin \alpha}$.

715. Fig. 189 shows the common perpendicular PQ to skew lines LL' and MM'. To obtain the angle at which the line-segment PQ is seen from the point A we have to draw the ray AP; then $\angle PAQ = \alpha$. Similarly, $\angle PBQ = \beta$. Through the point P draw the straight line PE parallel to MM'. Then the angle between MM' and LL' is (by definition) the angle $\varphi = \angle EPB$. Drop the perpendicular AE from A to PE and draw AB (the rest of the lines depicting a parallelepiped with the edges PQ, QA and PB are constructed to make the drawing more vivid). From the right-angled triangle BPQ we find

$$PB = PQ \cot \beta = h \cot \beta$$

Similarly,

$$PE = QA = h \cot \alpha$$

Then,

$$BE^2 = PB^2 + PE^2 - 2 \cdot PB \cdot PE \cos \varphi =$$
$$= h^2 (\cot^2 \alpha + \cot^2 \beta - 2 \cot \alpha \cot \beta \cos \varphi)$$

AE is perpendicular to the plane EPB, since it is parallel to PQ which is the common perpendicular to PB and PE. From the right-angled triangle AEB we find

$$AB^2 = AE^2 + BE^2 = h^2 + BE^2$$

Answer: $AB^2 = h^2 (1 + \cot^2 \alpha + \cot^2 \beta - 2 \cot \alpha \cot \beta \cos \varphi)$.

716. See the drawing to the preceding problem (in the present problem $\varphi = 90°$).

We have

$$BE = \sqrt{PE^2 + PB^2} = h \sqrt{\cot^2 \alpha + \cot^2 \beta}$$

The angle between AB and PQ is equal to the angle between AB and AE parallel to PQ. Denoting it by γ, we have

$$\tan \gamma = \frac{BE}{AE} = \frac{h \sqrt{\cot^2 \alpha + \cot^2 \beta}}{h}$$

Answer: $\tan \gamma = \sqrt{\cot^2 \alpha + \cot^2 \beta}$.

717. Let (Fig. 190)

$$\frac{DM}{MA} = \frac{m_1}{n_1}, \quad \frac{DN}{NB} = \frac{m_2}{n_2}, \quad \frac{DP}{PC} = \frac{m_3}{n_3}$$

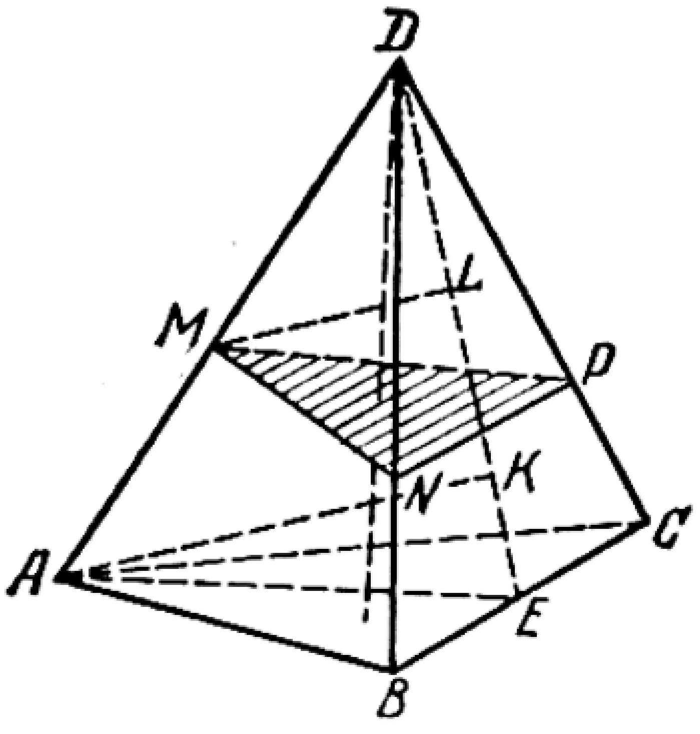


Fig. 190

Let us first find the ratio of the volume V_1 of the pyramid $DMNP$ to the volume V of the pyramid $DABC$. Let the face BDC be the base of the pyramid $DABC$ and the face NPD, the base of the pyramid $DMNP$. Let the edge DA be projected on the plane DBC as a line-segment lying on DE. Then the points A and M are projected into some points K and L lying on DE. Consequently, the altitudes $AK = h$ and $ML = h_1$ are contained in the plane ADE and the triangles DML and DAK are similar. Hence,

$$\frac{h_1}{h} = \frac{DM}{DA} = \frac{DM}{DM + MA} = \frac{m_1}{m_1 + n_1}$$

The area S_1 of the base NDP is to the area S of the base BDC as $DN \cdot DP$ to $DB \cdot DC$ (since the triangles NDP and BDC have the common angle D). Hence,

$$\frac{S_1}{S} = \frac{DN}{DB} \cdot \frac{DP}{DC} = \frac{m_2}{m_2 + n_2} \cdot \frac{m_3}{m_3 + n_3}$$

Hence,

$$\frac{V_1}{V}=\frac{h_1}{h}\cdot\frac{S_1}{S}=\frac{m_1m_2m_3}{(m_1+n_1)(m_2+n_2)(m_3+n_3)}$$

Now we find the ratio $\frac{V_1}{V-V_1}$, in which the volume of the pyramid $DABC$ is divided.

Answer: $\frac{V_1}{V-V_1}=\frac{m_1m_2m_3}{(m_1+n_1)(m_2+n_2)(m_3+n_3)-m_1m_2m_3}$.

718. *The plan of solution*: from the similarity of the triangles OEL and MEK (Fig. 191) let us express OL in terms of $MK=b$ and $ME=\frac{H}{2}$; from the similarity of the triangles OCE and MEN let us express OC in terms of $MN=h$ and $ME=\frac{H}{2}$. Substituting the expressions found into the relationship $OC^2=2\cdot OL^2$, we get the equation for finding H.

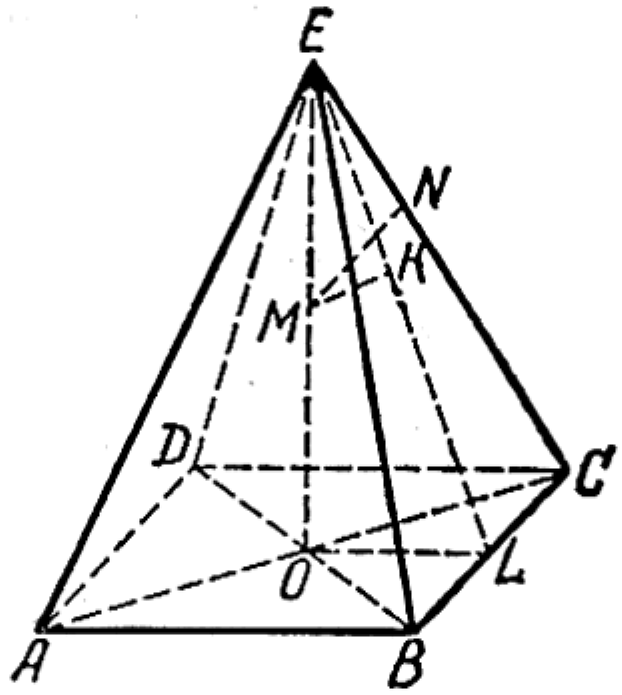


Fig. 191

Solution. We have

$$OL:H=MK:EK,$$

i.e.

$$OL:H=b:\sqrt{\left(\frac{H}{2}\right)^2-b^2}$$

whence

$$OL^2=\frac{4b^2H^2}{H^2-4b^2}$$

similarly, $OC^2=\frac{4h^2H^2}{H^2-4h^2}$. Hence,

$$\frac{4h^2H^2}{H^2-4h^2}=2\,\frac{4b^2H^2}{H^2-4b^2}$$

Dividing by H^2 and transforming, we obtain

$$H=\frac{2bh}{\sqrt{2b^2-h^2}}$$

Now we find

$$OL^2=\frac{4b^2H^2}{H^2-4b^2}=\frac{2b^2h^2}{h^2-b^2}$$

and

$$V=\frac{1}{3}(2OL)^2\cdot H$$

Answer: $V=\frac{16b^3h^3}{3(h^2-b^2)\sqrt{2b^2-h^2}}$.

CHAPTER X
SOLIDS OF REVOLUTION

719. *Answer*: $V=\frac{\pi l^3}{8\sqrt{3}}$

720. *Answer*: $V=\frac{c^2}{24\pi^2}\sqrt{4\pi^2 l^2-c^2}$

721. *Answer*: $V=\frac{a^3}{4\pi}$

722. *Answer*: $V=\frac{d^3\cos^2\alpha\sin\alpha}{4\pi}$

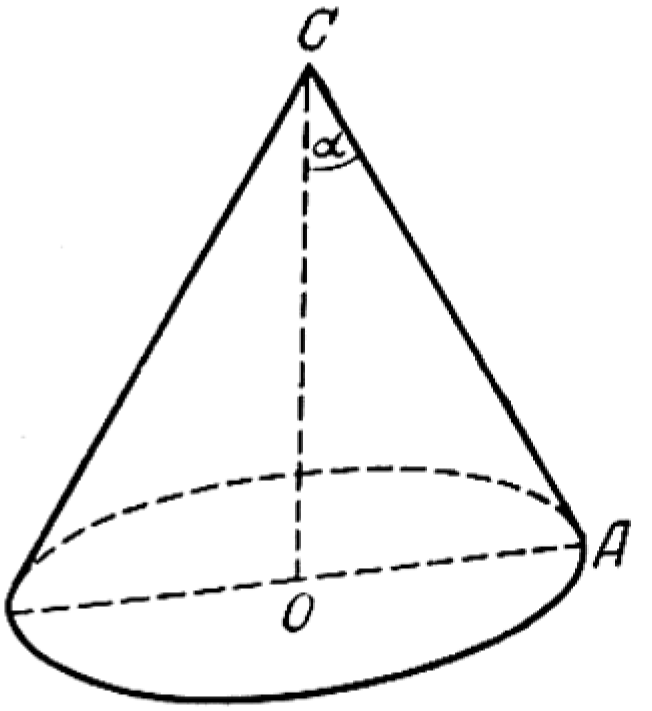


Fig. 192

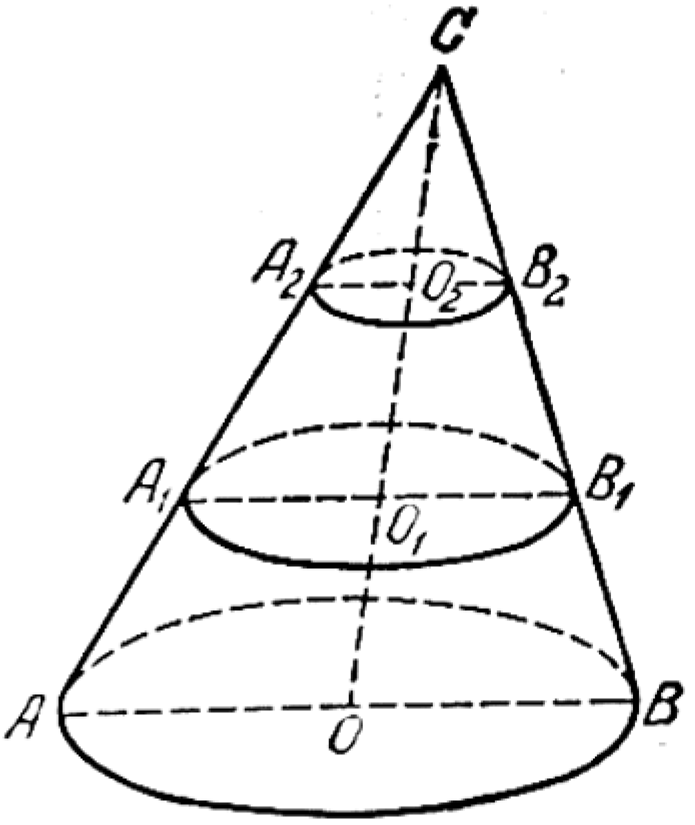


Fig. 193

723. The radius of the base $R=l\sin\alpha$ (Fig. 192)*, the altitude of the cone $H=l\cos\alpha$. The volume

$$V=\frac{\pi R^2 H}{3}=\frac{\pi l^3\sin^2\alpha\cos\alpha}{3}$$

The surface

$$S=\pi R(l+R)=\pi l^2\sin\alpha\,(1+\sin\alpha)$$

By hypothesis, $l+H=m$; hence,

$$l=\frac{m}{1+\cos\alpha}=\frac{m}{2\cos^2\frac{\alpha}{2}}$$

Answer: $V=\frac{\pi m^3\sin^2\alpha\cos\alpha}{24\cos^6\frac{\alpha}{2}}$

$$S=\frac{\pi m^2\sin\alpha\cos^2\left(45^\circ-\frac{\alpha}{2}\right)}{2\cos^4\frac{\alpha}{2}}$$

724. (Fig. 193). The planes A_1B_1 and A_2B_2 cut off the cone ACB the cones

---

* For drawing an ellipse (depicting the circle in the base of a cone) see page 254.

A_1CB_1 and A_2CB_2 which are similar to the given cone. And the volumes (V, V_1 and V_2) are in the same ratios as the cubes of the altitudes:

$$\frac{V_1}{V}=\frac{\left(\frac{2}{3}H\right)^3}{H^3} \quad \text{and} \quad \frac{V_2}{V}=\frac{\left(\frac{1}{3}H\right)^3}{H^3}$$

The volume V_{mid} of the mid-portion $A_1A_2B_2B_1$ is equal to V_1-V_2. Subtracting the second proportion from the first one, we find V_{mid}.

Answer: $V_{mid}=\frac{7}{27}V$.

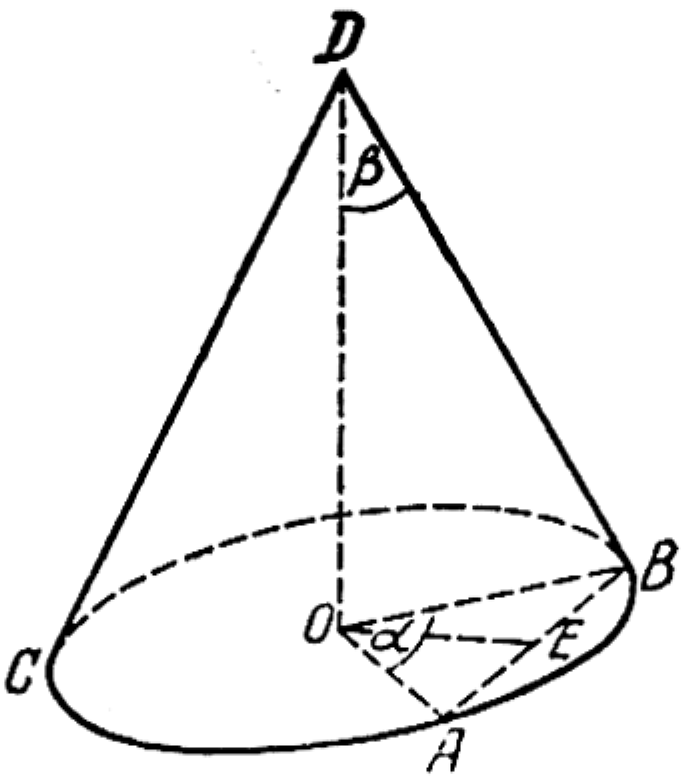


Fig. 194

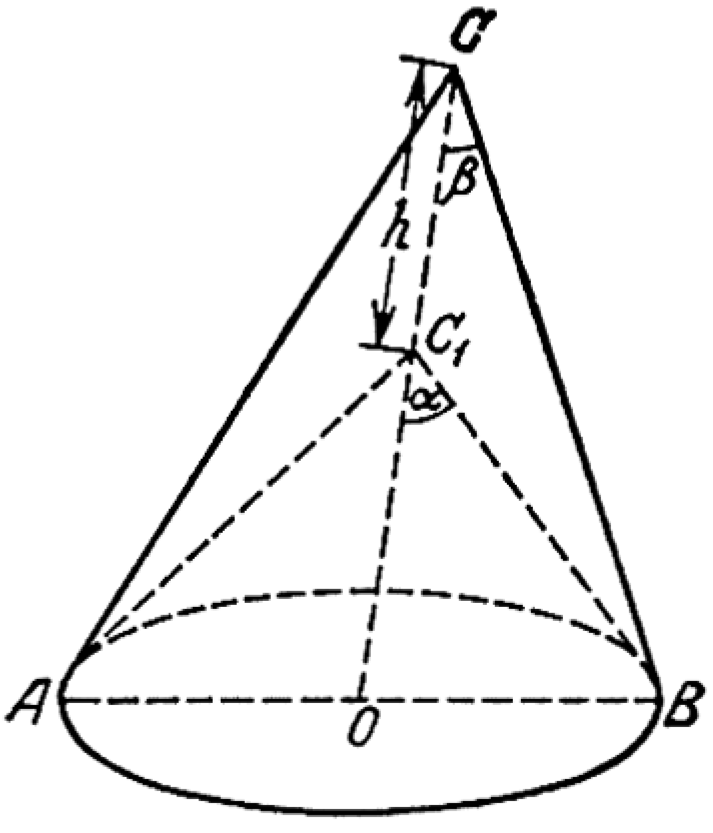


Fig. 195

725. From the triangle AOE (Fig. 194) we find

$$OA=R=\frac{AB}{2\sin\frac{\alpha}{2}}=\frac{a}{2\sin\frac{\alpha}{2}}$$

From the triangle OBD we have $H=R\cot\beta$.

Answer: $V=\frac{\pi a^3\cot\beta}{24\sin^3\frac{\alpha}{2}}$.

726. By hypothesis, $OC-OC_1=h$ (Fig. 195). We have

$$OC=R\cot\beta$$

and

$$OC_1=R\cot\alpha$$

Hence,

$$R=\frac{h}{\cot\beta-\cot\alpha}$$

The required volume V is equal to the difference between the volumes of the cones ACB and AC_1B. Hence,

$$V=\frac{1}{3}\pi R^2(OC-OC_1)=\frac{1}{3}\pi R^2h.$$

Answer: $V=\frac{\pi h^3}{3(\cot\beta-\cot\alpha)^2}=\frac{\pi h^3\sin^2\alpha\sin^2\beta}{3\sin^2(\alpha-\beta)}$

727. By hypothesis,

$$\pi Rl=S$$

The area of the base

$$S_{base}=\pi R^2$$

is equal to $P-S$. Dividing (by terms) the equality

$$\pi R^2=P-S$$

by the equality

$$\pi Rl=S,$$

we get $\frac{R}{l}=\frac{P-S}{S}$. Let us denote the required angle by β; from the triangle OBD (see Fig. 194) we have

$$\sin\beta=\frac{R}{l}$$

Answer: $\beta=\arcsin\frac{P-S}{S}$.

728. From the isosceles triangle ADA_1 (Fig. 196) we find $AD=\frac{a}{2\sin\frac{\alpha}{2}}$. If α is the radian measure of the angle ADA_1, then

$$\overset{\frown}{ABCA_1}=AD\cdot\alpha=\frac{a\alpha}{2\sin\frac{\alpha}{2}}$$

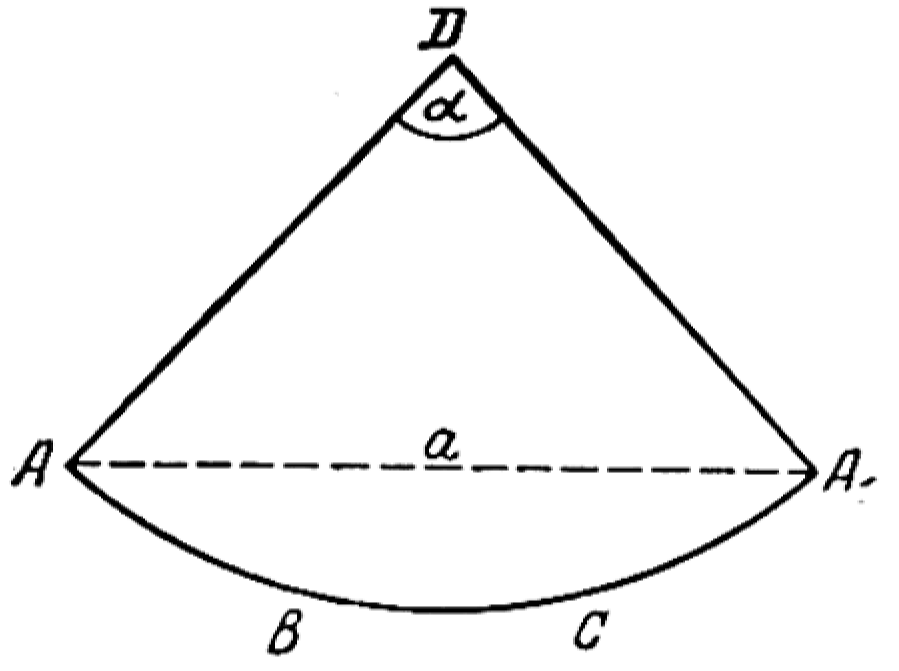


Fig. 196

Prior to developing the curved surface the line-segment AD was an element of the cone, thus

$$l=AD=\frac{a}{2\sin\frac{\alpha}{2}}$$

the arc $ABCA_1$ was the base circumference, hence

$$2\pi R=\frac{a\alpha}{2\sin\frac{\alpha}{2}}$$

The altitude of the cone is

$$H=\sqrt{l^2-R^2}=\frac{a}{4\pi\sin\frac{\alpha}{2}}\sqrt{4\pi^2-\alpha}$$

Answer: $V=\dfrac{\alpha^2 a^3\sqrt{4\pi^2-\alpha^2}}{192\pi^2\sin^3\dfrac{\alpha}{2}}$, where α is the radian measure of the given angle.

729. The angle DOM (Fig. 197) is equal to the angle $\varphi=\angle DEO$. From $\triangle ODM$ and $\triangle OEM$ we find

$$OD=H=\frac{a}{\cos\varphi}$$

and

$$OE=\frac{a}{\sin\varphi}$$

From $\triangle OCE$ we find

$$OC=R=\frac{OE}{\cos\dfrac{\alpha}{2}}$$

Answer: $V=\dfrac{\pi a^3}{3\sin^2\varphi\cos\varphi\cos^2\dfrac{\alpha}{2}}$.

730. The radius of the base circle of the cone is $R=\dfrac{a}{\sqrt{2}}$ (Fig. 198). From

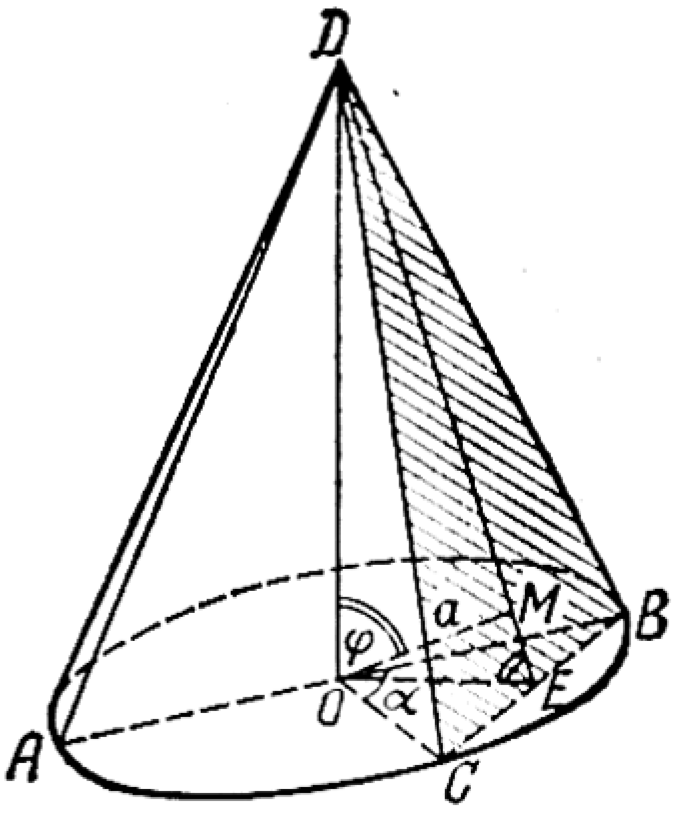


Fig. 197

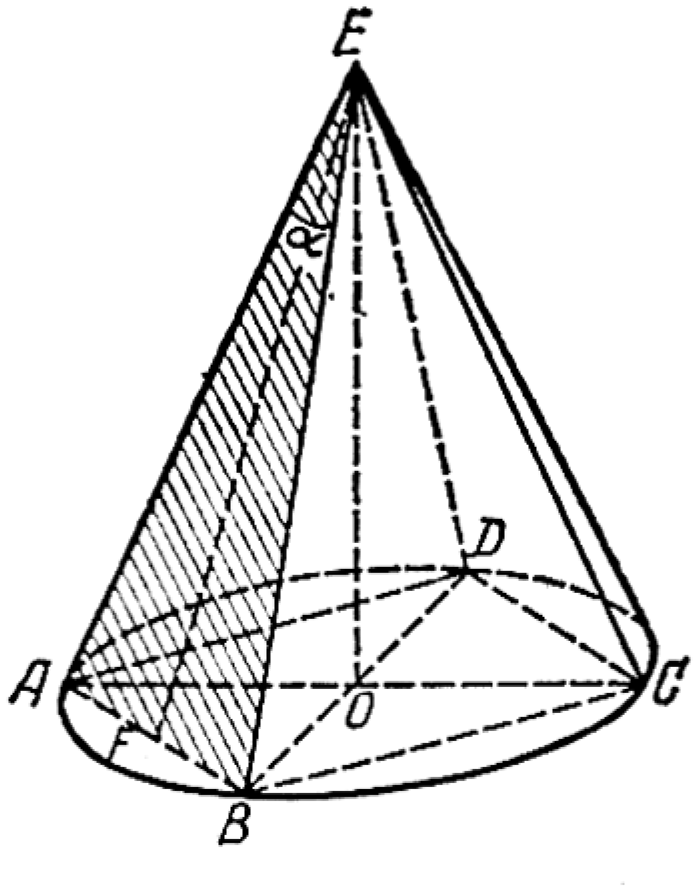


Fig. 198

$\triangle AEF$ we find $AE=l=\dfrac{a}{2\sin\dfrac{\alpha}{2}}$; from $\triangle AOE$ we find

$$H=\sqrt{AE^2-AO^2}=\frac{a}{2\sin\dfrac{\alpha}{2}}\sqrt{1-2\sin^2\frac{\alpha}{2}}=\frac{a\sqrt{\cos\alpha}}{2\sin\dfrac{\alpha}{2}}$$

The surface S is equal to

$$\pi R(l+R)=\frac{\pi a}{\sqrt{2}}\left(\frac{a}{2\sin\dfrac{\alpha}{2}}+\frac{a}{\sqrt{2}}\right)=\frac{\pi a^2}{2\sin\dfrac{\alpha}{2}}\left(\frac{1}{\sqrt{2}}+\sin\frac{\alpha}{2}\right)$$

The expression in parentheses may be transformed as follows:

$$\frac{1}{\sqrt{2}}+\sin\frac{\alpha}{2}=\sin 45^\circ+\sin\frac{\alpha}{2}=2\sin\frac{90^\circ+\alpha}{4}\cos\frac{90^\circ-\alpha}{4}$$

Answer: $V=\dfrac{\pi a^3\sqrt{\cos\alpha}}{12\sin\frac{\alpha}{2}}$

$$S=\frac{\pi a^2\left(1+\sqrt{2}\sin\frac{\alpha}{2}\right)}{2\sqrt{2}\sin\frac{\alpha}{2}}=\frac{\pi a^2\sin\frac{90^\circ+\alpha}{4}\cos\frac{90^\circ-\alpha}{4}}{\sin\frac{\alpha}{2}}$$

731. From the triangle AA_1C (Fig. 199) we have $AC=l\cos\alpha$. From the triangle AA_1B we find $AB=2R=\dfrac{l}{\cos\alpha}$; thus $AO=R=\dfrac{l}{2\cos\alpha}$. Hence,

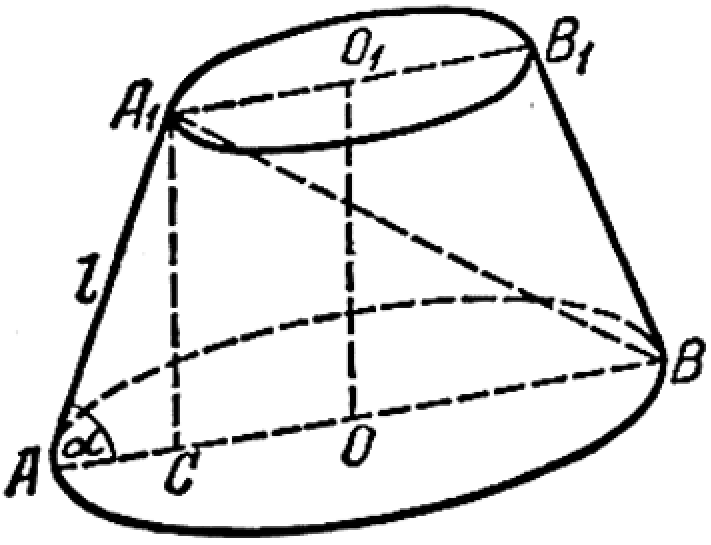


Fig. 199

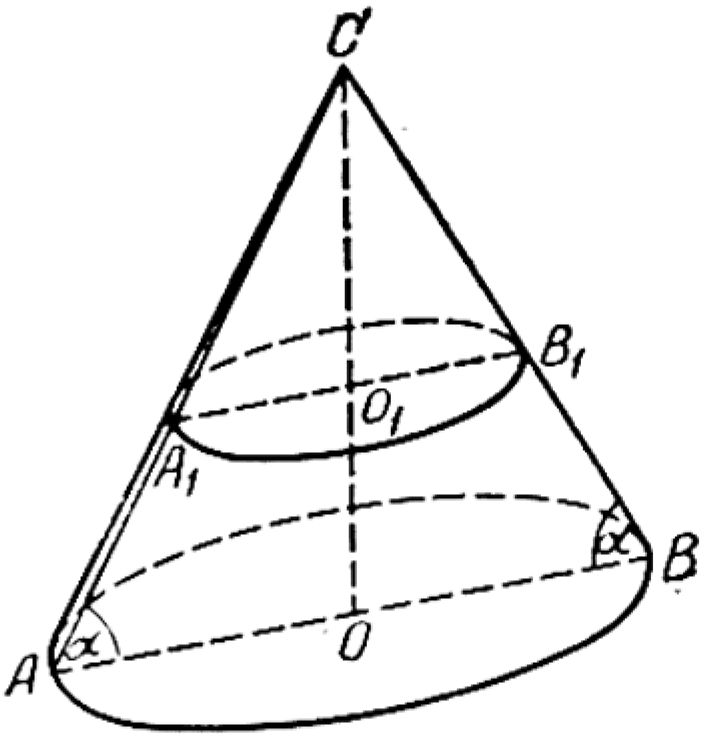


Fig. 200

$$A_1O_1=r=AO-AC=l\left(\frac{1}{2\cos\alpha}-\cos\alpha\right)$$

Now we find

$$S_{curved}=\pi l\,(R+r)=\pi l^2\left(\frac{1}{\cos\alpha}-\cos\alpha\right)$$

Answer: $S_{curved}=\dfrac{\pi l^2\sin^2\alpha}{\cos\alpha}=\pi l^2\tan\alpha\sin\alpha,$

732. From the expressions $V=\dfrac{1}{3}\pi R^2H$ and $R=H\cot\alpha$ we get

$$H=\sqrt[3]{\frac{3V\tan^2\alpha}{\pi}}\quad\text{and}\quad R=\sqrt[3]{\frac{3V\cot\alpha}{\pi}}$$

Let it be required to halve the *curved* surface area. Since the cones ABC and A_1B_1C (Fig. 200) are similar, their curved surface areas S and S_1 are in the same ratio as $H^2=OC^2$ to $H_1^2=O_1C^2$. Consequently,

$$H_1:H=\sqrt{S_1:S}=\sqrt{\frac{1}{2}}, \text{ i.e. } H_1=\frac{H}{\sqrt{2}}=\frac{1}{\sqrt{2}}\sqrt[3]{\frac{3V\tan^2\alpha}{\pi}}$$

Let it now be required to halve the *total* surface area. Then

$$\pi R_1l_1=\frac{1}{2}\pi(R^2+Rl)$$

Substituting $R_1=H_1\cot\alpha$; $l_1=\frac{H_1}{\sin\alpha}$ and $l=\frac{H}{\sin\alpha}$, we get

$$\pi H_1^2\frac{\cot\alpha}{\sin\alpha}=\frac{\pi}{2}\left(H^2\cot^2\alpha+\frac{H^2\cot\alpha}{\sin\alpha}\right)$$

whence $H_1=H\cos\frac{\alpha}{2}$.

Answer: if the curved surface area is halved, then $H_1=\frac{1}{\sqrt{2}}\sqrt[3]{\frac{3V\tan^2\alpha}{\pi}}$; if the total one, then $H_1=\cos\frac{\alpha}{2}\sqrt[3]{\frac{3V\tan^2\alpha}{\pi}}$.

733. Let us denote (Fig. 201) the radius of the sphere by R, the altitude of the segment ACB, by h and the line-segment DA, by r. The volume V of the sector is equal to $V=\frac{2}{3}\pi R^2h$. From the triangle ACD, wherein $\angle CAD=\frac{\alpha}{4}$ (as an inscribed angle subtended by the arc $\overset{\frown}{BC}=\frac{\alpha}{2}$), we find $h=r\tan\frac{\alpha}{4}$. From the triangle ADO we have $r=R\sin\frac{\alpha}{2}$. Consequently,

$$V=\frac{2}{3}\pi R^2h=\frac{2}{3}\pi R^2\cdot R\sin\frac{\alpha}{2}\tan\frac{\alpha}{4}$$

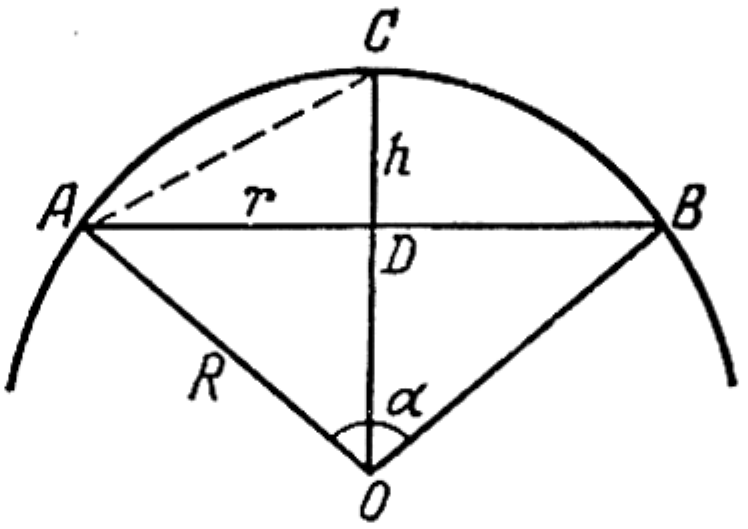


Fig. 201

The surface of the spherical sector is made up of the curved surface of the spherical segment ACB equal to $2\pi Rh$ and the curved surface of the cone AOB equal to πrR. Consequently, $S=2\pi Rh+\pi rR=\pi R(2h+r)$.

Answer: $V=\frac{4\pi}{3}R^3\sin^2\frac{\alpha}{4}$; $S=\pi R^2\sin\frac{\alpha}{2}\left(2\tan\frac{\alpha}{4}+1\right)$.

734. (See Fig. 201.) With the notation of the preceding problem we have $S=2\pi Rh+\pi r^2$. From the triangle ADO we have $AO^2=AD^2+OD$; since $OD=R-h$, then $R^2=r^2+(R-h)^2$ and $r^2=2Rh-h^2$. Hence, $S=4\pi Rh-\pi h^2$. Hence

$$h=2R\pm\frac{\sqrt{4\pi^2R^2-\pi S}}{\pi}$$

Since $h<R$, the plus sign is not suitable.

Answer: $h=2R-\sqrt{4R^2-\frac{S}{\pi}}$.

735. Figure 202 shows an axial section of the solid obtained by rotating the triangle ABC about the side AB. This solid is made up of two cones. Its volume

$$V=\frac{1}{3}\pi\cdot DC^2\cdot AD+\frac{1}{3}\pi\cdot DC^2\cdot DB=\frac{\pi}{3}DC^2(AD+DB)=\frac{\pi}{3}DC^2\cdot AB$$

Remember that $DC\cdot AB=2S$ and $DC=b\sin\alpha$.

Answer: $V=\frac{2\pi Sb\sin\alpha}{3}$.

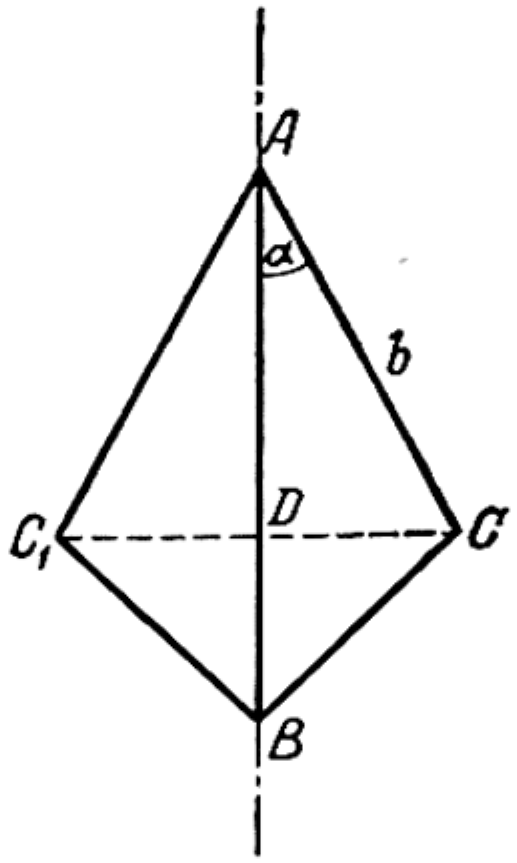


Fig. 202

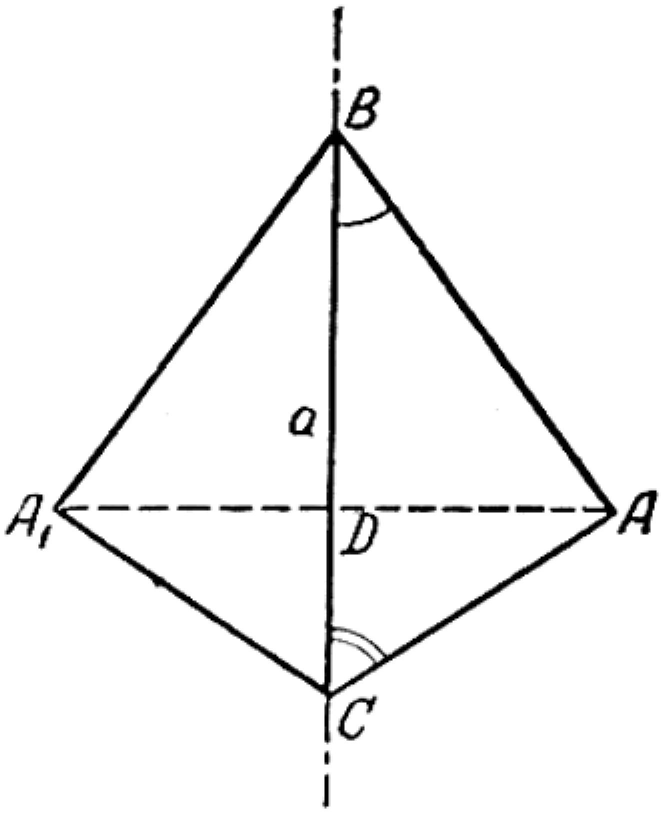


Fig. 203

736. (Fig. 203) The volume of the solid of revolution (see the preceding problem) is

$$V=\frac{1}{3}\pi\cdot DA^2(BD+DC)=\frac{1}{3}\pi a\cdot DA^2$$

To determine DA proceed in the following way: from the triangle BAD we find $BD=DA\cdot\cot B$, and from the triangle DAC we find

$$DC=DA\cdot\cot C$$

Consequently,

$$a=BD+DC=DA\,(\cot B+\cot C)$$

Hence we find DA.

Answer: $V=\frac{1}{3}\cdot\frac{\pi a^3}{(\cot B+\cot C)^2}=\frac{\pi a^3\sin^2 B\sin^2 C}{3\sin^2(B+C)}$.

737. The volume of the solid of revolution (whose section is shown in Fig. 204) is equal to the sum of the volumes of two equal frustums of cones, obtained by rotating the trapezoids $AMBC$, and $ANDC$ less the sum of the volumes of two equal cones obtained by rotating the triangles AMB and AND. The radius of one base of the frustum is $AC=d$, that of the other is $MB=\frac{d}{2}$.

We have

$$V=2\left[\frac{\pi\cdot BO}{3}\left(d^2+\frac{d^2}{4}+\frac{d^2}{2}\right)-\frac{\pi\cdot BO}{3}\cdot\frac{d^2}{4}\right)=\pi d^2 BO$$

From $\triangle AOB$ we find

$$BO=\frac{d}{2}\tan\frac{\gamma}{2}$$

Answer: $V=\dfrac{\pi d^3\tan\frac{\gamma}{2}}{2}$.

738. The volume V (Fig. 205) of the solid of revolution is equal to the volume of the frustum of a cone obtained by rotating the trapezoid OO_1BC less the volume of two cones generated by rotating the triangles AO_1B and AOC.

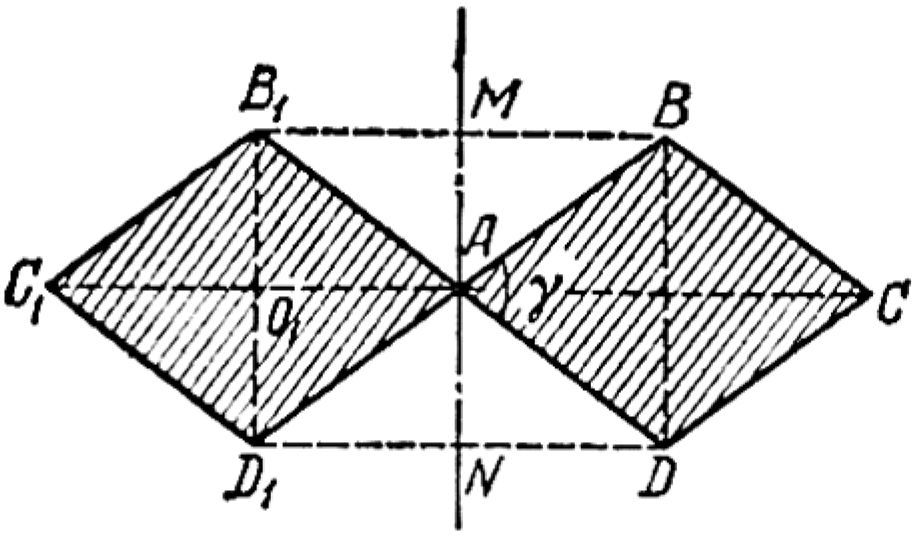


Fig. 204

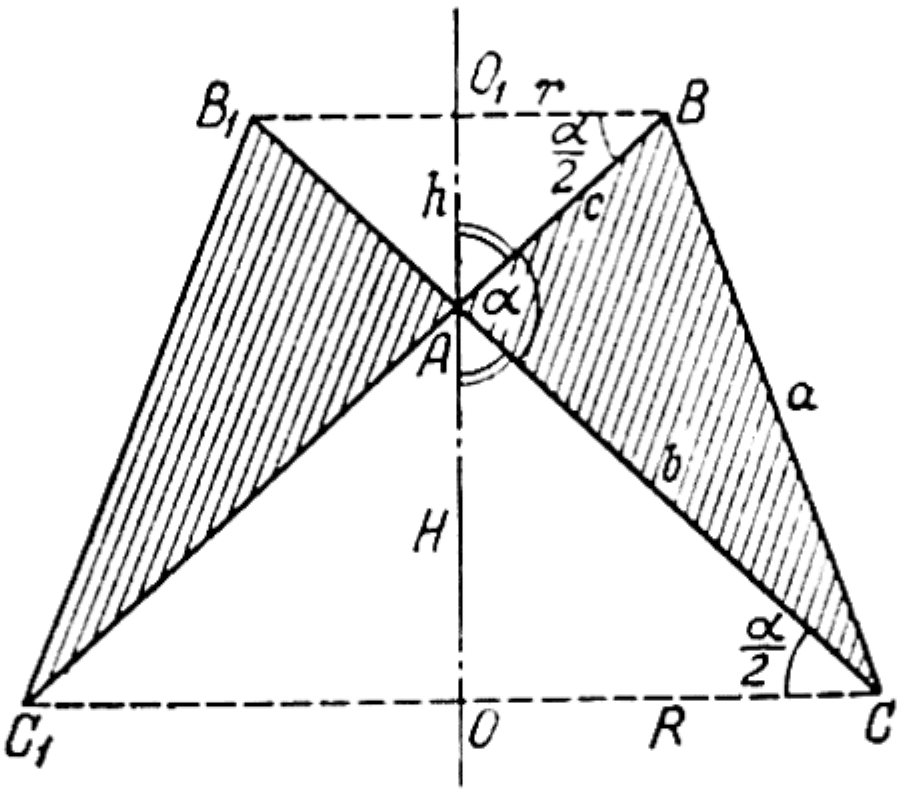


Fig. 205

Since, by hypothesis $\angle BAO_1=\angle CAO$, we have $\angle BAO_1=\angle CAO=90^\circ-\frac{\alpha}{2}$; thus,

$$\angle O_1BA=\angle OCA=\frac{a}{2}$$

We have (see Fig. 205): $H=b\sin\frac{\alpha}{2}$, $R=b\cos\frac{\alpha}{2}$ (found from the triangle AOC) and $h=c\sin\frac{\alpha}{2}$; $r=c\cos\frac{\alpha}{2}$ (found from the triangle AO_1B). Hence,

$$V=\frac{\pi}{3}(H+h)(R^2+Rr+r^2)-\frac{\pi}{3}HR^2-\frac{\pi}{3}hr^2=$$

$$=\frac{\pi}{3}\sin\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\left[(b+c)(b^2+bc+c^2)-b^3-c^3\right]$$

Answer: $V=\dfrac{\pi bc(b+c)\sin\alpha\cos\frac{\alpha}{2}}{3}$.

739. The surface S of the solid of revolution (Fig. 206) consists of the sum of the curved surfaces of two equal cones with the axial sections DAD_1 and CBC_1 and the curved surface of the cylinder with the axial section CDD_1C_1. With the notation adopted in Fig. 206 we have

$$r=b\sin\alpha,\quad h=MN=AB-2AM=\frac{b}{\cos\alpha}-2b\cos\alpha$$

Hence,

$$S=2\pi r(b+h)=\frac{2\pi b^2\sin\alpha}{\cos\alpha}(\cos\alpha+1-2\cos^2\alpha)$$

Answer: $S=2\pi b^2\tan\alpha(\cos\alpha+1-2\cos^2\alpha)=4\pi b^2\tan\alpha\sin\frac{\alpha}{2}\sin\frac{3\alpha}{2}$.

740. By rotating the given planes about the altitude of the cone without changing the angles α and β, we can bring them to a position (shown in Fig. 207) in which they intersect along the common element BD of the cone. From the triangles OBM and OBN we find

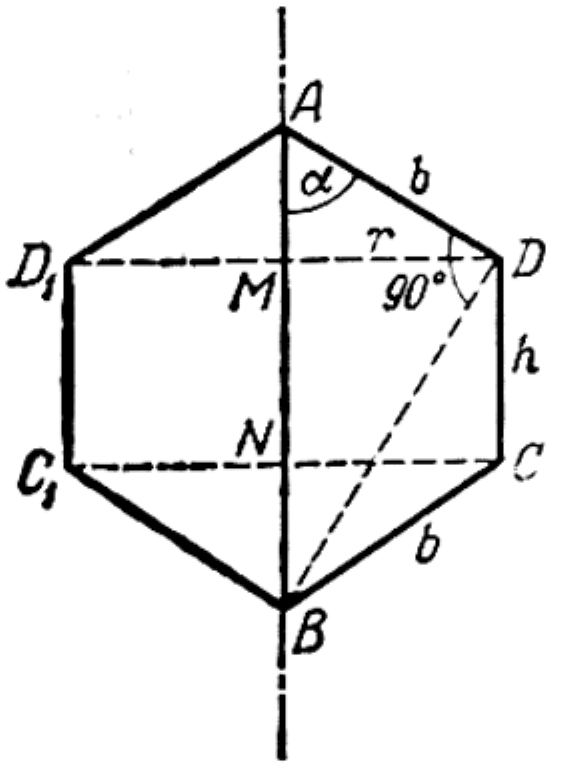


Fig. 206

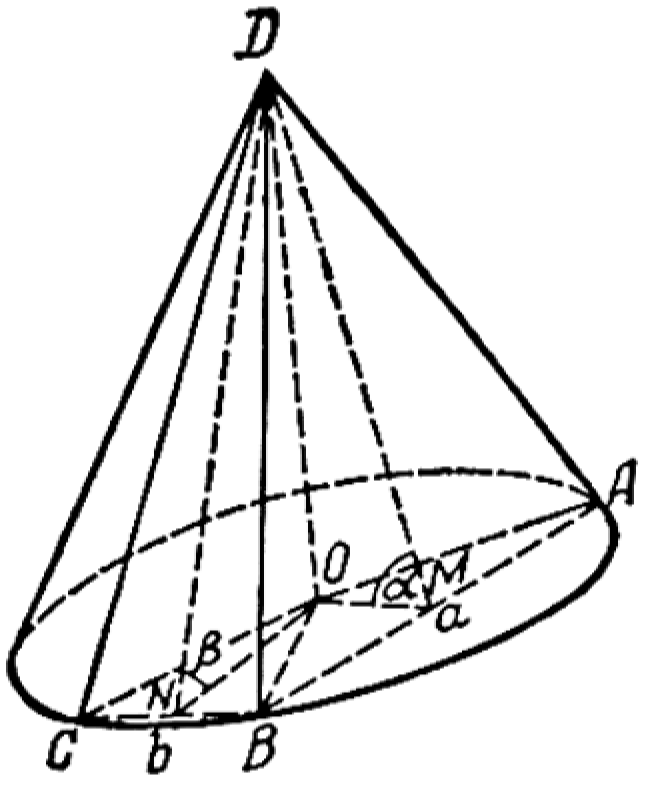


Fig. 207

$$OB^2=R^2=\frac{a^2}{4}+OM^2=\frac{b^2}{4}+ON^2$$

here $OM=H\cot\alpha$ and $ON=H\cot\beta$. Consequently,

$$R^2=\frac{a^2}{4}+H^2\cot^2\alpha\quad\text{and}\quad\frac{a^2}{4}+H^2\cot^2\alpha=\frac{b^2}{4}+H^2\cot^2\beta$$

The equations yield H and R.

Answer: $V=\dfrac{\pi(b^2\cot^2\alpha-a^2\cot^2\beta)\sqrt{b^2-a^2}}{24(\cot^2\alpha-\cot^2\beta)^{3/2}}$.

741. Shown in Fig. 208 is an axial section of the cone. Intersecting the sphere of radius r it yields a circle of radius $OD=r$, inscribed in the triangle ABC.

We have

$$r = R\tan\frac{\alpha}{2} = l\cos\alpha\tan\frac{\alpha}{2}$$

Answer: $V = \dfrac{4\pi l^3\cos^3\alpha\tan^3\dfrac{\alpha}{2}}{3}$.

742. Through the point M (Fig. 209) on the curved surface of the cone the tangent line MB is drawn, forming the angle $\theta = \angle BMA$ with the element CMA. Another angle $\alpha = \angle OAM$ is also known; it is required to find the angle φ formed by MB with the plane P of the base of the cone.

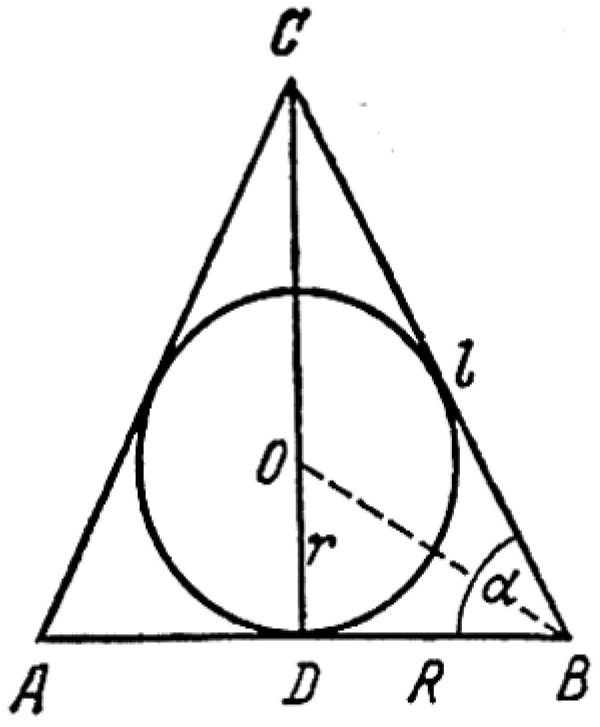


Fig. 208

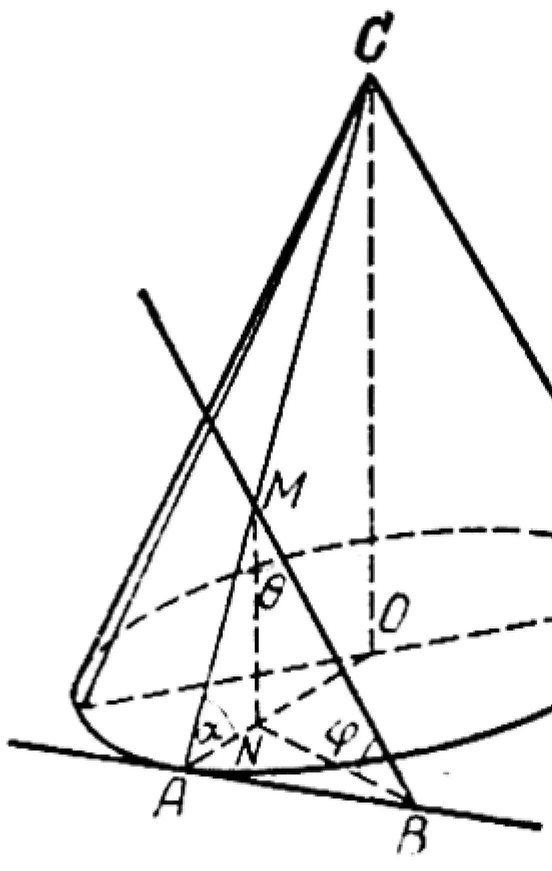


Fig. 209

The line MB, tangent to the cone, intersects the plane at a point B lying on the tangent AB to the base circle *. Dropping from the point M the perpendicular MN to the radius OA, we get the projection BN of BM on the plane P. Hence, $\varphi = \angle NBM$. From the triangle AMN we have

$$AM = \frac{MN}{\sin\alpha}$$

from $\triangle MAB$ we have

$$MB = \frac{AM}{\cos\theta} = \frac{MN}{\sin\alpha\cos\theta}$$

from $\triangle MNB$ we have

$$\sin\varphi = \frac{MN}{MB} = \sin\alpha\cos\theta$$

Answer: $\varphi = \arcsin(\sin\alpha\cos\theta)$.

743. The surface S of the solid of revolution is equal to the sum of the curved surfaces of two cones with the axial sections BAB_1 and BCB_1. With the notation adopted in Fig. 210 we have $S = \pi Rc + \pi Ra$. From the triangle CBE we have

---

* This can be proved only on the basis of the definition of the tangent to the curved surface of a cone. But such a definition is not included in the textbooks on elementary geometry.

$$a = \frac{h}{\sin\beta}$$

by the law of sines we have

$$\frac{c}{\sin[180^\circ - (\alpha+\beta)]} = \frac{a}{\sin\alpha}$$

hence,

$$c = \frac{a\sin(\alpha+\beta)}{\sin\alpha}$$

From $\triangle BCD$, wherein $\angle BCD = \alpha+\beta$, we have $R = a\sin(\alpha+\beta)$. Hence,

$$S = \frac{\pi h^2 \sin(\alpha+\beta)\,[\sin(\alpha+\beta)+\sin\alpha]}{\sin^2\beta\sin\alpha}$$

The expression in brackets may be transformed according to the formula for the sum of sines.

$$\textit{Answer}:\ S = \frac{2\pi h^2 \sin(\alpha+\beta)\sin\left(\alpha+\frac{\beta}{2}\right)\cos\frac{\beta}{2}}{\sin\alpha\sin^2\beta}.$$

744. Figure 211 shows an axial section of the conic vessel; ADB is the water level. The triangle ABC is an equilateral one; the circle DKL (the great circle of the sphere) is inscribed in it. With the notation adopted in Fig. 211 $R = OD\cdot\tan 60^\circ = r\sqrt{3}$ and $H = CD = 3r$*. The volume V of water in the vessel is equal to the volume of the cone ABC less the volume of the sphere, i.e.

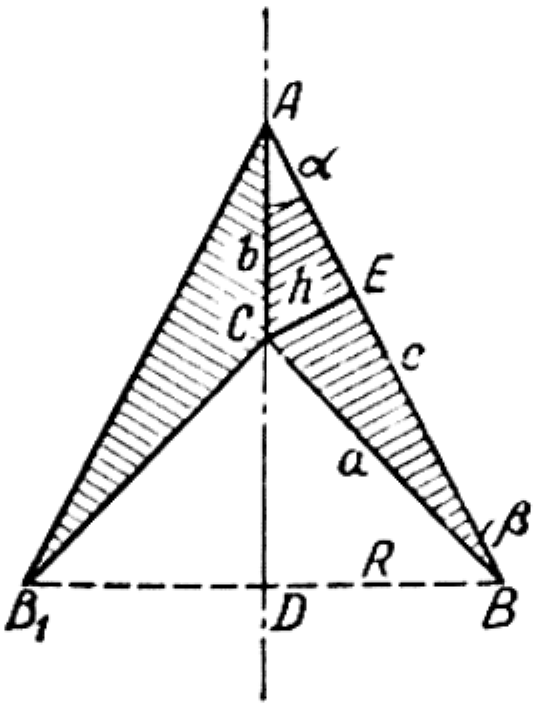


Fig. 210

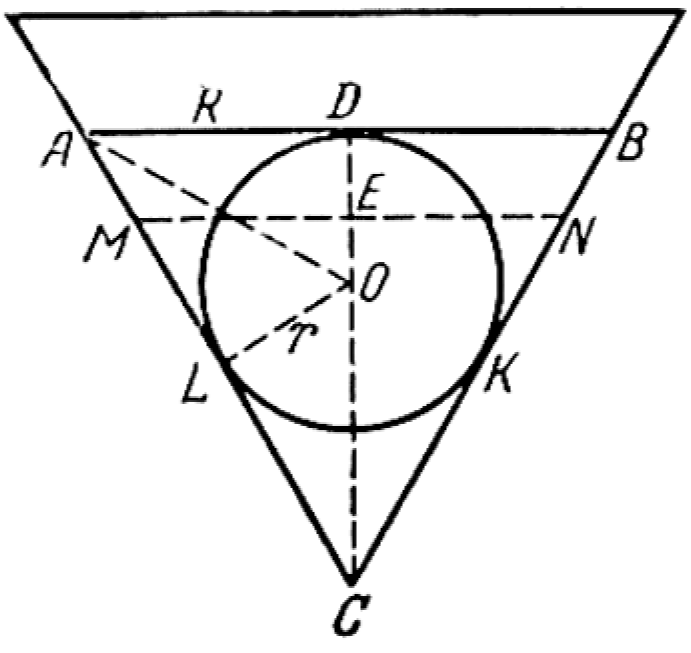


Fig. 211

$$V = \frac{1}{3}\pi\,(R^2H - 4r^3) = \frac{5}{3}\pi r^3$$

---

[*] The radius of a circle inscribed in an equilateral triangle is equal to one third of the altitude of this triangle; it follows from the fact that the point of intersection of medians in any triangle divides each median in the ratio 1 : 2.

When the sphere is removed, the water drops to a level MN and fills the cone MNC. Let $CE=h$, then $ME=CE\cdot\tan 30°=\frac{h}{\sqrt{3}}$, thus

$$V=\frac{\pi}{3}\cdot ME^2\cdot CE=\frac{\pi h^3}{9}$$

We get the equation

$$\frac{\pi h^3}{9}=\frac{5}{3}\pi r^3$$

Answer: $h=r\sqrt[3]{15}$.

745. If the radius O_1A_1 (Fig. 212) is denoted by r, then the altitude A_1M of the prism is also equal to r, and from the triangle $A_1B_1C_1$, wherein $A_1B_1=$ $=2r$, we have

$A_1C_1=2r\cos\alpha$ and $B_1C_1=2r\sin\alpha$

The quantity r is found from the triangle AA_1M, where $AM=R-r$. We have $R-r=r\cot\frac{\alpha}{2}$. Hence

$$r=\frac{R}{1+\cot\frac{\alpha}{2}}$$

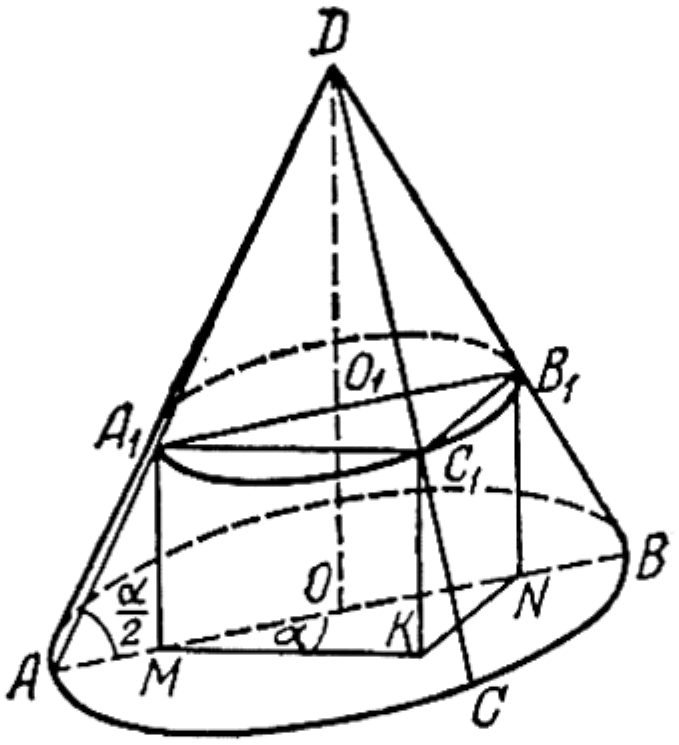


Fig. 212

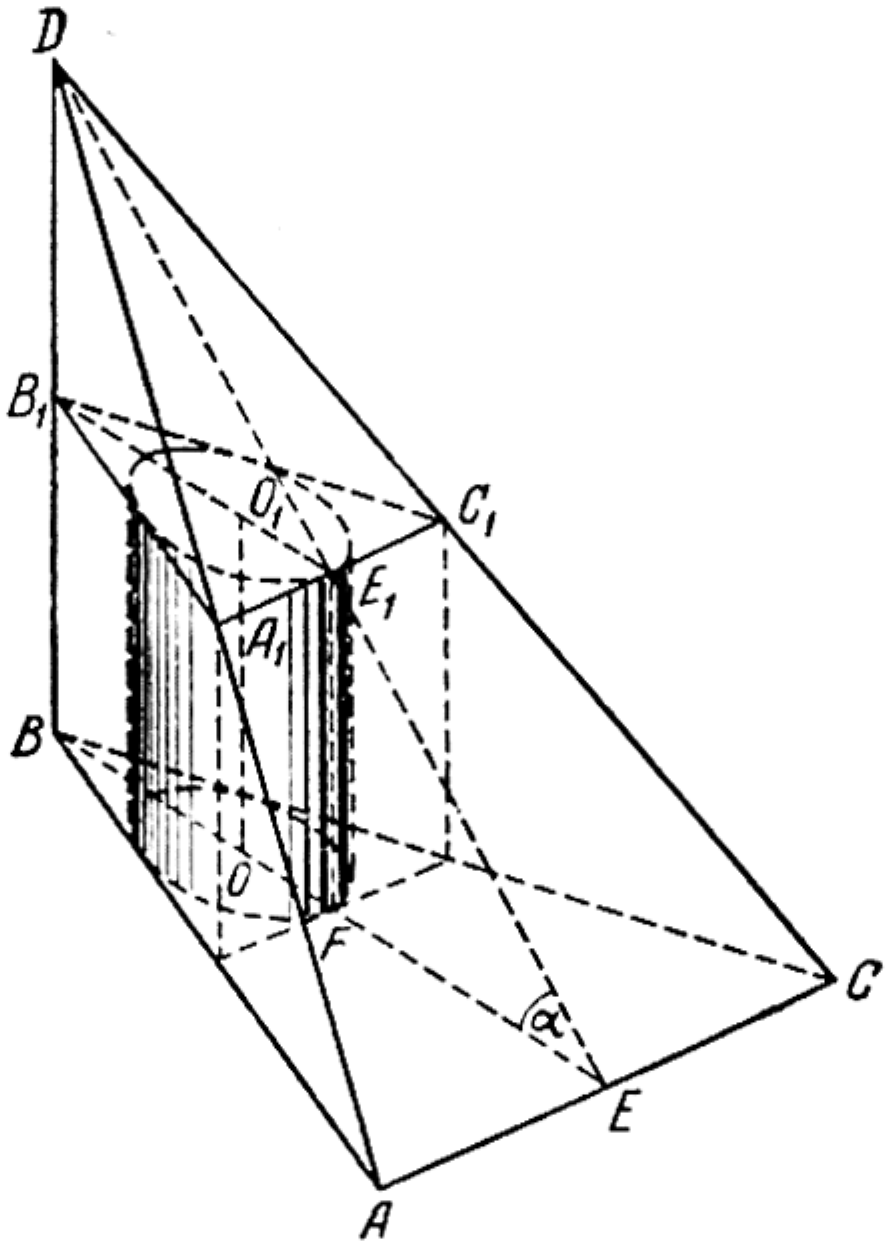


Fig. 213

Now we find the lateral area of the prism:

$$S_{lat}=(2r+2r\cos\alpha+2r\sin\alpha)\cdot r=\frac{2R^4}{\left(1+\cot\frac{\alpha}{2}\right)^2}(1+\cos\alpha+\sin\alpha)$$

Answer: $S_{lat}=\dfrac{2R^2(1+\cos\alpha+\sin\alpha)}{\left(1+\cot\frac{\alpha}{2}\right)^2}=\dfrac{\sqrt{2}\,R^2\sin\frac{\alpha}{2}\sin\alpha}{\cos\left(45°-\frac{\alpha}{2}\right)}$.

746. The radius $R=OF$ (Fig. 213) of the cylinder is equal to $\frac{1}{3}BF$*. But

$$BF=BE-FE=BE-FE_1\cdot\cot\alpha=\frac{a\sqrt{3}}{2}-\frac{a}{2}\cot\alpha=\frac{a}{2}(\cot 30^\circ-\cot\alpha)=$$
$$=\frac{a\sin(\alpha-30^\circ)}{2\sin\alpha\sin 30^\circ}=\frac{a\sin(\alpha-30^\circ)}{\sin\alpha}$$

Therefore the volume of the cylinder is

$$V_1=\pi\cdot OF^2\cdot FE_1=\pi\left[\frac{a\sin(\alpha-30^\circ)}{3\sin\alpha}\right]^2\frac{a}{2}$$

The volume of the pyramid $DA_1B_1C_1$ is

$$V_2=\frac{1}{3}\,\frac{A_1C_1}{2}\,B_1E_1\cdot B_1D$$

Here

$$B_1E_1=FB=\frac{a\sin(\alpha-30^\circ)}{\sin\alpha}$$

and

$$B_1D=B_1E_1\tan\alpha;\quad B_1E_1=\frac{A_1C_1\cdot\sqrt{3}}{2}$$

hence

$$\frac{A_1C_1}{2}=\frac{B_1E_1}{\sqrt{3}},\quad V_2=\frac{B_1E_1^3\cdot\tan\alpha}{3\sqrt{3}}.$$

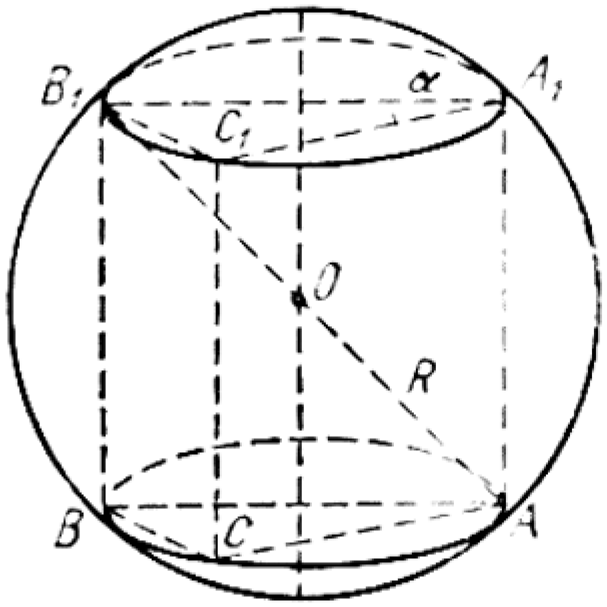


Fig. 214

The problem is possible if $BE>FE$, i.e. if $\frac{a\sqrt{3}}{2}>\frac{a}{2}\cot\alpha$ or $\cot 30^\circ>$ $>\cot\alpha$; hence, $\alpha>30^\circ$.

Answer: $V_1=\frac{\pi a^3\sin^2(\alpha-30^\circ)}{18\sin^2\alpha}$

$$V_2=\frac{a^3\sin^3(\alpha-30^\circ)\tan\alpha}{3\sqrt{3}\sin^3\alpha}$$

Preliminary Notes to Problems **747-780**

The methods of correct graphical representation of a sphere and its sections, as also of various solids inscribed in and circumscribed about a sphere are rather involved. Therefore, the problems below are supplied with schematic plane drawings which are much simpler to construct and still present a clear picture quite sufficient for understanding and solving the given problems. When a plane drawing fails to serve these purposes it is accompanied by a three-dimensional drawing.

747. Intersecting the sphere, the planes containing the bases of the prism (triangles BAC and $B_1A_1C_1$ in Fig. 214) yield two circles in which the right-angled triangles ABC and $A_1B_1C_1$ are inscribed. Therefore, the hypotenuses AB and A_1B_1 are the diameters of the obtained circles. The plane ABB_1A_1

---

* See footnote on page 336.

passes through the centre of the sphere. Since, by hypothesis, ABB_1A_1 is a square, we have $H = AA_1 = R\sqrt{2}$ and $AB = R\sqrt{2}$.

Answer: $V = \dfrac{R^3 \sin 2\alpha}{\sqrt{2}}$.

748. Intersecting the sphere, the plane containing the base of the pyramid yields the circle $ABCD$ (Fig. 215) circumscribed about this base. The altitude of the pyramid passes through the centre O_1 of this circle (since all the edges are inclined to the base at equal angles) and also through the centre O of the sphere. A cutting plane drawn through the diagonal AC of the base and vertex E yields a great circle circumscribed about the diagonal section of the pyramid AEC. From the triangle AEC, wherein the angle AEC is equal to $180° - 2\varphi$, we find $AC = 2R \sin (180° - 2\varphi) = 2R \sin 2\varphi$ (by the law of sines); hence, $AO_1 = R \sin 2\varphi$. From the triangle AEO_1 we find the altitude of the pyramid

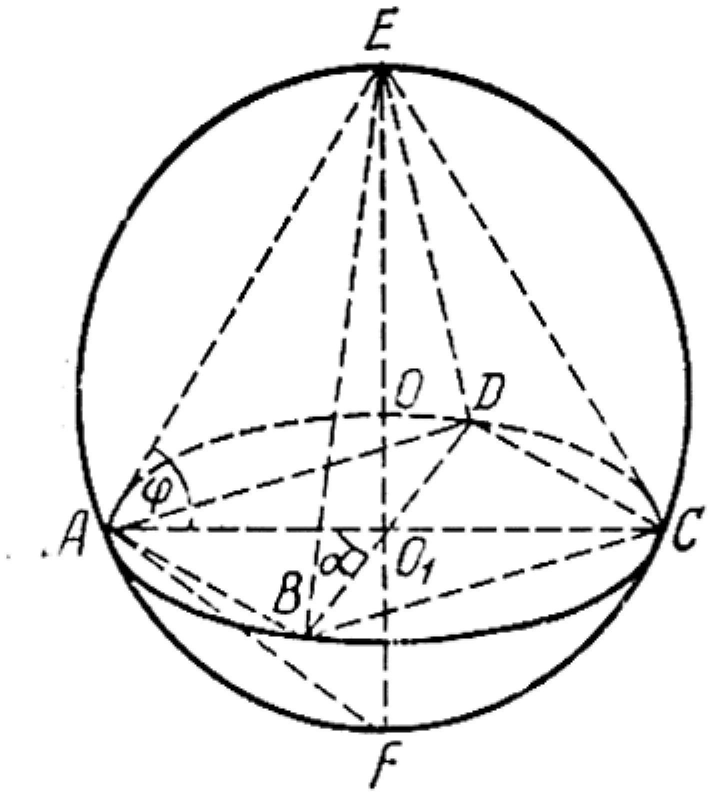


Fig. 215

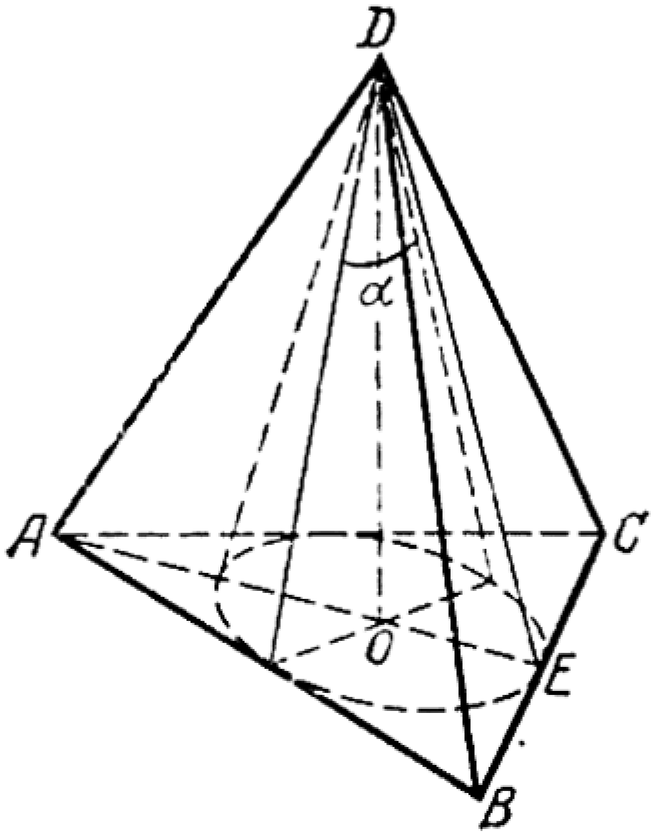


Fig. 216

$$EO_1 = H = AO_1 \cdot \tan \varphi = R \sin 2\varphi \tan \varphi$$

Answer: $V = \dfrac{2}{3} R^3 \sin^3 2\varphi \sin \alpha \tan \varphi$.

749. Since the radius OE (Fig. 216) of the circle inscribed in the base is equal to R, $AB = 2R\sqrt{3}$. From $\triangle DOE$ we find $DO = H = R \cot \dfrac{\alpha}{2}$.

Answer: $V = \sqrt{3} R^3 \cot \dfrac{\alpha}{2}$.

750. Fig. 217 shows an axial section. We have

$$S_{curved} = \pi l (r_1 + r_2) = \pi \cdot AD \cdot (AM + DN)$$

But $AM + DN = AF + DF = AD$. Therefore $S_{curved} = \pi \cdot AD^2$. From the triangle AED, wherein $DE = MN = 2r$, we find $AD = \dfrac{2r}{\sin \alpha}$.

Answer: $S_{curved} = \dfrac{4\pi r^2}{\sin^2 \alpha}$.

751. See the preceding problem. We have $S=S_{curved}+\pi(r_1^2+r_2^2)$. From the triangle AOM (see Fig. 217) we find

$$AM=r_1=OM\cdot\cot\frac{\alpha}{2}=r\cot\frac{\alpha}{2}$$

From the triangle DON, wherein $\angle ODN=\frac{180^\circ-\alpha}{2}$, we have

$$DN=r_2=r\cot\left(90^\circ-\frac{\alpha}{2}\right)=r\tan\frac{\alpha}{2}$$

Calculations become simpler if the expression $r_1^2+r_2^2$ is transformed as follows $r_1^2+r_2^2=(r_1+r_2)^2-2r_1r_2$. Since $r_1+r_2=l$, $S_{curved}=\pi l^2$ (see the preceding problem) and, from the right-angled triangle AOD, $AF\cdot FD=OF^2$ or $r_1r_2=r^2$, we have

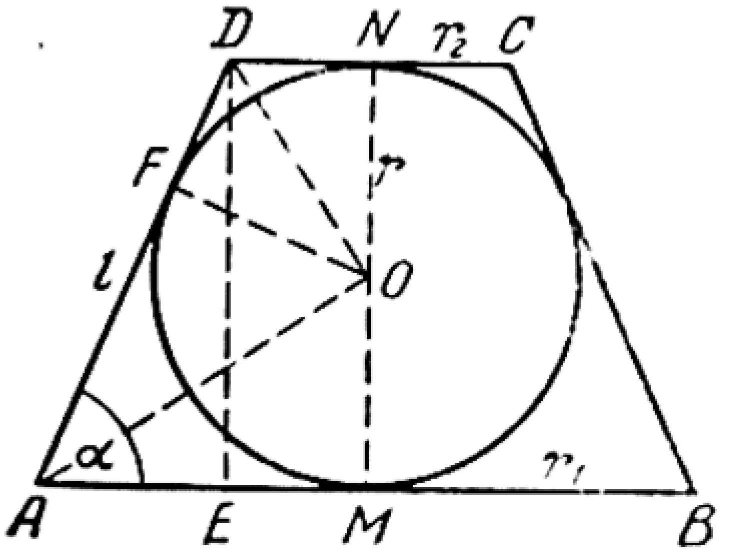


Fig. 217

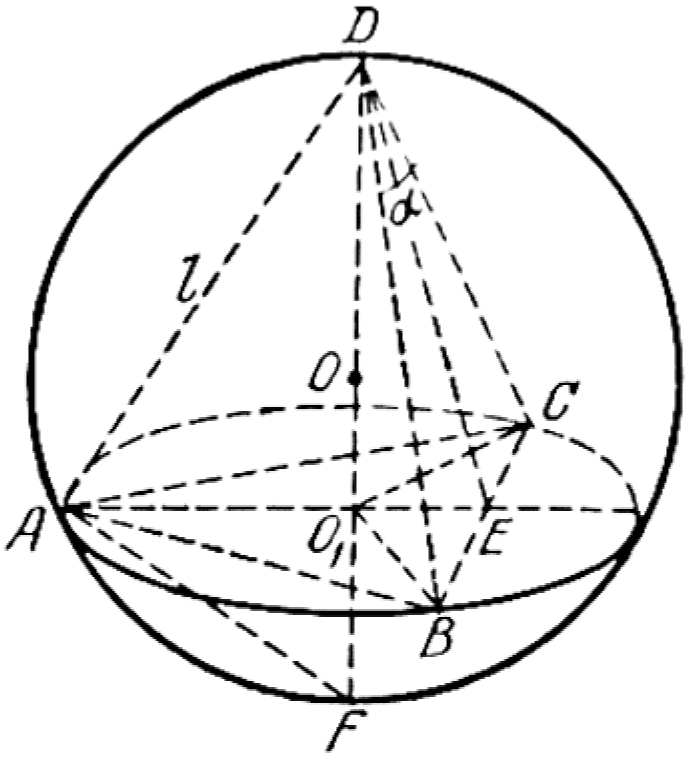


Fig. 218

$$S=\pi l^2+\pi l^2-2\pi r^2=2\pi(l^2-r^2)$$

Substitute $l=\frac{2r}{\sin\alpha}$ into this expression.

Answer: $S=2\pi r^2\left(\frac{4}{\sin^2\alpha}-1\right)$.

752. See the preceding problem. We have

$$V=\pi\frac{2r}{3}(r_1^2+r_1r_2+r_2^2)=\frac{2\pi r}{3}\Big[(r_1+r_2)^2\Big]-r_1r_2=\frac{2\pi r}{3}(l^2-r^2)$$

Substitute $l=\frac{2r}{\sin\alpha}$.

Answer: $V=\frac{2\pi r^3}{3}\left(\frac{4}{\sin^2\alpha}-1\right)$.

753. Let us denote the length of the equal chords DA, DB, DC (Fig. 218) by l. From the isosceles triangle DBC we find $BC=2l\sin\frac{\alpha}{2}$. Similarly, $AB=$

$= AC = 2l \sin\frac{\alpha}{2}$. Consequently, the triangle ABC is an equilateral one.

Dropping the perpendicular DO_1 to the plane ABC and revealing congruence of the triangles DO_1A, DO_1B, DO_1C, let us prove that $AO_1 = BO_1 = CO_1$, i.e. that O_1 is the centre of the base (thus, the pyramid $DABC$ is a regular one). Since the points A, B, C lie on the surface of the sphere, $OA = OB = OC$ (O is the centre of the sphere). Dropping a perpendicular from O to the plane ABC, let us prove that the foot of the perpendicular is the centre of the triangle ABC, i.e. coincides with the point O_1. Consequently, OO_1 (and, hence, DO_1) lies on a diameter of the sphere (DF in Fig. 218). From the right-angled triangle DAF, wherein $DF = 2R$, we find $l^2 = DA^2 = 2R \cdot DO_1$. The line-segment DO_1 may be related to l by another formula. Namely,

$$DO_1 = \sqrt{AD^2 - AO_1^2}$$

where

$$AD = l \quad \text{and} \quad AO_1 = \frac{BC}{\sqrt{3}} = \frac{2l \sin\frac{\alpha}{2}}{\sqrt{3}}$$

Hence,

$$DO_1 = l\sqrt{1 - \frac{4\sin^2\frac{\alpha}{2}}{3}}$$

Substitute this expression into the equality $l^2 = 2R \cdot DO_1$. We find

$$l = 2R\sqrt{1 - \frac{4\sin^2\frac{\alpha}{2}}{3}}$$

Reduce this expression to a form convenient for taking logarithms. We have

$$l = 2R\sqrt{1 - \frac{2(1-\cos\alpha)}{3}} = \frac{2R}{\sqrt{3}}\sqrt{1 + 2\cos\alpha} =$$

$$= \frac{2R}{\sqrt{3}}\sqrt{2(\cos 60^\circ + \cos\alpha)} = \frac{4R}{\sqrt{3}}\sqrt{\cos\left(30^\circ + \frac{\alpha}{2}\right)\cos\left(30^\circ - \frac{\alpha}{2}\right)}$$

Answer: $l = 2R\sqrt{1 - \frac{4\sin^2\frac{\alpha}{2}}{3}} =$

$$= \frac{4R}{\sqrt{3}}\sqrt{\cos\left(30^\circ + \frac{\alpha}{2}\right)\cos\left(30^\circ - \frac{\alpha}{2}\right)}.$$

754. The isosceles trapezoid $ABCD$ (Fig. 219) represents an axial section of the frustum of a cone. By hypothesis, $\angle AOB = \alpha$ and $\angle DOC = \beta$. Therefore

$$R_1 = AE = AO\sin\frac{\alpha}{2} = R\sin\frac{\alpha}{2} \quad \text{and} \quad R_2 = DF = R\sin\frac{\beta}{2}$$

The angle $AOD=\frac{360^\circ-(\alpha+\beta)}{2}=180^\circ-\frac{\alpha+\beta}{2}$. Therefore $l=AD=$ $=2R\cos\frac{\alpha+\beta}{2}$. We have

$$S_{curved}=\pi l\,(R_1+R_2)=2\pi R^2\cos\frac{\alpha+\beta}{4}\left(\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}\right)$$

Answer: $S_{curved}=2\pi R^2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{4}$.

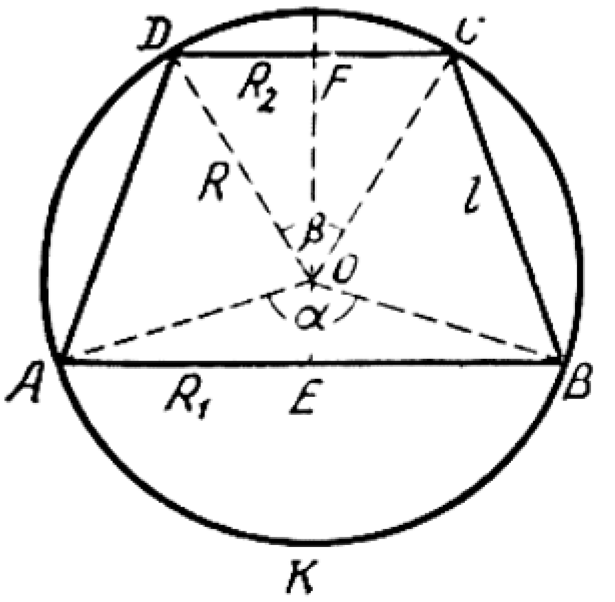


Fig. 219

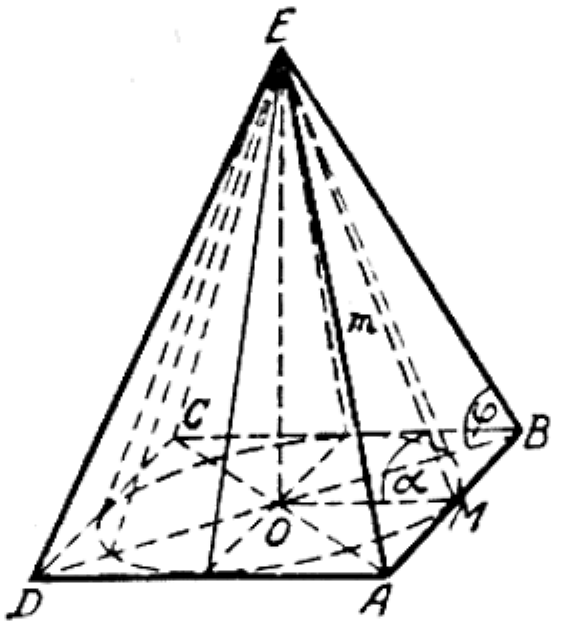


Fig. 220

755. From the triangle OME (Fig. 220) we have

$$OM=r=m\cos\alpha;\qquad EO=H=m\sin\alpha$$

By the formula $S=\pi r\,(r+l)$, where $l=m$, we find

$$S=\pi m^2\cos\alpha\,(1+\cos\alpha)$$

or

$$S=2\pi m^2\cos\alpha\cos^2\frac{\alpha}{2}$$

The angle $\varphi=\angle EBO$, at which the lateral edge BE is inclined to the base, is determined from the triangle EBO, wherein $OB=OM\sqrt{2}=m\sqrt{2}\cos\alpha$. We have $\tan\varphi=\frac{EO}{OB}=\frac{m\sin\alpha}{m\sqrt{2}\cos\alpha}=\frac{\tan\alpha}{\sqrt{2}}$

Answer: $S=2\pi m^2\cos\alpha\cos^2\frac{\alpha}{2}$; $\varphi=\arctan\frac{\tan\alpha}{\sqrt{2}}$.

756. From $\triangle ASB$ (Fig. 221) we find $AB=2l\sin\frac{\alpha}{2}$; hence, $R=OA=AB=$ $=2l\sin\frac{\alpha}{2}$. From $\triangle ASO$ we find

$$SO=H=\sqrt{l^2-R^2}=l\sqrt{1-4\sin^2\frac{\alpha}{2}}$$

We get

$$V=\frac{1}{3}\pi R^2H=\frac{\pi}{3}\cdot 4l^2\sin^2\frac{\alpha}{2}\cdot l\sqrt{1-4\sin^2\frac{\alpha}{2}}$$

The radicand may be reduced to a form convenient for taking logarithms in the same way as in Problem **753**: we get

$$1-4\sin^2\frac{\alpha}{2}=4\sin\left(30^\circ+\frac{\alpha}{2}\right)\sin\left(30^\circ-\frac{\alpha}{2}\right)$$

Answer: $V=\frac{8}{3}\pi l^3\sin^2\frac{\alpha}{2}\sqrt{\sin\left(30^\circ+\frac{\alpha}{2}\right)\sin\left(30^\circ-\frac{\alpha}{2}\right)}$.

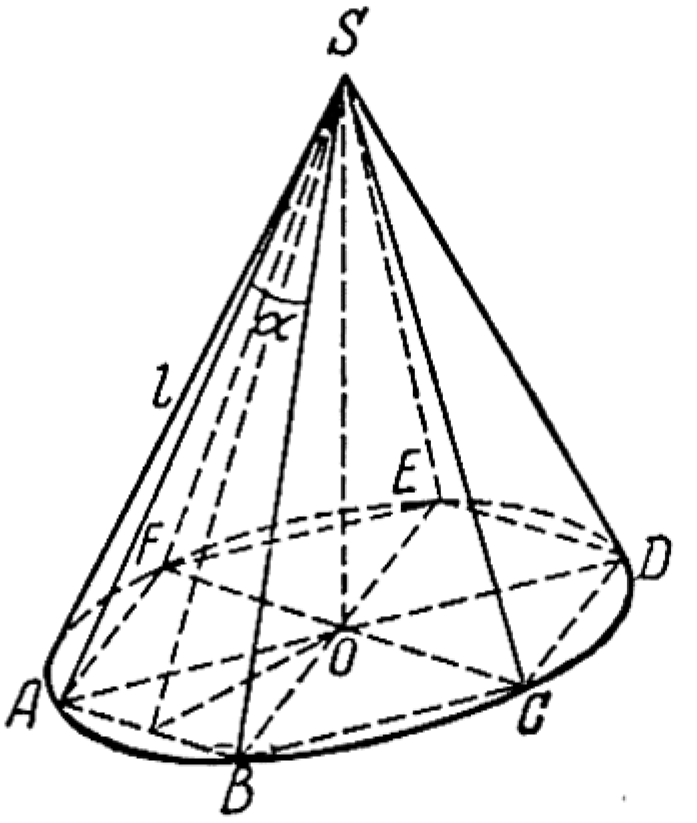


Fig. 221

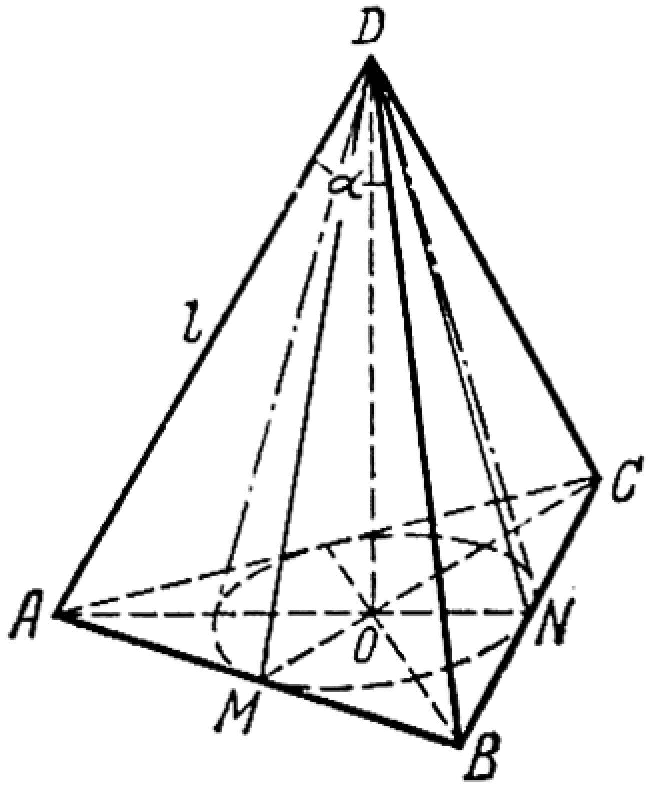


Fig. 222

757. From $\triangle ADM$ (Fig. 222) we find $AM=l\sin\frac{\alpha}{2}$. From $\triangle AMO$ we find

$$r=OM=AM\cdot\tan 30^\circ=l\sin\frac{\alpha}{2}\cdot\frac{1}{\sqrt{3}}$$

and

$$R=AO=\frac{AM}{\cos 30^\circ}=\frac{2l\sin\frac{\alpha}{2}}{\sqrt{3}}$$

From $\triangle AOD$ we find

$$OD=H=\sqrt{l^2-R^2}=\frac{l}{\sqrt{3}}\sqrt{3-4\sin^2\frac{\alpha}{2}}$$

The volume of the cone is

$$V=\frac{\pi}{3}r^2H=\frac{\pi}{3}\cdot\frac{l^2\sin^2\frac{\alpha}{2}}{3}\cdot\frac{l}{\sqrt{3}}\sqrt{3-4\sin^2\frac{\alpha}{2}}$$

The radicand may be transformed in the same way as in Problem 753.

$$\textit{Answer}:\ V=\frac{\pi l^3\sin^2\frac{\alpha}{2}}{9\sqrt{3}}\sqrt{3-4\sin^2\frac{\alpha}{2}}=$$

$$=\frac{2\pi l^3\sin^2\frac{\alpha}{2}\sqrt{\cos\left(30^\circ+\frac{\alpha}{2}\right)\cos\left(30^\circ-\frac{\alpha}{2}\right)}}{9\sqrt{3}}$$

758. The volume of the sphere (see Fig. 223) is equal to $\frac{4}{3}\pi R^3$, and the volume of the cone ACB, to $\frac{1}{3}\pi r^2\cdot CO_1=\frac{1}{3}\pi r^2 H$. By hypothesis,

$$\frac{1}{3}\pi r^2 H=\frac{1}{4}\cdot\frac{4}{3}\pi R^3$$

i.e. $r^2H = R^3$. Another relationship between r and R we obtain from the right-angled triangle CAD; namely, $AO_1^2 = CO_1\cdot DO_1$, i.e. $r^2 = H(2R - H)$. Substituting this expression into the preceding equality, we get $R^3 - 2H^2R + H^3 = 0$. Though this equation in the unknown R is of the third degree, its one solution $R = H$ is quite obvious (it could be guessed immediately by the given conditions, since the volume of a cone, in which both the radius of the base and the altitude are equal to the radius of the sphere, is equal to a quarter of the volume of the sphere). Consequently (according to the remainder theorem), the left member may be factorized, one of the factors being $R - H$. For this purpose it is sufficient to divide $R^3 - 2H^2R + H^3$ by $R - H$ or accomplish the following transformation:

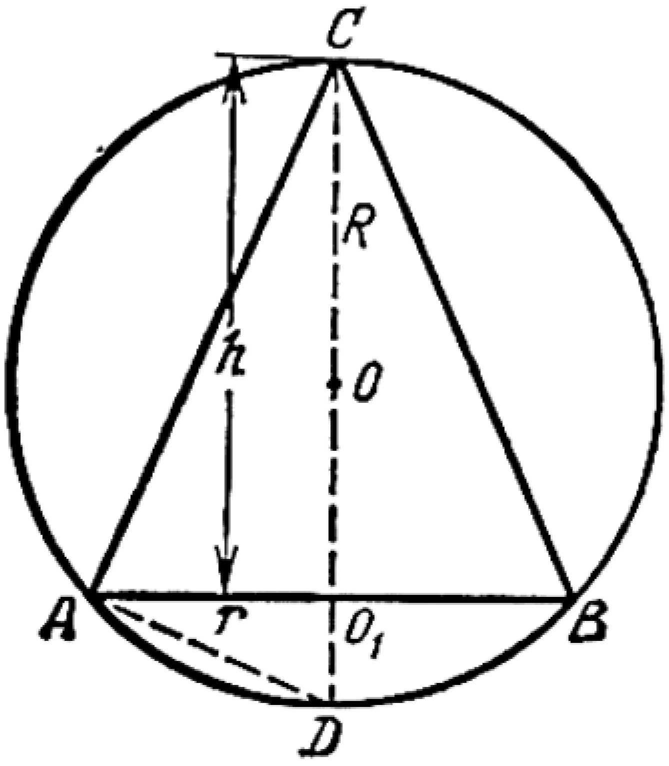


Fig. 223

$$R^3-2H^2R+H^3=(R^3-H^2R)-(H^2R-H^3)=$$
$$=R(R-H)(R+H)-H^2(R-H)=(R-H)(R^2+RH-H^2)=0$$

The equation $R^2 + RH - H^2 = 0$ has one positive root $R=\frac{H(\sqrt{5}-1)}{2}$ $\left(\text{the negative root } R=-\frac{H(\sqrt{5}+1)}{2} \text{ is not suitable}\right)$. Geometrically, this means that the radius of the sphere is equal to the larger portion of the altitude of the cone divided in extreme and mean ratios.

Answer: the problem has two solutions:

$$V=\frac{4}{3}\pi H^3 \quad\text{and}\quad V=\frac{4}{3}\pi(\sqrt{5}-2)H^3$$

759. The altitude of the prism is equal to the diameter $2R$ of the inscribed sphere. If a plane is drawn through the centre of the sphere and parallel to the bases of the prism, then the section of the prism by this plane yields an equilateral triangle (KLM in Figs 224, 224a) equal to the base of the prism, while the section of the sphere is a great circle (PNQ) inscribed in the triangle (KLM).

From the triangle LON, wherein $ON = R$ and $\angle NLO = 30°$, we find $LN = R\sqrt{3}$. Consequently, $LM = a = 2R\sqrt{3}$. The lateral area of the prism is $S_{lat} = 3aH = 12R^2\sqrt{3}$. The area of the base $S_{base} = \frac{a^2\sqrt{3}}{4} = 3R^2\sqrt{3}$. Hence,

$$S_{total} = 12R^2\sqrt{3} + 6R^2\sqrt{3} = 18R^2\sqrt{3}$$

The surface of the sphere is equal to $4\pi R^2$.

Answer: the required ratio is $\frac{2\pi}{9\sqrt{3}}$.

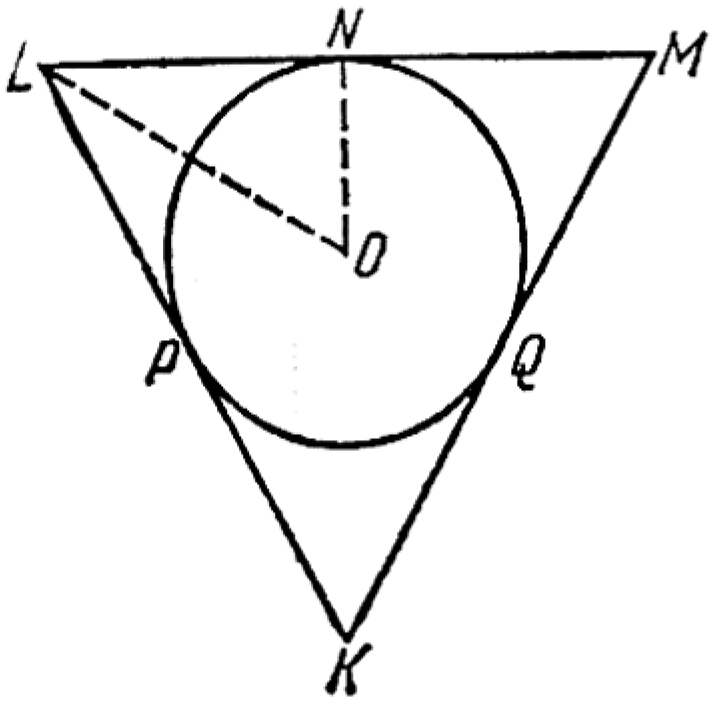


Fig. 224

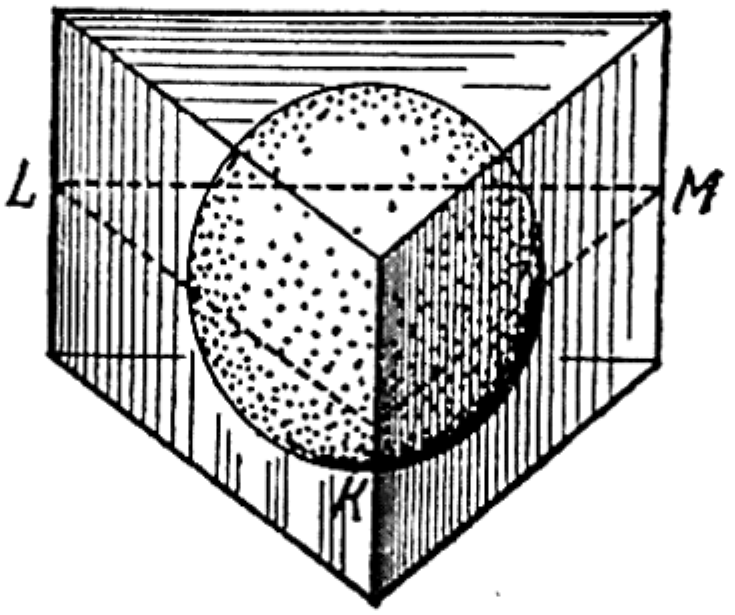


Fig. 224a

760. (a) *Drawing*. The centre O_1 of the sphere inscribed in the pyramid (if it is possible to inscribe a sphere in this pyramid) must be equidistant from the lateral face BEC and the base $ABCD$ (Fig. 225). Therefore, it must lie in the bisector plane of the dihedral angle φ at the edge BC. Similarly, O_1 lies in the bisector planes of the dihedral angles φ at the edges AB, AD, DC. Hence, all the lateral faces of the pyramid O_1ABCD (it is not shown in the drawing) are inclined to the base at one and the same angle $\frac{\varphi}{2}$. Consequently, the altitude O_1O of the pyramid O_1ABCD passes through the centre O of the circle inscribed in the rhombus $ABCD$ (see the Preliminary Notes to Problem 617). The altitude EO of the given pyramid passes through the same centre. Hence, the centre O_1 of the sphere lies on the altitude EO.

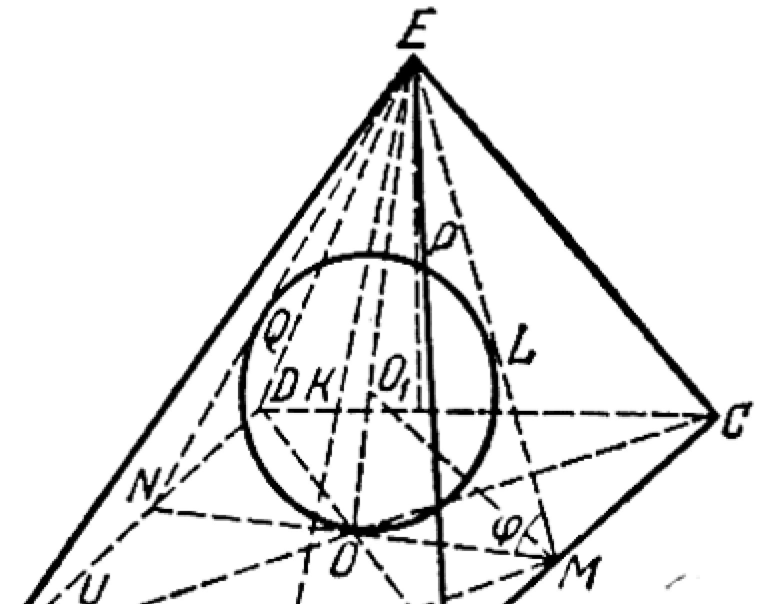


Fig. 225

The point of tangency of the sphere and the face BEC is the foot L of the perpendicular dropped from the centre O_1 of the sphere to the plane BEC. Hence, the plane O_1EL is perpendicular to the face BEC (prove it!). At the same time the plane O_1EL is perpendicular to the base $ABCD$ (since it passes through the altitude EO). Consequently, the plane O_1EL is perpendicular to the edge BC. Hence, the straight line MN, along which the planes O_1EL and $ABCD$ intersect, is the altitude of the rhombus (drawn through its centre O). The same thing is with the remaining three points (K, Q and P), at which the lateral faces touch the sphere.

Hence, the following constructions: draw the altitude NOM of the rhombus $ABCD$ (it is desirable to make it horizontal), construct the section NEM (an isosceles triangle) and depict the circle inscribed in the triangle NEM. The points L and Q at which this circle touches the sides ME and NE are the points of tangency of the sphere and the faces BEC and AED. To find the point K draw $MS \parallel AC$. Then OS (not shown in the drawing) represents the other altitude of the rhombus (prove it!). Draw ES and through the point L draw $LK \parallel MS$ (not shown in the drawing). The fourth point P is found in a similar way. As follows from this construction, the sphere with O_1 as the centre and radius $R = O_1L$ is actually inscribed in the pyramid.

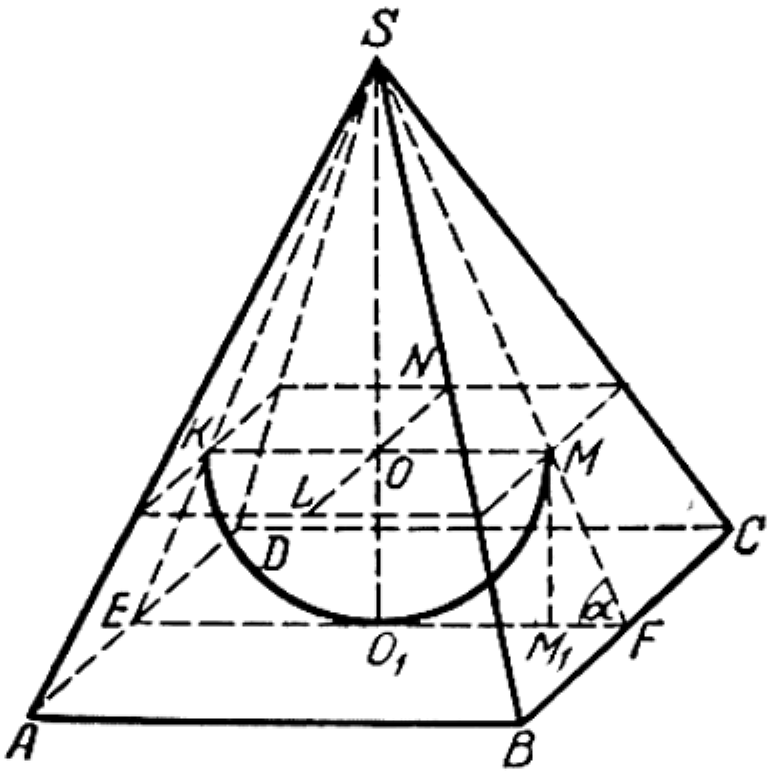


Fig. 226

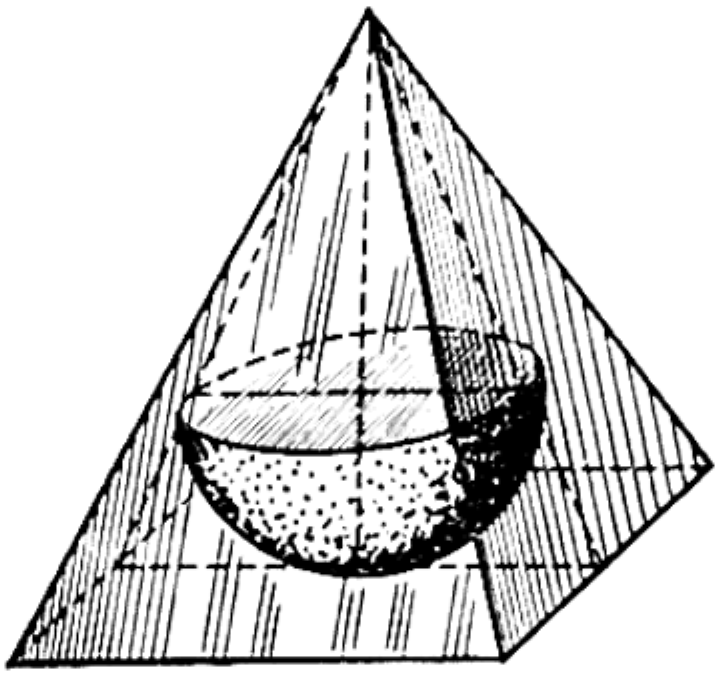


Fig. 226a

(b) *Solution*. From the triangle MOO_1 we find

$$OM = OO_1 \cot \frac{\varphi}{2} = R \cot \frac{\varphi}{2}$$

so that

$$H = OE = OM \cdot \tan \varphi = R \cot \frac{\varphi}{2} \tan \varphi$$

Then from the triangle BUA (where $BU \parallel MN$) we find

$$AB = a = \frac{BU}{\sin \alpha} = \frac{2 \cdot OM}{\sin \alpha} = \frac{2R \cot \frac{\varphi}{2}}{\sin \alpha}$$

Hence,

$$S_{base} = a^2 \sin \alpha = \frac{4R^2 \cot^2 \frac{\varphi}{2}}{\sin \alpha}$$

Answer: $V = \dfrac{4R^3 \tan \varphi \cot^3 \frac{\varphi}{2}}{3 \sin \alpha}$.

761. (a) *Drawing*. The centre O of the equator of the hemisphere (Fig. 226) lies on the altitude SO_1 of the pyramid. Since

$$OM = OO_1 = r,$$

the point M lies on the bisector O_1M of the angle OO_1M. Marking M as the point of intersection of O_1M and SF, draw the section $KLMN$ parallel to the base. The midpoints K, L, M, N of the sides of the section are the points of tangency of the equator and the lateral faces. The simicircle KO_1M is the section of the hemisphere by the plane ESF.

(b) *Solution.* The side of the base is

$$a = EF = 2 \cdot O_1F = 2\,(O_1M_1 + M_1F)$$

But $O_1M_1 = OM = r$, and $M_1F = MM_1 \cdot \cot\alpha = r\cot\alpha$. Hence,

$$a = 2r\,(1 + \cot\alpha)$$

We have

$$S_{total} = \frac{2S_{base}\cos^2\frac{\alpha}{2}}{\cos\alpha}$$

(see the Note to Problem 619). Here $S_{base} = a^2 = 4r^2\,(1 + \cot\alpha)^2$.

Answer: $$S_{total} = \frac{8r^2\,(1+\cot\alpha)^2\cos^2\frac{\alpha}{2}}{\cos\alpha} = \frac{4r^2\sin^2(45^\circ + \alpha)}{\cos\alpha\sin^2\frac{\alpha}{2}}.$$

762. Intersecting the hemisphere, the plane ESF (Fig. 227) yields the semicircle NPM touching the slant heights of the pyramid at the points Q and G.

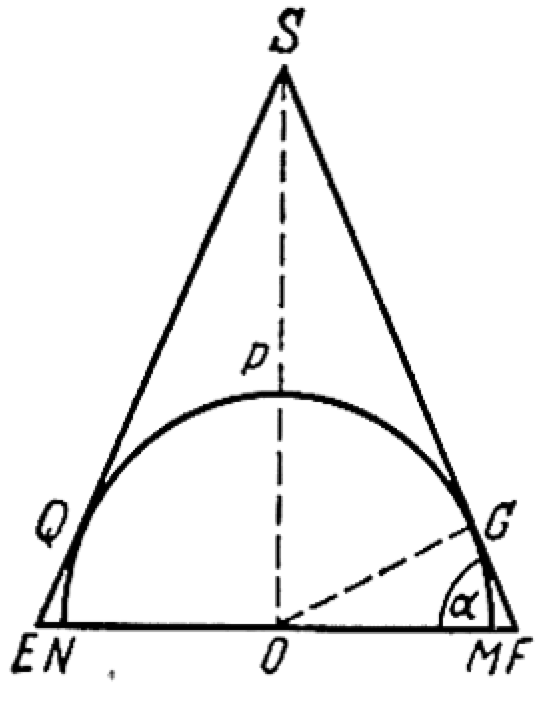


Fig. 227

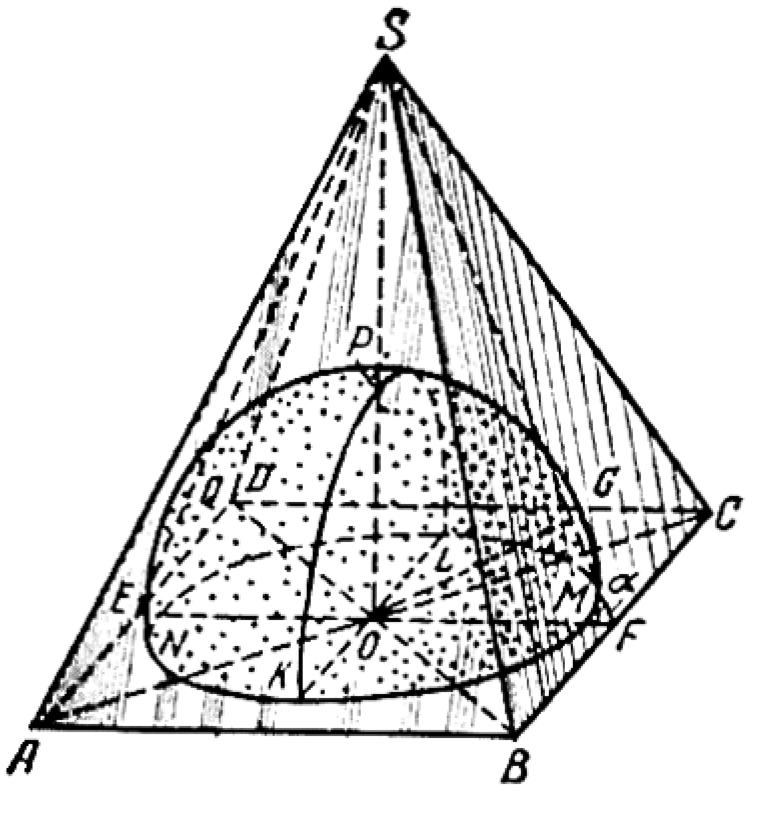


Fig. 227a

If we denote the side of the base of the pyramid by a and the radius of the hemisphere, by r, then the surface of the hemisphere is

$$S_1 = 2\pi r^2 + \pi r^2 = 3\pi r^2$$

and the total surface area of the pyramid

$$S_2 = \frac{2a^2\cos^2\frac{\alpha}{2}}{\cos\alpha}$$

(see the Note to Problem 619); their ratio is

$$q=\frac{3\pi r^2\cos\alpha}{2a^2\cos^2\frac{\alpha}{2}}$$

From $\Delta\,OGF$ we have $OG = OF\cdot\sin\alpha$, i.e. $r=\frac{a}{2}\sin\alpha$. Substitute this expression into the preceding equality.

To determine the volume V of the hemisphere find r proceeding from the condition $a-2r=m$ and from the above equality $r=\frac{a}{2}\sin\alpha$. We get

$$r=\frac{m\sin\alpha}{2(1-\sin\alpha)}=\frac{m\sin\alpha}{4\sin^2\left(45^\circ-\frac{\alpha}{2}\right)}$$

Answer: $q=\frac{3\pi}{8}\sin 2\alpha\tan\frac{\alpha}{2}$

$$V=\frac{\pi m^3\sin^3\alpha}{96\sin^6\left(45^\circ-\frac{\alpha}{2}\right)}$$

763. Figure 228 shows an axial section of the cone and the sphere inscribed in it. The required volume V is obtained by subtracting the volume of the spherical segment MEN from the volume of the cone MCN. Hence,

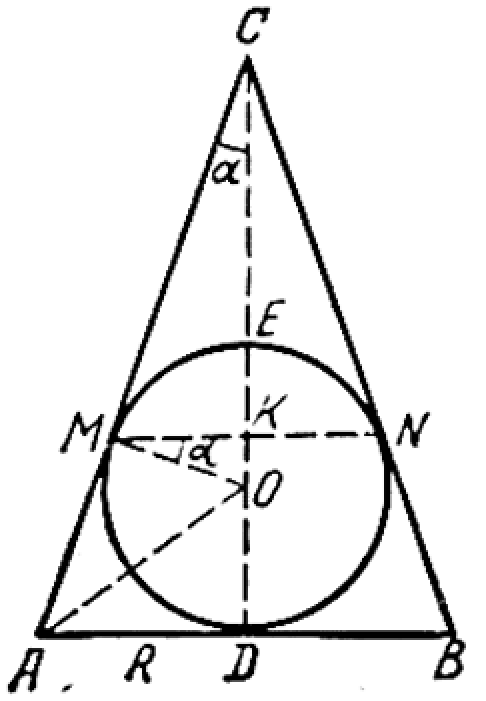


Fig. 228

$$V=\frac{\pi}{3}\cdot MK^2\cdot KC-\pi\cdot KE^2\left(r-\frac{1}{3}KE\right)$$

where r is the radius of the sphere. From the triangle AOD we find

$$r=OD=AD\cdot\tan\frac{\angle DAC}{2}=R\tan\left(45^\circ-\frac{\alpha}{2}\right)$$

Now, from the triangle OMK, wherein $\angle OMK=\alpha$ (the sides of the angles OMK and MCK are mutually perpendicular), we have

$$MK=OM\cdot\cos\alpha=r\cos\alpha \text{ and } OK=r\sin\alpha$$

Hence, $KE=OE-OK=r(1-\sin\alpha)$. Finally, $KC=MK\cdot\cot\alpha=r\cos\alpha\cot\alpha$. Consequently,

$$V=\frac{\pi}{3}r^3\cos^3\alpha\cot\alpha-\pi r^2(1-\sin\alpha)^2\left[r-\frac{r(1-\sin\alpha)}{3}\right]=$$

$$=\frac{\pi}{3}r^3\left[\frac{\cos^4\alpha}{\sin\alpha}-(1-\sin\alpha)^2(2+\sin\alpha)\right]$$

This expression may be simplified. Factor out $(1-\sin\alpha)^2$ on having transformed $\cos^4\alpha$ beforehand; namely,

$$\cos^4\alpha=(1-\sin^2\alpha)^2=(1-\sin\alpha)^2(1+\sin\alpha)^2$$

Now we have

$$V=\frac{\pi r^3(1-\sin\alpha)^2}{3\sin\alpha}\,[(1+\sin\alpha)^2-(2+\sin\alpha)\sin\alpha]$$

The expression in brackets is equal to unity. We get

$$V=\frac{\pi r^3(1-\sin\alpha)^2}{3\sin\alpha}$$

Substitute into it the found expression

$$r=R\tan\left(45^\circ-\frac{\alpha}{2}\right)$$

We may also use the formula

$$1-\sin\alpha=2\sin^2\left(45^\circ-\frac{\alpha}{2}\right)$$

Answer: $V=\dfrac{4\pi R^3\tan^3\left(45^\circ-\frac{\alpha}{2}\right)\sin^4\left(45^\circ-\frac{\alpha}{2}\right)}{3\sin\alpha}$.

764. Using the notation adopted in Fig. 229, the given condition is expressed by the equality $\pi R(l+R)=n\cdot 4\pi r^2$. From the triangle OBO_1 we find $r=R\tan\frac{\alpha}{2}$, and from the triangle BOC we have $BC=l=\frac{R}{\cos\alpha}$. When reduced by πR^2, the above equality takes the form

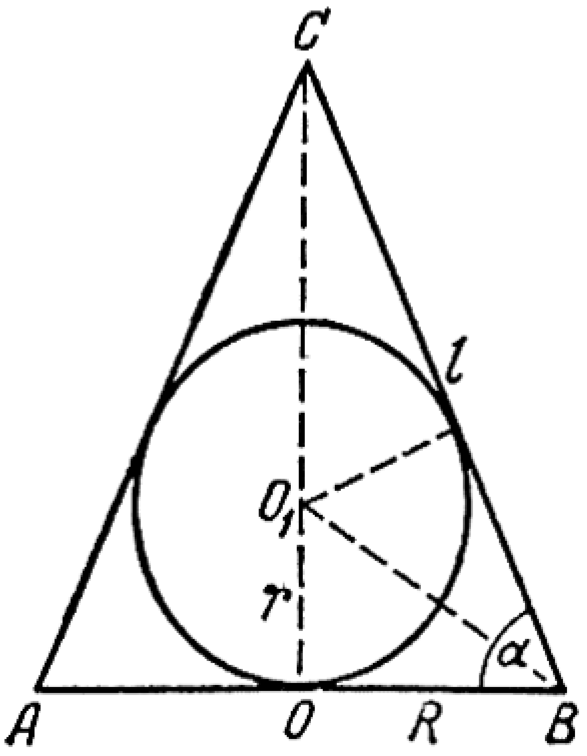


Fig. 229

$$1+\frac{1}{\cos\alpha}=4n\tan^2\frac{\alpha}{2}$$

Apply the formula

$$\cos\alpha=\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

We have the equation

$$\frac{2}{1-\tan^2\frac{\alpha}{2}}=4n\tan^2\frac{\alpha}{2}$$

Putting $\tan\frac{\alpha}{2}=z$, we get*

$$z^4-z^2+\frac{1}{2n}=0$$

---

* Getting rid of the denominator, we could introduce an extraneous solution $\left(\tan^2\frac{\alpha}{2}=1\right)$, but we do not get such a solution, since it does not satisfy the original equation.

whence

$$z^2=\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{1}{2n}}$$

Hence it is clear that for $n<2$ the problem has no solution (since the radicand is negative). For $n \geqslant 2$ both values of the quantity z^2 are positive $\left(\text{since } \sqrt{\frac{1}{4}-\frac{1}{2n}}<\sqrt{\frac{1}{4}}, \text{ i.e. } \sqrt{\frac{1}{4}-\frac{1}{2n}}<\frac{1}{2}\right)$. Since the quantity $\tan\frac{\alpha}{2}$ must be positive, only two solutions are possible:

$$z=\tan\frac{\alpha}{2}=+\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{1}{2n}}}$$

and

$$z=\tan\frac{\alpha}{2}=+\sqrt{\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{1}{2n}}}$$

Since the angle $\frac{\alpha}{2}$ is less than 45°, $\tan\frac{\alpha}{2}$ must be less than unity; hence, there must be $z^2<1$. But this inequality is always satisfied, because

$$\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{1}{2n}}<\frac{1}{2}+\sqrt{\frac{1}{4}}=1$$

and

$$\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{1}{2n}}<\frac{1}{2}$$

Answer: the problem is solvable only if $n \geqslant 2$. For $n>2$ there are two solutions:

$$\tan\frac{\alpha}{2}=\sqrt{\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{1}{2n}}}$$

at $n=2$ both solutions coincide $\left(\tan\frac{\alpha}{2}=\sqrt{\frac{1}{2}}\right)$.

765. Using the notation adopted in Fig. 229, we have

$$\frac{1}{3}\pi R^2H=n\cdot\frac{4}{3}\pi r^3$$

Substituting

$$r=R\tan\frac{\alpha}{2} \quad \text{and} \quad H=R\tan\alpha$$

we get the equation

$$\tan\alpha=4n\tan^3\frac{\alpha}{2}$$

Applying the formula

$$\tan\alpha=\frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$$

and denoting $\tan\frac{\alpha}{2}$ by z, we get the equation

$$z\left(\frac{1}{1-z^2}-2nz^2\right)=0$$

It decomposes into two equations, but one of them ($z=0$) disagrees with two given conditions (the angle α must be non-zero). The other equation is reduced to the form $z^4-z^2+\frac{1}{2n}=0$, i.e. it coincides with the equation in the preceding problem. We obtain the following two solutions:

$$\tan\frac{\alpha}{2}=\sqrt{\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{1}{2n}}}$$

At $n=4$ one solution is

$$\tan\frac{\alpha}{2}=\sqrt{\frac{1}{2}+\sqrt{\frac{1}{8}}}=\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}=\sqrt{\frac{1+\cos 45^\circ}{2}}=$$
$$=\cos 22^\circ 30' \approx 0.9239;$$

the other is

$$\tan\frac{\alpha}{2}=\sin 22^\circ 30' \approx 0.3827$$

(hence $\alpha_1 \approx 85^\circ 28'$ and $\alpha_2 \approx 41^\circ 53'$).

Answer: the same as in the preceding problem.

At $n=4$ we have $\alpha_1=2\arctan(\cos 22^\circ 30')$ $(\approx 85^\circ 28')$
$\alpha_2=2\arctan(\sin 22^\circ 30')$ $(\approx 41^\circ 53')$

766. The area of the axial section is RH. The surface is $\pi Rl+\pi R^2$. By hypothesis, $\frac{\pi(l+R)}{H}=n$. If β is the angle between the axis and generator, then $R=l\sin\beta$ and $H=l\cos\beta$. Substituting these expressions, we get

$$\frac{1+\sin\beta}{\cos\beta}=\frac{n}{\pi}$$

This equation may be solved in several ways; the shortest one is to apply the formula $\frac{1+\cos\alpha}{\sin\alpha}=\cot\frac{\alpha}{2}$; we get

$$\frac{1+\sin\beta}{\cos\beta}=\frac{1+\cos(90^\circ-\beta)}{\sin(90^\circ-\beta)}=\cot\left(45^\circ-\frac{\beta}{2}\right)$$

Consequently,

$$\cot\left(45^\circ-\frac{\beta}{2}\right)=\frac{n}{\pi}$$

wherefrom we determine the angle $45^\circ-\frac{\beta}{2}$, and then the angle β.

But the problem is solvable not at any n. Indeed, the angle β is within the range between 0 and 90°; hence, the angle $45^\circ-\frac{\beta}{2}$ is between 0 and 45°, i.e. the quantity $\frac{n}{\pi}=\cot\left(45^\circ-\frac{\beta}{2}\right)$ must exceed unity by all means, i.e. n must be greater than π. At $n=1, 2, 3$ the problem has no solution.

Note. The equation

$$\frac{1+\sin\beta}{\cos\beta}=\frac{n}{\pi}$$

may be solved in a different way. Let us rewrite it in the form $\frac{n}{\pi}\cos\beta-1=$ $=\sin\beta$, square both members and replace $\sin^2\beta$ by $1-\cos^2\beta$. We obtain two solutions: one of them, $\cos\beta=0$, turns out to be an extraneous one $\left(\text{it is the solution of the equation } \frac{n}{\pi}\cos\beta-1=-\sin\beta\right)$; the other, $\cos\beta=\frac{2n\pi}{\pi^2+n^2}$, coincides with the preceding one.

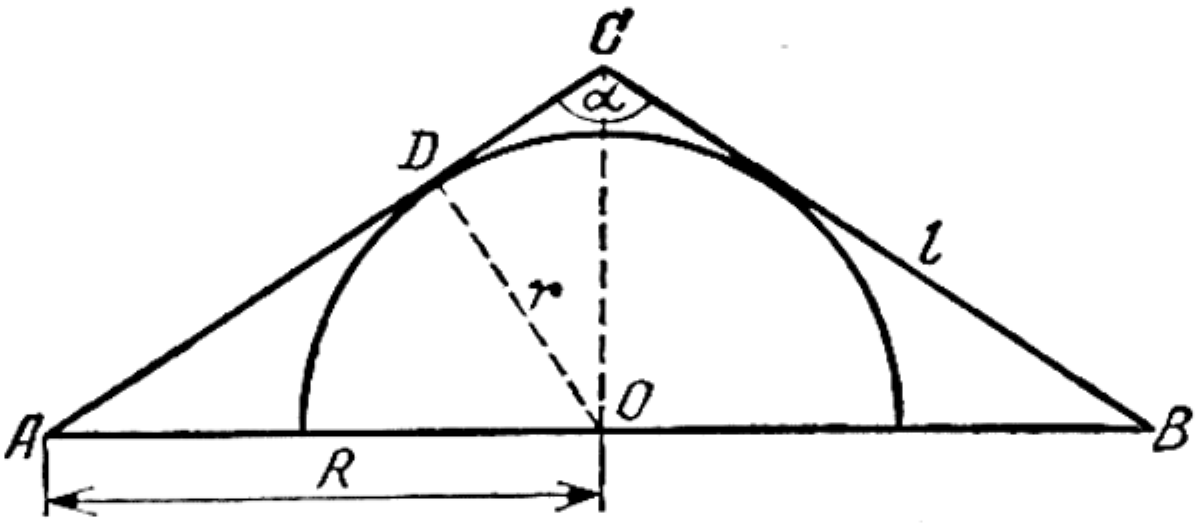


Fig. 230

But now one may easily arrive at an erroneous conclusion that the problem is solvable at $n=1, 2, 3$ as well. Indeed, at any positive value of n the quantity $\frac{2n\pi}{\pi^2+n^2}$ ranges from 0 to 1 $\left(\text{we have } 1-\frac{2n\pi}{\pi^2+n^2}=\frac{(\pi-n)^2}{\pi^2+n^2}>0\right)$. Therefore within the range between 0 and 90° one can always find an angle, whose cosine is equal to $\frac{2n\pi}{\pi^2+n^2}$.

The error of this reasoning consists in the following. From the relationship $\cos\beta=\frac{2n\pi}{\pi^2+n^2}$ and from the given equation it follows that $\sin\beta=\frac{n^2-\pi^2}{n^2+\pi^2}$, whence it is obvious that n must be greater than π (otherwise the angle β will be negative, which is impossible).

Answer: if $n<\pi$, the problem has no solution. If $n>\pi$, then

$$\beta=90^\circ-2\,\mathrm{arccot}\,\frac{n}{\pi}$$

or

$$\beta=\arccos\frac{2n\pi}{\pi^2+n^2}=\arcsin\frac{n^2-\pi^2}{n^2+\pi^2}$$

767. Using the notation adopted in Fig. 230, we have

$$\frac{R(l+R)}{2r^2}=\frac{18}{5}$$

We find (from the triangle AOD)

$$r = R\cos\angle AOD = R\cos\angle ACO = R\cos\frac{\alpha}{2}$$

and (from the triangle AOC)

$$l = \frac{R}{\sin\frac{\alpha}{2}}$$

The preceding equality takes the form

$$\frac{1+\sin\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos^2\frac{\alpha}{2}} = \frac{18}{5}, \quad \text{i.e.} \quad \frac{1+\sin\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\left(1-\sin^2\frac{\alpha}{2}\right)} = \frac{18}{5}$$

Reduce the fraction by $1+\sin\frac{\alpha}{2}$ (this quantity is non-zero). The equation is reduced to the form

$$\sin^2\frac{\alpha}{2} - \sin\frac{\alpha}{2} + \frac{5}{36} = 0$$

Answer: $\alpha_1 = 2\arcsin\frac{5}{6}$ $(\approx 112°53')$ and $\alpha_2 = 2\arcsin\frac{1}{6}$ $(\approx 19°11')$.

768. Using the notation adopted in the preceding problem, we have $\frac{\pi}{3}R^2H = \frac{4}{3}\cdot\frac{2}{3}\pi r^3$. Denote the required angle by β $\left(\text{in Fig. 230 } \beta = \frac{\alpha}{2}\right)$. Then $r = R\cos\beta$ and $H = R\cot\beta$. From the preceding relationship we get $3\cot\beta - 8\cos^3\beta = 0$. Multiplying both members of this equation by $\tan\beta$ (which, by the sense of the problem, cannot be zero), we get the equation

$$3 - 8\sin\beta\cos^2\beta = 0$$

whence

$$8\sin^3\beta - 8\sin\beta + 3 = 0$$

To solve this cubic equation we have to apply an artificial method. Thus, the left member may be factorized in the following way:

$$\begin{aligned} 8\sin^3\beta - 8\sin\beta + 3 &= (8\sin^3\beta - 1) - (8\sin\beta - 4) = \\ &= [(2\sin\beta)^3 - 1] - 4(2\sin\beta - 1) = \\ &= (2\sin\beta - 1)[(2\sin\beta)^2 + 2\sin\beta + 1 - 4] \end{aligned}$$

Consequently, the found equation decomposes into two equations. From the first one we find $\sin\beta = \frac{1}{2}$, and from the second, $\sin\beta = \frac{\sqrt{13}-1}{4}$. (The other solution of the quadratic equation is not suitable.) A check shows that both of the found solutions are suitable.

Answer: $\beta_1 = 30°$;

$$\beta_2 = \arcsin\frac{\sqrt{13}-1}{4}.$$

769. By hypothesis, the curved surface of the cone MCN (see Fig. 231) must be equal to one half of the curved surface of the cone ACB. But the curved surfaces of these cones are in the same ratio as the squares of their elements, i.e. $\frac{CN^2}{CB^2}=\frac{1}{2}$. And since

$$CN = CO,$$

we have

$$\left(\frac{CO}{CB}\right)^2=\frac{1}{2},$$

i.e.

$$\cos^2\alpha=\frac{1}{2}$$

Answer: $\alpha=45°$.

770. By hypothesis, the volume V of the spherical sector $CMKN$ (Fig. 232) must be equal to one half of the volume of the cone ACB. Let us denote the line-segment KL by h, and the altitude of the cone CO, by H. Then $V=\frac{2}{3}\pi r^2h$. We get the equality $\frac{2}{3}\pi r^2h=\frac{1}{2}\cdot\frac{1}{3}\pi R^2H$, i.e. $4r^2h=R^2H$ or $4r^2h=$ $=H^3\tan^2\alpha$. Expressing h in terms of r, we have

$$h=LK=CK-CL=r-r\cos\alpha=2r\sin^2\frac{\alpha}{2}$$

We get the equation

$$8r^3\sin^2\frac{\alpha}{2}=H^3\tan^2\alpha$$

Answer: $r=\frac{H}{2}\sqrt[3]{\frac{\tan^2\alpha}{\sin^2\frac{\alpha}{2}}}$.

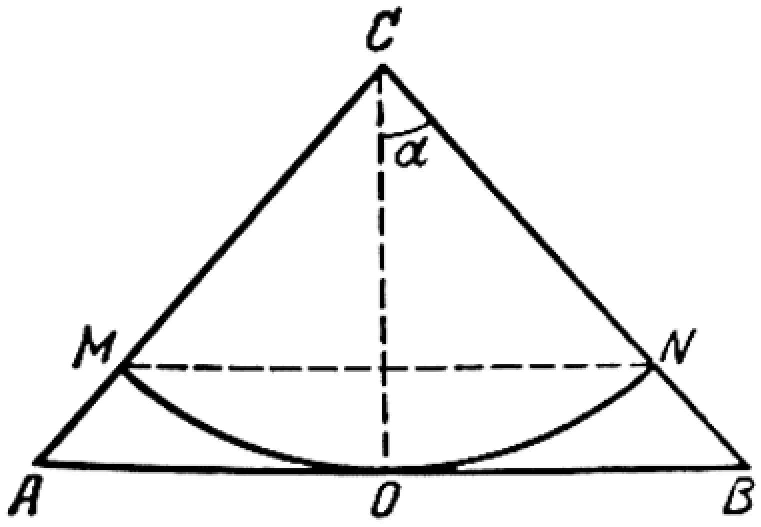


Fig. 231

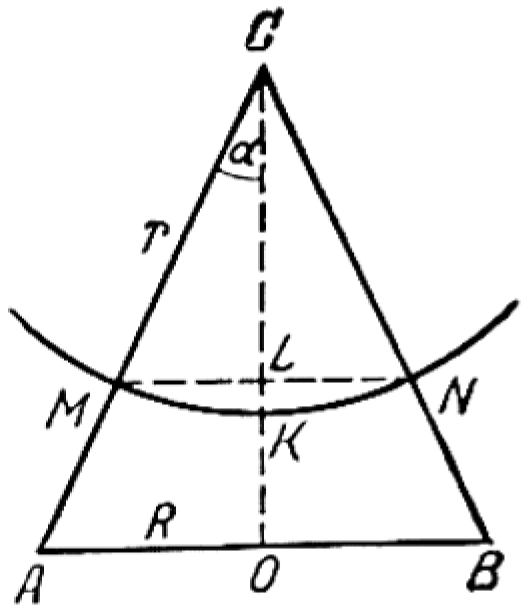


Fig. 232

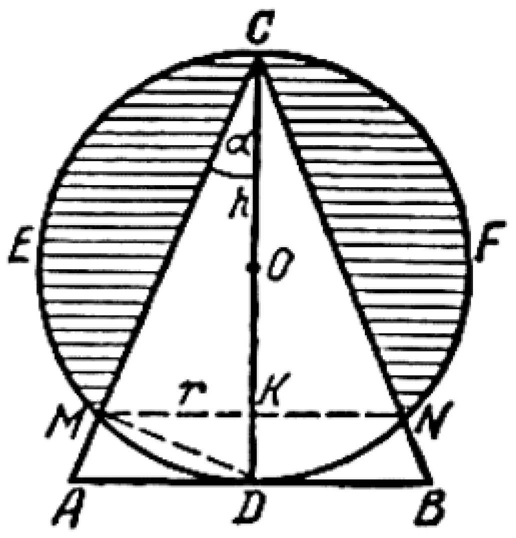


Fig. 233

771. In Fig. 233 the axial section of the portion of the sphere, whose volume must be determined, is hatched. This volume V is obtained by subtracting the volume V_1 of the cone MCN from the volume V_2 of the spherical segment

CEMKNF. Let us introduce the following notation: $MK = r$ and $KC = h$. Since the radius of the sphere is $OC = \frac{1}{2} CD = \frac{H}{2}$, we have

$$V = V_2 - V_1 = \pi h^2 \left(\frac{H}{2} - \frac{h}{3}\right) - \frac{\pi r^2 h}{3}$$

Substitute the expressions $h = MC \cdot \cos\alpha = H\cos^2\alpha$ and $r = MC \cdot \sin\alpha = H\cos\alpha\sin\alpha$ [the computation is simplified if $r^2 = MK^2$ is replaced by $CK \cdot KD = h(H-h)$]; then

$$V = \frac{\pi h^2 H}{6}$$

Answer: $V = \frac{\pi H^3 \cos^4\alpha}{6}$.

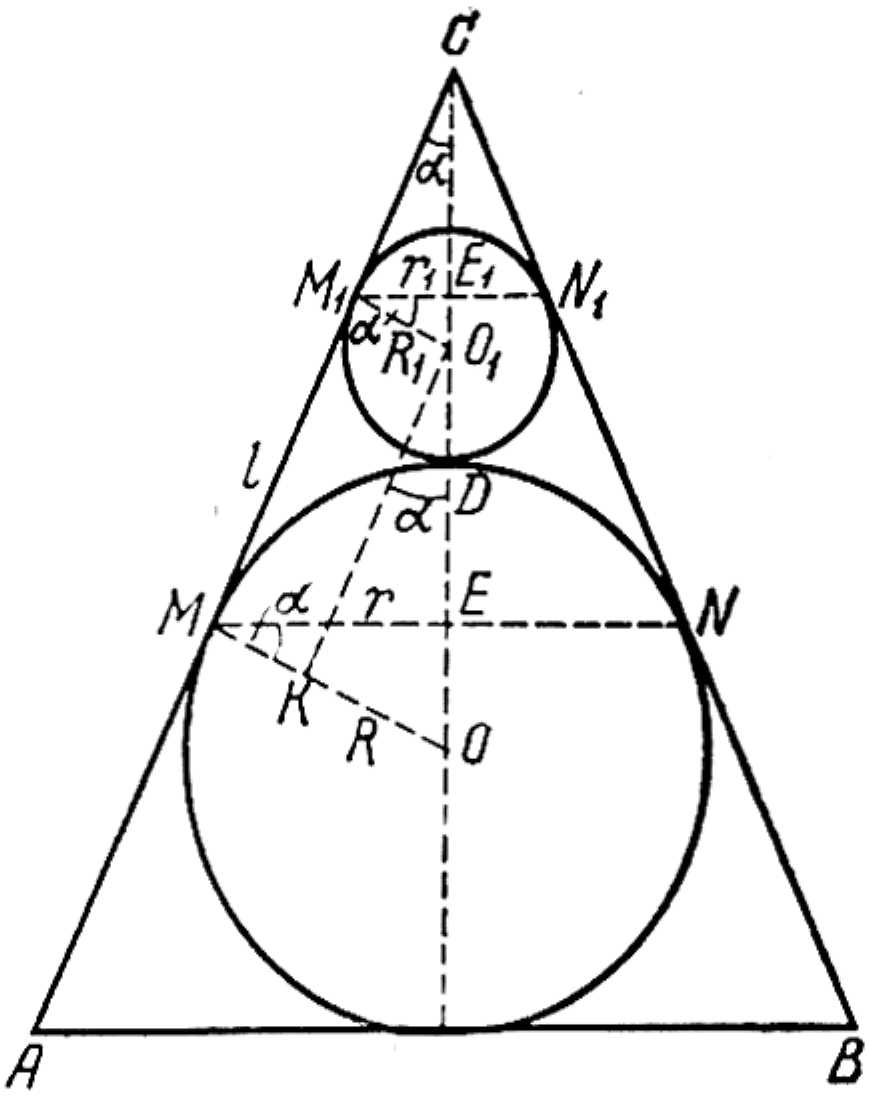


Fig. 234

772. With the notation adopted in Fig. 234 we have: $S_{curved} = \pi(r + r_1)l$. Draw radii $OM = R$ and $O_1M_1 = R_1$ to the points of tangency and the straight line O_1K perpendicular to OM. We get the triangles $O_1M_1E_1$, OME and O_1KO, which are similar to one another (as the right-angled triangles with an equal angle α). In the triangle O_1KO we have

$$O_1O = R + R_1;\ OK = R - R_1;$$
$$O_1K = MM_1 = l$$

Hence,

$$l = \sqrt{(R+R_1)^2 - (R-R_1)^2} = 2\sqrt{RR_1}$$

From the similar triangles OME and O_1KO we have $\frac{r}{l} = \frac{R}{R+R_1}$, whence

$$r = \frac{lR}{R+R_1} = \frac{2R\sqrt{RR_1}}{R+R_1}$$

From the triangles $O_1M_1E_1$ and O_1KO we have $\frac{r_1}{l} = \frac{R_1}{R+R_1}$; hence $r_1 = \frac{2R_1\sqrt{RR_1}}{R+R_1}$.

Answer: $S_{curved} = 4\pi RR_1$.

773. Four balls of the radius r lie on the plane P (Fig. 235), touching it at the points M, N, K and L. Their centres O_1, O_2, O_3, O_4 are equidistant from the plane: $O_1M = O_2N = O_3K = O_4L = r$. The distance between the centres of two contacting balls is equal to $2r$, i.e. $O_1O_2 = O_2O_3 = O_3O_4 = O_4O_1 = 2r$. The fifth ball is in contact with each of the four balls; consequently, its centre O_5 is situated also at a distance of $2r$ from the centres O_1, O_2, O_3, O_4, i.e. $O_1O_5 = O_2O_5 = O_3O_5 = O_4O_5 = 2r$. Therefore, the figure $O_5O_1O_2O_3O_4$ is a regular quadrangular pyramid with equal edges. The distance between the centre of the fifth ball and the plane P is equal to $OO_5 + OA_1 = OO_5 + r$. The topmost point A of the fifth ball is found on the extension of the perpendicular A_1O_5 at a distance of $O_5A = r$ from the centre O_5. Thus, the distance AA_1 between the topmost point of the fifth ball and the plane P is equal to $2r + OO_5$. The line-segment OO_5 is found from the right-angled triangle O_1OO_5, wherein

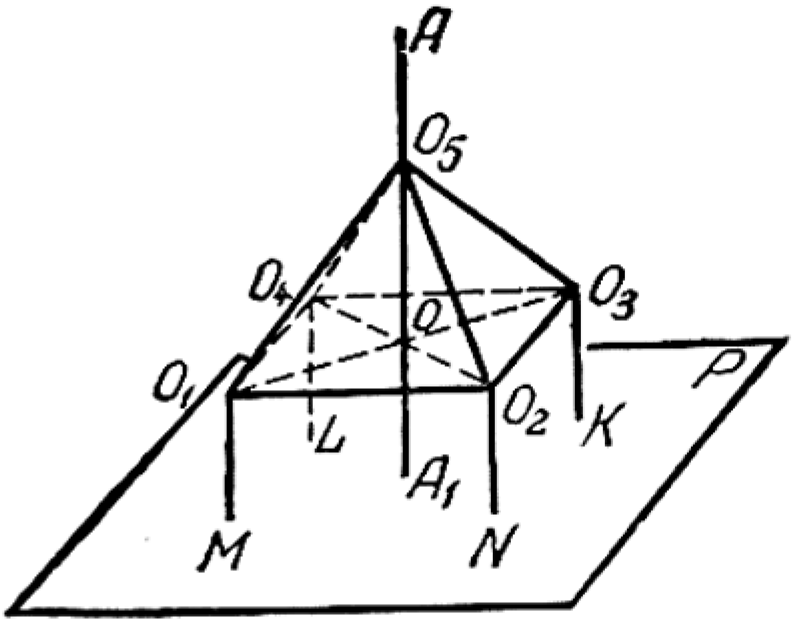


Fig. 235

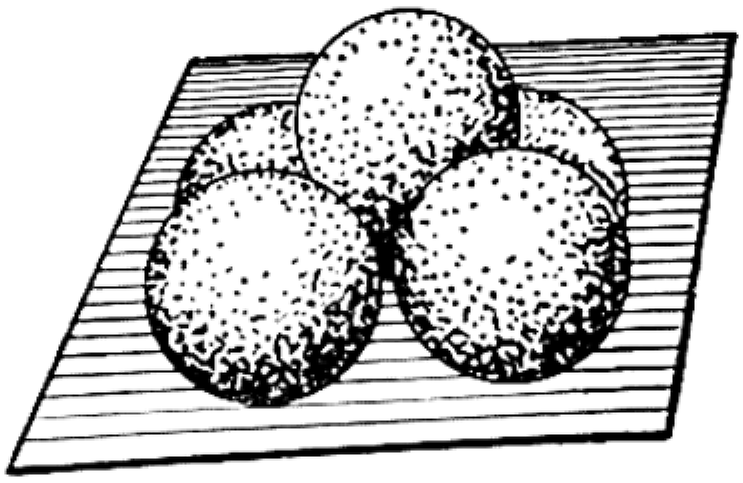


Fig. 235a

$$O_1O_5 = 2r \quad \text{and} \quad OO_1 = \frac{O_1O_2}{\sqrt{2}} = \frac{2r}{\sqrt{2}}$$

Answer: $AA_1 = r(2 + \sqrt{2})$.

774. The centres O_1, O_2, O_3, O_4 of the four balls must be at a distance of $2r$ from one another (see the preceding problem). Hence, the figure $O_1O_2O_3O_4$ is a regular tetrahedron with the edge equal to $2r$. The cone ACB (Fig. 236), circumscribed about the four balls, contacts one of them (O_4) along the circle NT and each one of the remaining three balls (for instance O_1) at two points: one of which, K, lies on the base, the other, M, on the curved surface. The axis

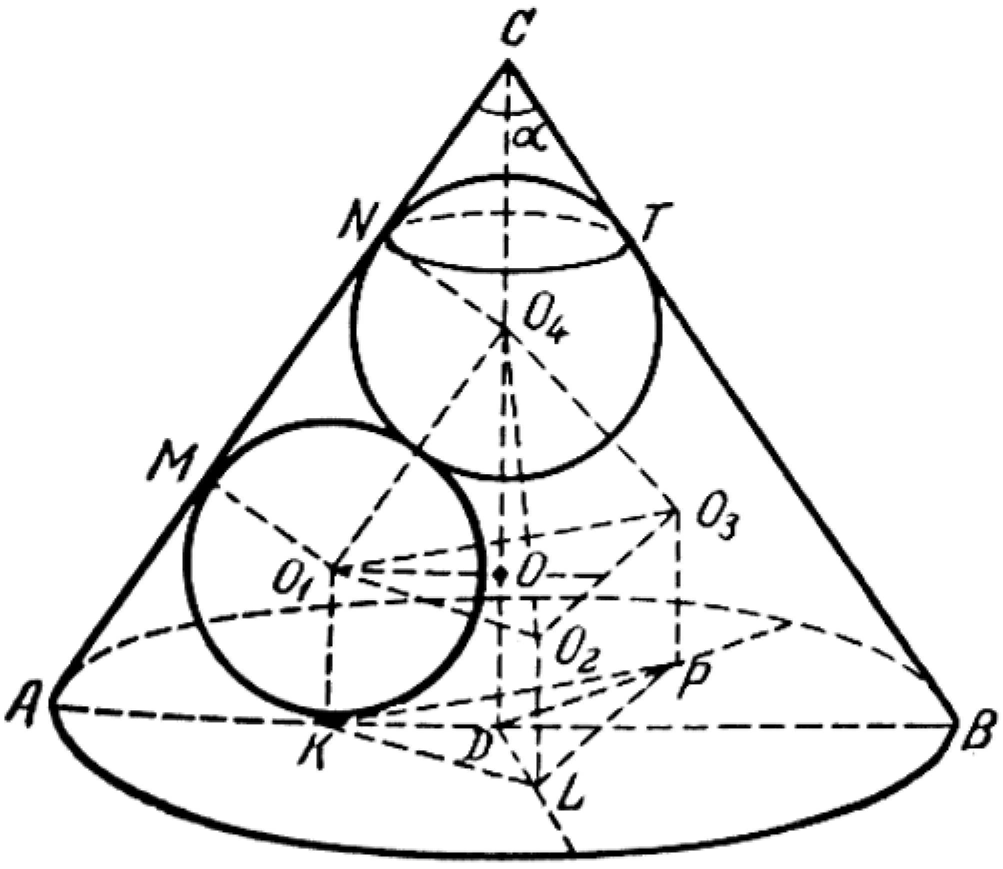


Fig. 236

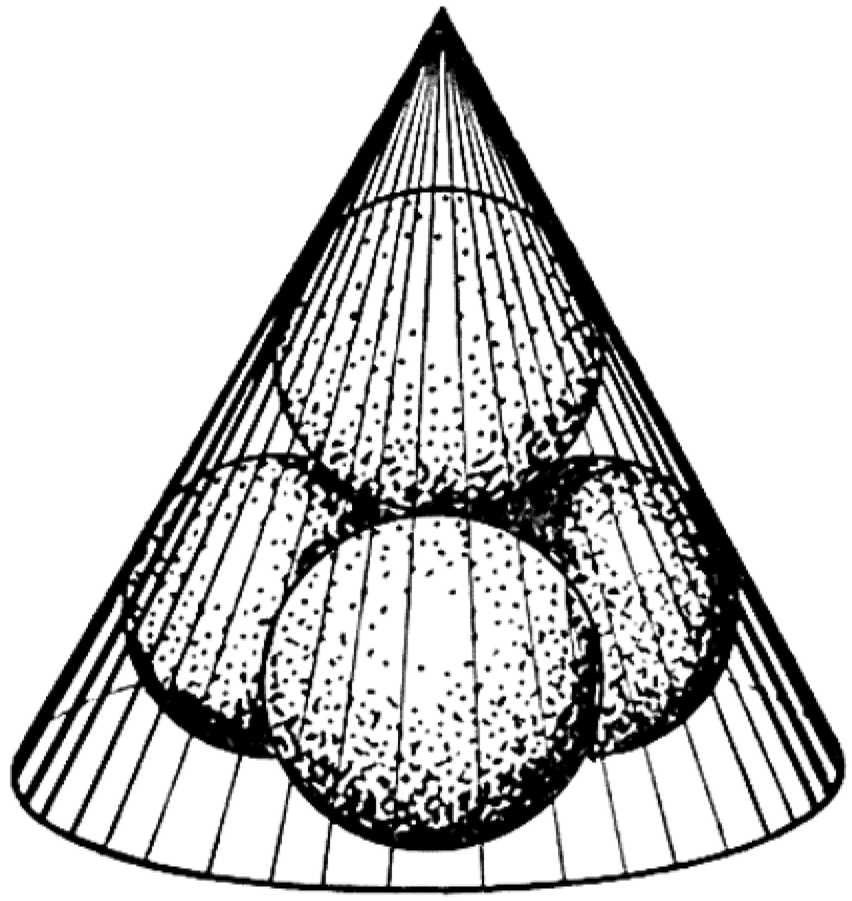


Fig. 236a

of the cone coincides with the altitude O_4O of the tetrahedron. The centre O_1 lies in the plane of the axial section ACD passing through the point of tangency M (since O_1M is perpendicular to the common tangent plane to the cone and the ball, and the plane of the axial section ACD is perpendicular to this tangent plane). Hence, the plane ACD intersects the balls O_1 and O_4 along their great circles, the element AC being the common tangent to these circles. Consequently, $AC \parallel O_1O_4$ and $\angle O_1O_4O = \angle ACD = \frac{\alpha}{2}$ (α is the required angle at the vertex C of the axial section). Hence, $\sin\frac{\alpha}{2} = \frac{OO_1}{O_1O_4}$. But $O_1O_4 = 2r$, and the line-segment OO_1 (the radius of the circle circumscribed about the triangle $O_1O_2O_3$) is equal to $\frac{O_1O_2}{\sqrt{3}} = \frac{2r}{\sqrt{3}}$. We get $\sin\frac{\alpha}{2} = \frac{1}{\sqrt{3}}$. Hence, $\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \frac{1}{3}$.

Answer: $\alpha = 2\arcsin\frac{1}{\sqrt{3}} = \arccos\frac{1}{3}$.

775. The plane bisecting the dihedral angle at the edge A_1A_2 (Fig. 237) of the frustum of a pyramid passes through the altitude O_1O_2 and is perpendicular to the face $B_1C_1C_2B_2$ (prove it!). The same thing is with the other two lateral edges. Therefore, the centre of the sphere touching the faces of the pyramid is situated on the altitude (namely, at its midpoint, since the sphere is also in contact with the bases) and the point K of tangency of the sphere and the face $B_1C_1C_2B_2$ lies on the slant height D_1D_2 of this face. The same is true for other lateral faces. We have

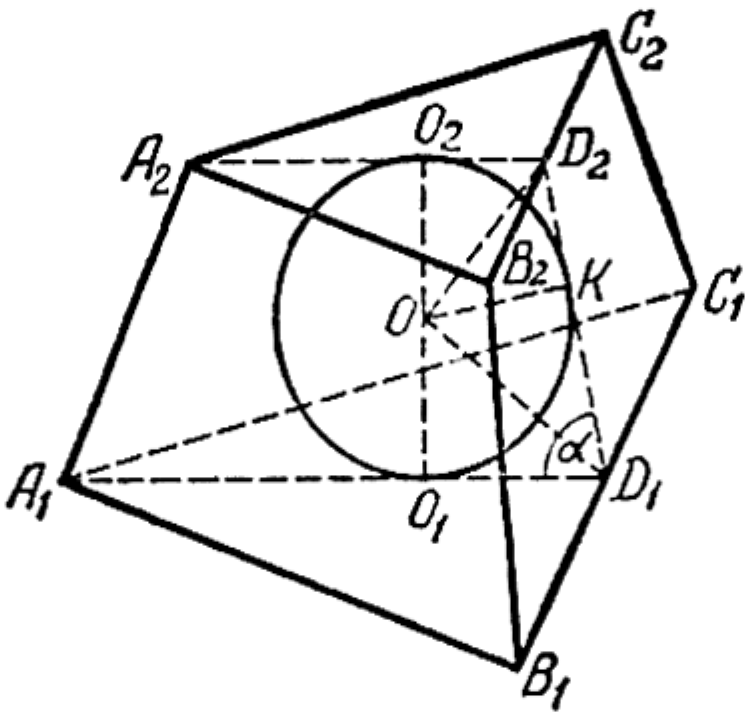


Fig. 237

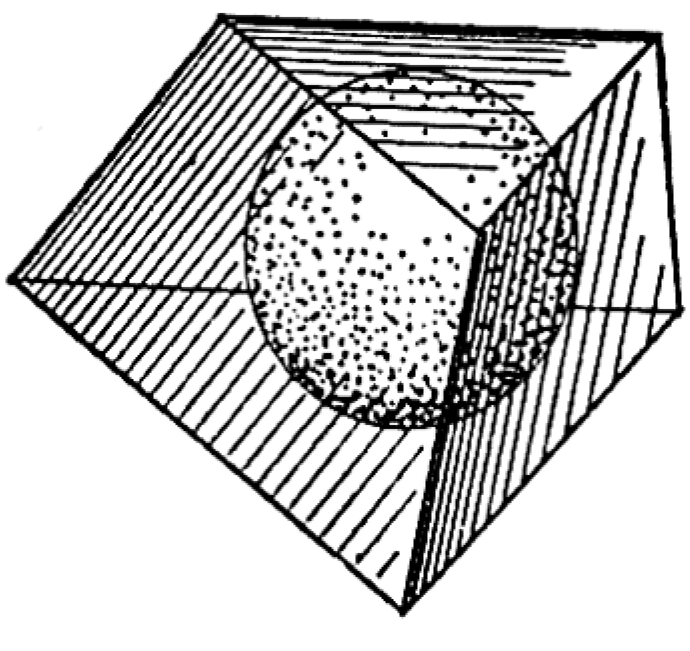


Fig. 237a

$$S_{pyr} = \frac{\sqrt{3}}{4}(a_1^2 + a_2^2) + 3\,\frac{(a_1 + a_2)\,l}{2}$$

($a_1 = B_1C_1$ and $a_2 = B_2C_2$ are the sides of the bases and $l = D_1D_2$ is the slant height of the lateral face). If $r_1 = O_1D_1$ and $r_2 = O_2D_2$ are the radii of the circles inscribed in the bases, then $a_1 = 2r_1\sqrt{3}$ and $a_2 = 2r_2\sqrt{3}$. Therefore

$$S_{pyr} = 3\sqrt{3}\,(r_1^2 + r_2^2) + 3\sqrt{3}\,(r_1 + r_2)\,l$$

In the same way as in Problem 751 we find that $r_1+r_2=l$ and $r_1^2+r_2^2=l^2-2r^2$. Then we get

$$S_{pyr}=6\sqrt{3}\,(l^2-r^2)=6\sqrt{3}\left(\frac{4r^2}{\sin^2\alpha}-r^2\right)$$

Answer: $S_{sph}:S_{pyr}=\dfrac{2\pi\sin^2\alpha}{3\sqrt{3}\,(4-\sin^2\alpha)}$.

776. Denote the radius OL of the cylinder (Fig. 238) by x, and the radius OB of the base of the cone by R. Since, by hypothesis, $ML=R$, the surface of the cylinder $S=2\pi x^2+2\pi xR$. By hypothesis, $2\pi x^2+2\pi xR=\frac{3}{2}\pi R^2$ or $x^2+Rx-\frac{3}{4}R^2=0$, whence $x=\frac{R}{2}$ (the negative solution $x=-\frac{3}{2}R$ is not suitable). From the triangle LMB we find

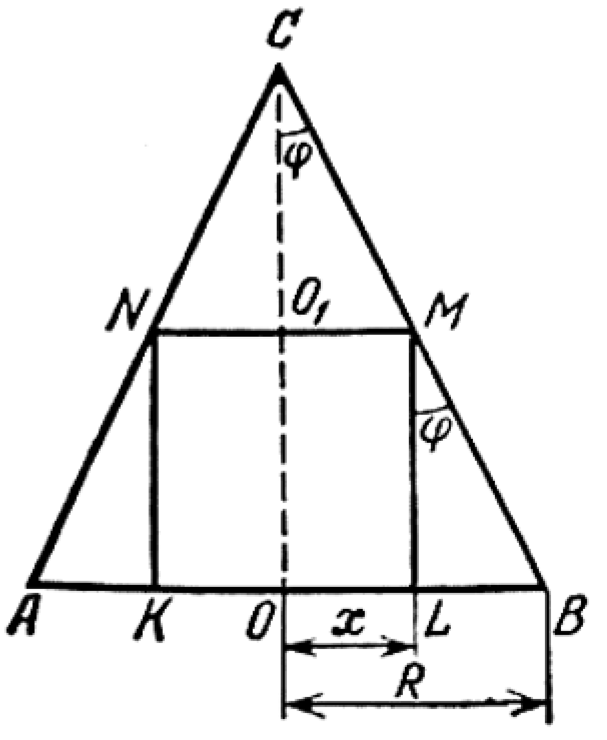


Fig. 238

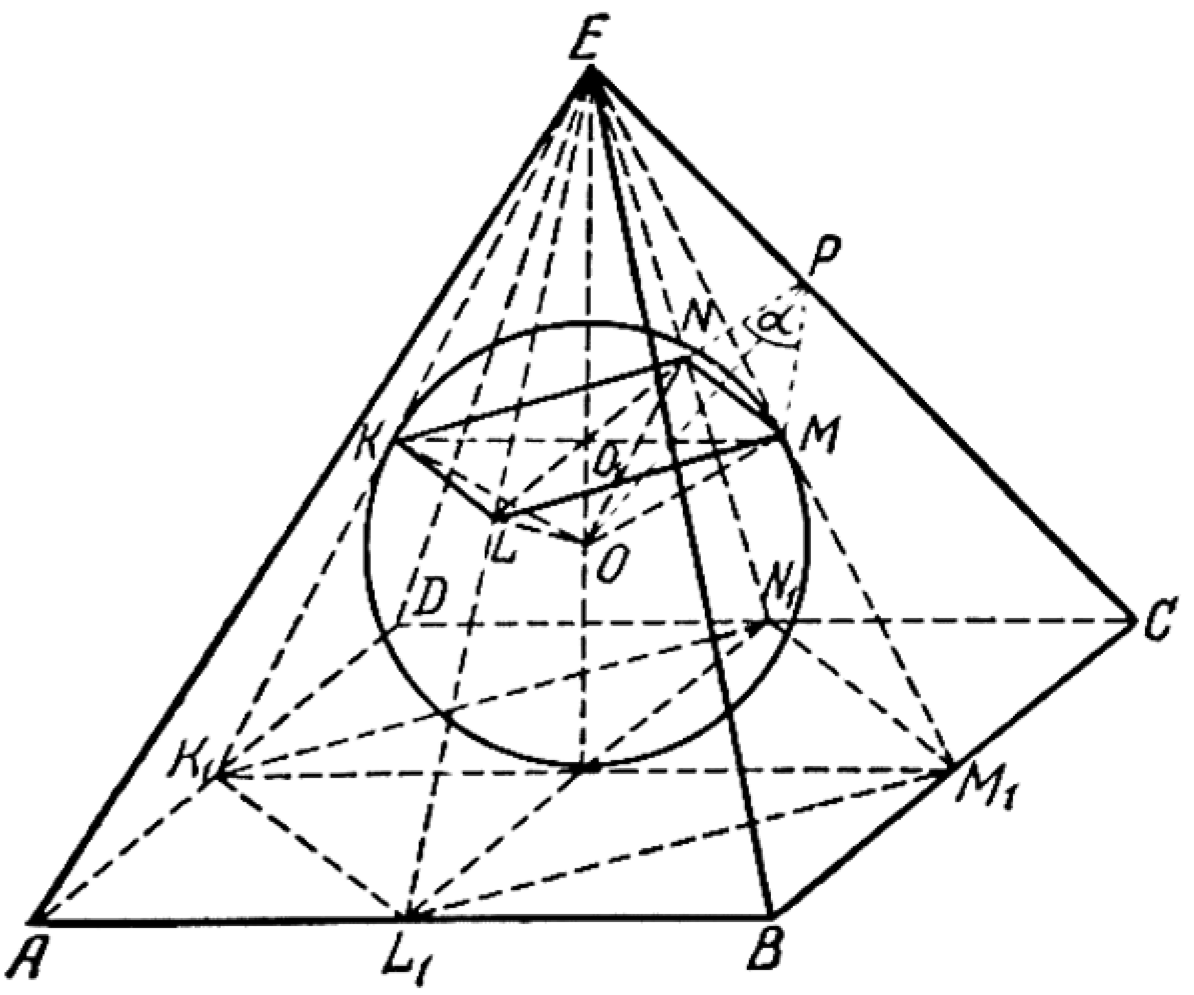


Fig. 239

$$\tan\varphi=\frac{LB}{LM}=\frac{R-x}{R}=\frac{1}{2}$$

Answer: $\varphi=\arctan\frac{1}{2}$.

777. The centre O of the inscribed sphere (Fig. 239) lies on the altitude of the pyramid and the points K, L, M, N, at which the sphere touches the lateral faces, are found on the slant heights EK_1, EL_1, EM_1, EN_1 (cf. Problem 775). The quadrilateral $KLMN$ is a square which is the base of the pyramid, whose volume is to be determined.

Through the radii OM and ON draw a plane NOM which turns out to be perpendicular to the face BEC (since it passes through the line OM perpendicular to the plane BEC) and also to the face DEC (since it passes through ON). Consequently, the plane NOM is perpendicular to the edge EC.

Let P be the point of intersection of the plane NOM and the edge EC. Then the angle NPM is a plane angle of the dihedral angle α. In the quadrilateral

$OMPN$ two angles (namely, at the vertices M and N) are right ones. Consequently, $\angle NOM = 180° - \alpha$. Hence

$$a = NM = 2 \cdot OM \cdot \sin \frac{180° - \alpha}{2} = 2r \cos \frac{\alpha}{2}$$

From the triangle OO_1M, where $O_1M = \frac{a}{2}$, we find

$$h = OO_1 = \sqrt{OM^2 - O_1M^2} = r\sqrt{1 - 2\cos^2\frac{\alpha}{2}}$$

Answer: $V = \frac{4}{3} r^3 \cos^2 \frac{\alpha}{2} \sqrt{1 - 2\cos^2 \frac{\alpha}{2}} = \frac{4}{3} r^3 \cos^2 \frac{\alpha}{2} \sqrt{-\cos \alpha}$.

778. We can draw two planes perpendicular to the given element of the cone (CA in Fig. 240) and tangent to the inscribed sphere, the points of tangency (N and N_1) lying on the diameter NN_1 parallel to CA. Let us first take the plane ND touching the sphere at the point N. The quadrilateral $ONDK$ (K is the point of tangency of the element CA and the sphere) is a square, hence $DK = ON = r$. By hypothesis $CD = d$. Consequently, $CK = d + r$. From the triangle KOC we find

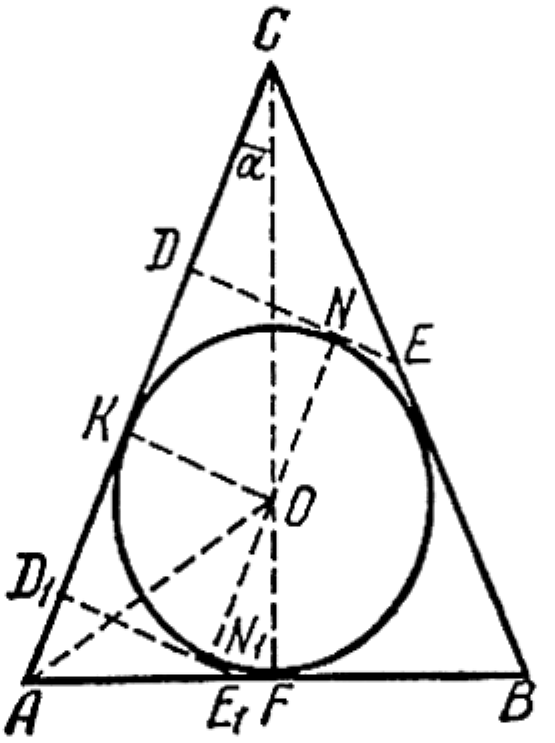


Fig. 240

$$CO = \sqrt{(d+r)^2 + r^2}$$

Hence,

$$H = CF = OF + OC = r + \sqrt{(d+r)^2 + r^2}$$

From similarity of the triangles AFC and KOC we find

$$AF : H = OK : KC$$

whence

$$R = AF = \frac{OK \cdot H}{KC} = \frac{r\left[r + \sqrt{(d+r)^2 + r^2}\right]}{d+r}$$

If we take the plane N_1D_1, then $d = CD_1$, and we get in the same way

$$H = r + \sqrt{(d-r)^2 + r^2}$$

$$\text{and } R = \frac{r\left[r + \sqrt{(d-r)^2 + r^2}\right]}{d-r}$$

Answer:

$$V = \frac{\pi r^2 \left[r + \sqrt{(d+r)^2 + r^2}\right]^3}{3(d+r)^2} \quad \text{or} \quad V = \frac{\pi r^2 \left[r + \sqrt{(d-r)^2 + r^2}\right]^3}{3(d-r)^2}.$$

779. The centre O of the sphere (Fig. 241) lies on the diagonal AB. Indeed, the point O is equidistant from the faces AA_1N_1N and AA_1Q_1Q. Hence, it lies on the plane bisecting the dihedral angle at the edge AA_1. Similarly, the point O must lie on the plane bisecting the dihedral angle at the edge AN. And the two planes intersect along the diagonal AB.

Let C and D be the points of tangency of the sphere and the faces $ANUQ$ and AA_1N_1N, and r the radius of the sphere. Then $OC = OD = r$, and the plane $ODGC$ is perpendicular to the edge AN, and also to the edge BQ_1.

Since, by hypothesis, the edge BQ_1 is tangent to the sphere, the plane $ODGC$ intersects the edge at the point E of its tangency with the sphere; consequently, $OE = r$. On the other hand, the point E is a vertex of the square $FGKE$ obtained in the section of the cube by the plane $ODGC$; hence the quadrilateral $MOLE$ (OL and OM are the extensions of OC and OD) is a square. Consequently, $OM = \frac{r}{\sqrt{2}}$. Since $OM + OD = MD = a$, $\frac{r}{\sqrt{2}} + r = a$; whence $r = (2 - \sqrt{2})\, a$.

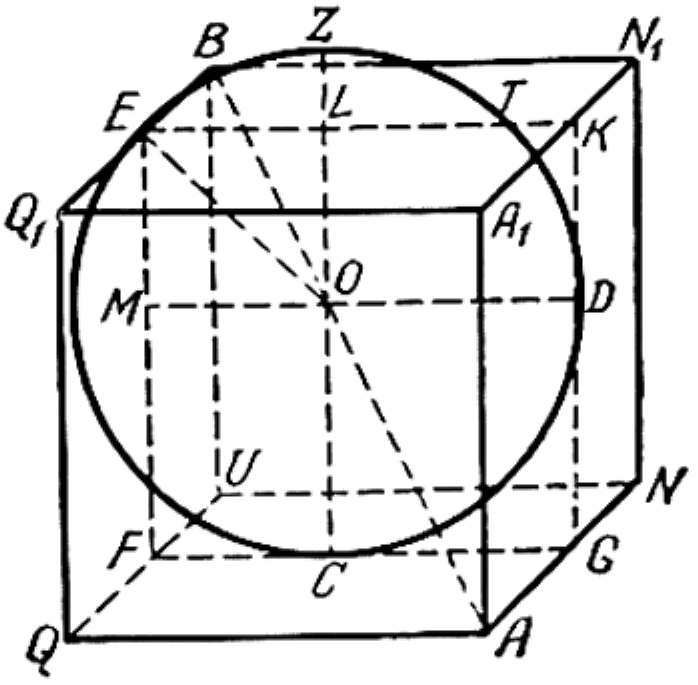


Fig. 241

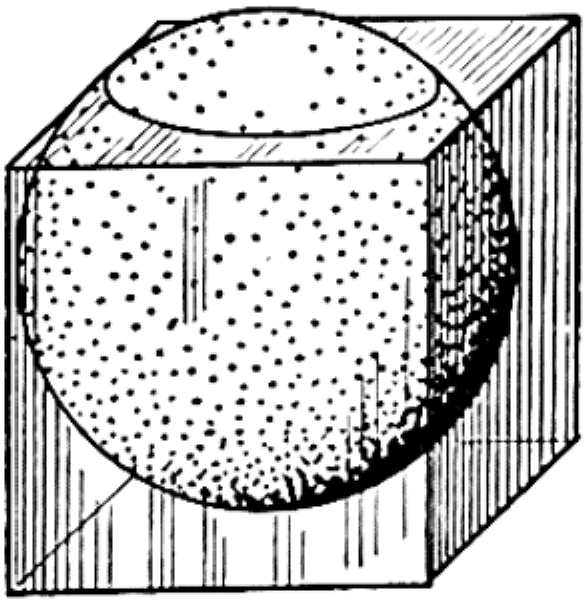


Fig. 241a

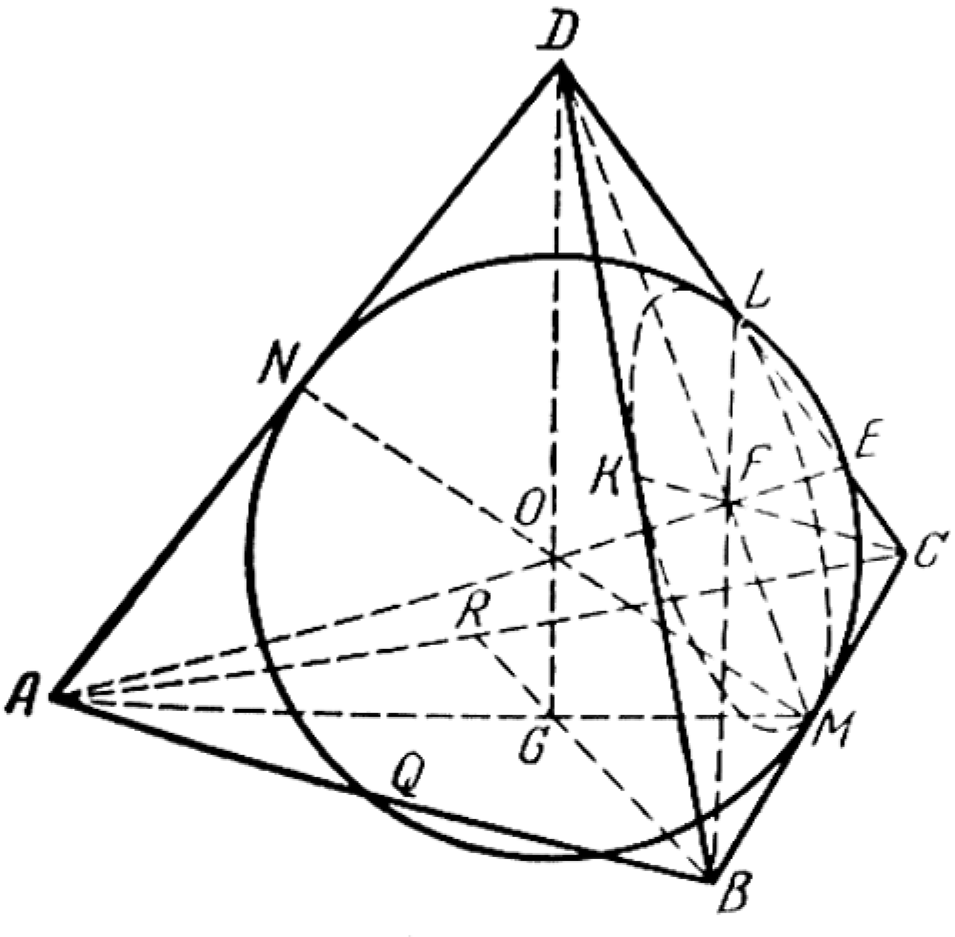


Fig. 242

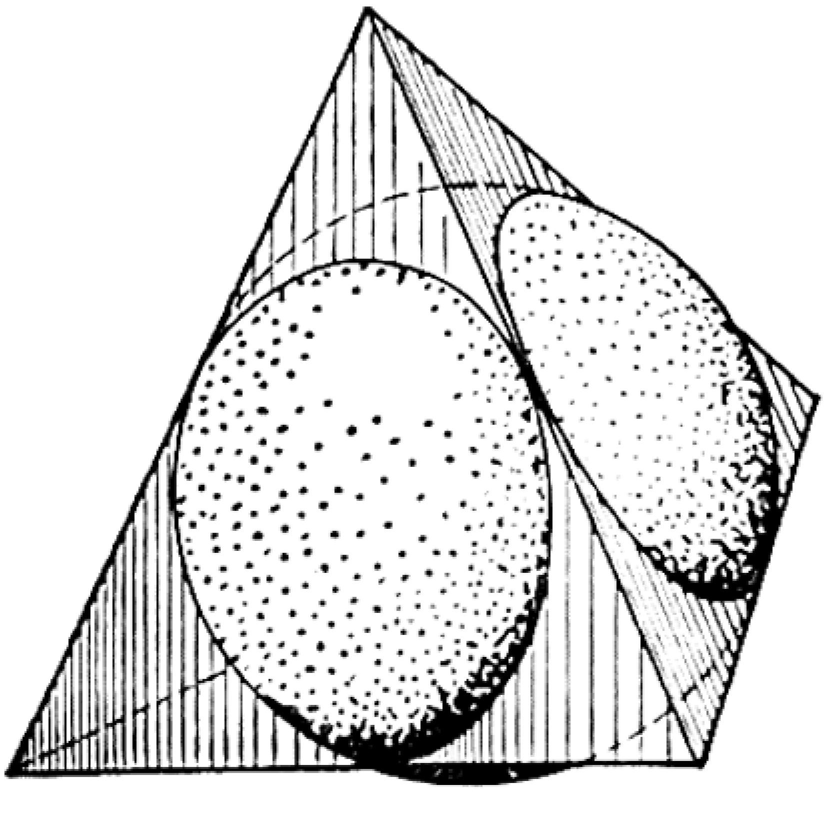


Fig. 242a

The portion of the surface area of the sphere found outside the cube is made up of three equal segments, $EZTL$ being one of them. The area of this segment is equal to

$$2\pi r \cdot LZ = 2\pi r\,(CZ - CL) = 2\pi r\,(2r - a)$$

Answer: $r = (2 - \sqrt{2})\, a$; $S = 6\pi a^2 (10 - 7\sqrt{2})$.

780. The centre of the sphere contacting the edges of the tetrahedron $ABCD$ (Fig. 242) coincides with the centre of the tetrahedron (i.e. with the point O

which is equidistant from the vertices A, B, C, D), and the points of tangency of the sphere and edges are the midpoints of the edges. For instance, the point of tangency N is the midpoint of the edge AD. Indeed, all six isosceles triangles AOB, BOC, COA, BOD, COD and AOD (only the triangle AOD is shown in the accompanying drawing) are congruent (having three equal sides). Consequently, their altitudes OM, ON, etc. are equal. Therefore, if a sphere of radius $ON = r$ is described, it passes through the midpoints L, M, N, Q, K, R of the edges and is tangent to them at these points (since $ON \perp AD$, and so on).

Through the altitude of the tetrahedron DG and edge AD draw a plane ADG which is perpendicular to the edge BC (the proof is given in Problem 652) and intersects this edge at its midpoint M. The section yields an isosceles triangle AMD ($AM = MD$). Draw the altitude MN of this triangle (N is the midpoint of AD). The centre O lies on MN (since it is equidistant from A and D). Consequently, $MO = NO$. Hence, $r = \frac{MN}{2}$. The altitude MN is determined from the triangle ANM, where $AN = \frac{a}{2}$ and $AM = \frac{a\sqrt{3}}{2}$ (as the apothem of an equilateral triangle ABC). We have

$$MN = \sqrt{\left(\frac{a\sqrt{3}}{2}\right)^2 - \left(\frac{a}{2}\right)^2} = \frac{a}{\sqrt{2}}$$

Hence,

$$r = \frac{MN}{2} = \frac{a}{2\sqrt{2}} = \frac{a\sqrt{2}}{4}$$

The portion of the sphere situated outside tetrahedron is made up of four equal segments cut off the sphere by the faces of the tetrahedron. Consider one of the faces—BDC. The circle LMK serving as the base of the spherical segment is inscribed in the equilateral triangle BDC (since the sides of the triangle are tangent to the sphere; hence, they are also tangent to the small circle LMK contained in the plane BDC). The radius of this circle $FM = \frac{a\sqrt{3}}{6}$.

Consequently,

$$OF = \sqrt{OM^2 - FM^2} = \sqrt{r^2 - FM^2} = \sqrt{\left(\frac{a\sqrt{2}}{4}\right)^2 - \left(\frac{a\sqrt{3}}{6}\right)^2} = \frac{a}{2\sqrt{6}}$$

Hence, the altitude of the segment

$$h = FE = OE - OF = \frac{a}{2\sqrt{2}} - \frac{a}{2\sqrt{6}} = \frac{a\sqrt{2}}{12}(3 - \sqrt{3})$$

The volume of one segment

$$V_{segm} = \pi h^2 \left(r - \frac{h}{3}\right) =$$

$$= \pi \left[\frac{a\sqrt{2}}{12}(3 - \sqrt{3})\right]^2 \cdot \left[\frac{a\sqrt{2}}{4} - \frac{a\sqrt{2}}{36}(3 - \sqrt{3})\right] = \frac{\pi a^3 \sqrt{2}\,(9 - 4\sqrt{3})}{432}$$

The required volume

$$V = 4V_{segm}$$

Note. The circle LKM inscribed in the triangle BCD is depicted as an ellipse, which is readily constructed without any French curve, if in addition to the points K, L, M, another three points are marked respectively symmetrical to them about F which is the point of intersection of the medians in the triangle BDC.

Answer: $r=\frac{a\sqrt{2}}{4}$; $V=\frac{\pi a^3\sqrt{2}\,(9-4\sqrt{3})}{108}$.

CHAPTER XI

TRIGONOMETRIC TRANSFORMATIONS

781. Express secants in terms of cosines to get in the left member

$$\frac{1}{\cos\left(\frac{\pi}{4}+\alpha\right)\cos\left(\frac{\pi}{4}-\alpha\right)}$$

Since

$$\cos\left(\frac{\pi}{4}+\alpha\right)=\sin\left[\frac{\pi}{2}-\left(\frac{\pi}{4}+\alpha\right)\right]=\sin\left(\frac{\pi}{4}-\alpha\right),$$

the left member is equal to

$$\frac{2}{2\sin\left(\frac{\pi}{4}-\alpha\right)\cos\left(\frac{\pi}{4}-\alpha\right)}=\frac{2}{\sin\left(\frac{\pi}{2}-2\alpha\right)}=\frac{2}{\cos 2\alpha}=2\sec 2\alpha$$

782. Reduce the left member to a common denominator and bring $2\sin\alpha\cos(\alpha+\beta)$ to the form

$$\sin[\alpha+(\alpha+\beta)]+\sin[\alpha-(\alpha+\beta)]=\sin(2\alpha+\beta)+\sin(-\beta)$$

783. The left member is equal to

$$\frac{2(1+\cos 2\alpha)}{\sin 2\alpha}=\frac{2\cdot 2\cos^2\alpha}{2\sin\alpha\cos\alpha}=2\cot\alpha$$

To pass over to the angle $\frac{\alpha}{2}$, use the formula $\cot 2\varphi=\frac{\cot^2\varphi-1}{2\cot\varphi}$ (the angle $\frac{\alpha}{2}$ is denoted by φ). We obtain

$$2\cot\alpha=2\,\frac{\cot^2\frac{\alpha}{2}-1}{2\cot\frac{\alpha}{2}}=\cot\frac{\alpha}{2}-\tan\frac{\alpha}{2}$$

784. Dividing both the numerator and denominator of the fraction in the left member of the equality by $\cos\alpha$, we get

$$\frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}=\frac{1+\tan\alpha}{1-\tan\alpha}$$

Since $1=\tan 45^\circ$, let us represent the obtained expression in the form

$$\frac{1+\tan\alpha}{1-\tan\alpha}=\frac{\tan 45^\circ+\tan\alpha}{1-\tan\alpha\cdot\tan 45^\circ}=\tan(45^\circ+\alpha)$$

which completes the proof.

785. Multiply both the numerator and denominator of the left member by $\cos\alpha+\sin\alpha$. After simplifications we get $\frac{1+\sin 2\alpha}{\cos 2\alpha}$ or

$$\frac{1}{\cos 2\alpha}+\frac{\sin 2\alpha}{\cos 2\alpha}=\sec 2\alpha+\tan 2\alpha$$

786. Since $\sin^2\varphi=\frac{1-\cos 2\varphi}{2}$, let us represent the left member as

$$\frac{1-\cos\left(\frac{\pi}{4}+2\alpha\right)-1+\cos\left(\frac{\pi}{4}-2\alpha\right)}{2}=\frac{\cos\left(\frac{\pi}{4}-2\alpha\right)-\cos\left(\frac{\pi}{4}+2\alpha\right)}{2}$$

Using the formula for a difference of cosines (or representing the expressions $\cos\left(\frac{\pi}{4}-2\alpha\right)$ and $\cos\left(\frac{\pi}{4}+2\alpha\right)$ by the formulas for the cosine of the sum and the difference, we get

$$\frac{2\sin\frac{\pi}{4}\sin 2\alpha}{2}=\frac{\sin 2\alpha}{\sqrt{2}}$$

787. The numerator is equal to $\cos 2\alpha$; the denominator is transformed to the form

$$2\tan\left(\frac{\pi}{4}-\alpha\right)\sin^2\left[\frac{\pi}{2}-\left(\frac{\pi}{4}-\alpha\right)\right]=2\tan\left(\frac{\pi}{4}-\alpha\right)\cos^2\left(\frac{\pi}{4}-\alpha\right)$$

With the aid of the formula

$$\tan\left(\frac{\pi}{4}-\alpha\right)=\frac{\sin\left(\frac{\pi}{4}-\alpha\right)}{\cos\left(\frac{\pi}{4}-\alpha\right)}$$

we get

$$2\sin\left(\frac{\pi}{4}-\alpha\right)\cos\left(\frac{\pi}{4}-\alpha\right)=\sin\left(\frac{\pi}{2}-2\alpha\right)$$

this expression is equal to $\cos 2\alpha$, hence the left member is equal to 1.

788. We have

$$\tan^2\left(\frac{\pi}{4}-\alpha\right)=\frac{\sin^2\left(\frac{\pi}{4}-\alpha\right)}{\cos^2\left(\frac{\pi}{4}-\alpha\right)}$$

Considering the angle $\frac{\pi}{4}-\alpha$ as half the angle $\frac{\pi}{2}-2\alpha$ and using the half-angle formulas, we obtain

$$\tan^2\left(\frac{\pi}{4}-\alpha\right)=\frac{1-\cos\left(\frac{\pi}{2}-2\alpha\right)}{1+\cos\left(\frac{\pi}{2}-2\alpha\right)}=\frac{1-\sin 2\alpha}{1+\sin 2\alpha}$$

789. Expressing tangent and cotangent in terms of sines and cosines, we get

$$\cot^2\alpha-\tan^2\alpha=\frac{\cos 2\alpha}{\sin^2\alpha\cos^2\alpha}$$

Substitute the obtained expression into the denominator of the left member, then the left member yields

$$\sin^2\alpha\cos^2\alpha=\frac{4\sin^2\alpha\cos^2\alpha}{4}=\frac{1}{4}\sin^2 2\alpha$$

790. Replace $\sin\alpha$ by $\cos\left(\frac{\pi}{2}-\alpha\right)$ and $\cos\alpha$ by $\sin\left(\frac{\pi}{2}-\alpha\right)$, and use the formulas for a sum of cosines and a difference of sines.

791. Replace (in the numerator) unity by $\sin^2\alpha+\cos^2\alpha$, and $\sin 2\alpha$ by $2\sin\alpha\cos\alpha$. We get in the numerator $(\sin\alpha+\cos\alpha)^2$; the denominator being equal to

$$\cos^2\alpha-\sin^2\alpha=(\cos\alpha+\sin\alpha)(\cos\alpha-\sin\alpha)$$

Reducing the fraction by $\cos\alpha+\sin\alpha$, we obtain $\frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}$. Dividing both the numerator and denominator by $\cos\alpha$, we find $\frac{1+\tan\alpha}{1-\tan\alpha}$. As is shown in Problem 784, this expression is transformed to $\tan\left(\frac{\pi}{4}+\alpha\right)$.

792. In the same way as in Problem 790, transform the left member to the form $\cot\left(\frac{\pi}{4}-y\right)$. Now apply the formula $\cot\frac{\alpha}{2}=\frac{1+\cos\alpha}{\sin\alpha}$ $\left(\text{putting } \frac{\pi}{4}-y=\frac{\alpha}{2}\right)$. We obtain

$$\frac{1+\cos\left(\frac{\pi}{2}-2y\right)}{\sin\left(\frac{\pi}{2}-2y\right)}=\frac{1+\sin 2y}{\cos 2y}$$

793. Expressing the left member of the given identity in terms of sine and cosine, performing subtraction of the obtained fractions and using the formula for a difference of squares, we get the left member in the form

$$\frac{(\sin\alpha\cos\beta-\sin\beta\cos\alpha)(\sin\alpha\cos\beta+\sin\beta\cos\alpha)}{\cos^2\alpha\cos^2\beta}$$

and this expression yields immediately the right-hand member.

794. Use the formula

$$\tan\frac{\varphi}{2}=\frac{\sin\varphi}{1+\cos\varphi}$$

$\left(\text{putting } \frac{\pi}{4}-\frac{\alpha}{2}=\frac{\varphi}{2}\right)$. We get

$$\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{1+\cos\left(\frac{\pi}{2}-\alpha\right)}=\frac{\cos\alpha}{1+\sin\alpha}$$

and then the left member is transformed into the right one.

795. Solved in the same way as the preceding problem.

796. Replace $2\cos^2\alpha$ by $1+\cos 2\alpha$; then the numerator takes the form: $2(\sin 2\alpha+\cos 2\alpha)$. Group the terms of the denominator in the following way: $(\cos\alpha-\cos 3\alpha)+(\sin 3\alpha-\sin\alpha)$ and use the formula for a difference of cosines and sines. Taking $2\sin\alpha$ outside the brackets, we obtain $2\sin\alpha(\sin 2\alpha+\cos 2\alpha)$. On reducing by $2(\sin 2\alpha+\cos 2\alpha)$ we get the right member.

797. Transform the numerator of the fraction in the left member of the identity:

$$\sin\alpha+\sin 5\alpha-\sin 3\alpha=2\sin 3\alpha\cos 2\alpha-\sin 3\alpha=\sin 3\alpha(2\cos 2\alpha-1)$$

Carrying out similar transformations in the denominator we get $\cos 3\alpha(2\cos 2\alpha-1)$.

798. Transform the sum of the first two terms in the left member of the identity using the formula for a sum of sines, and consider the third addend $\sin(b-c)$ as a double-angle sine. We get

$$2\sin\frac{2a-b-c}{2}\cos\frac{b-c}{2}+2\sin\frac{b-c}{2}\cos\frac{b-c}{2}=$$

$$=2\cos\frac{b-c}{2}\left[\sin\frac{2a-b-c}{2}+\sin\frac{b-c}{2}\right]$$

Now apply the formula for a sum of sines to the bracketed expression.

799. Considering the expression $\sin^6 x+\cos^6 x$ as a sum of cubes, factorize it and take into account that $\sin^2 x+\cos^2 x=1$. Then the left member of the equality is brought to the form

$$-\sin^4 x-2\sin^2 x\cos^2 x-\cos^4 x+1=1-(\sin^2 x+\cos^2 x)^2=0$$

800. Transform the sum of the last two terms as a sum of sines. We get

$$\sin\left(\alpha+\frac{2\pi}{3}\right)+\sin\left(\alpha+\frac{4\pi}{3}\right)=2\sin(\pi+\alpha)\cos\frac{\pi}{3}=-2\cdot\frac{1}{2}\sin\alpha=-\sin\alpha$$

Hence, the left member is equal to zero.

801. Taking into account that

$$\sin^2\alpha-\sin^2\beta=\frac{1-\cos 2\alpha}{2}-\frac{1-\cos 2\beta}{2}=$$

$$=\frac{\cos 2\beta-\cos 2\alpha}{2}=\sin(\alpha+\beta)\sin(\alpha-\beta)$$

the left member of the equation can be represented in the form

$$\sin(45^\circ + \alpha + 30^\circ - \alpha)\sin(45^\circ + \alpha - 30^\circ + \alpha) - \sin 15^\circ \cos(15^\circ + 2\alpha)$$

and since $\sin 75^\circ = \cos 15^\circ$, the expression takes the form

$$\cos 15^\circ \sin(15^\circ + 2\alpha) - \sin 15^\circ \cos(15^\circ + 2\alpha) = \\ = \sin(15^\circ + 2\alpha - 15^\circ) = \sin 2\alpha$$

which completes the proof.

802. Transform the numerator of the left member as follows

$$(\sin^2 \varphi + \cos^2 \varphi) - 2\cos^2 \varphi = \sin^2 \varphi - \cos^2 \varphi$$

803. Replace $\sin 2\alpha$ by $2 \sin\alpha \cos\alpha$ in the right member. Reducing the fraction by $2 \sin \alpha$, we get in the right member the expression $\frac{1-\cos\alpha}{1+\cos\alpha}$ equal to $\tan^2 \frac{\alpha}{2}$.

804. Grouping the second and third terms, take outside the brackets $\cos(\alpha + \varphi) = \cos\alpha\cos\varphi - \sin\alpha\sin\varphi$. The left member takes the form

$$\cos^2 \varphi - (\cos\alpha\cos\varphi - \sin\alpha\sin\varphi)(\cos\alpha\cos\varphi + \sin\alpha\sin\varphi)$$

Transforming the product of sum by difference, we find:

$$\cos^2 \varphi - \cos^2 \alpha \cos^2 \varphi + \sin^2 \alpha \sin^2 \varphi = \\ = \cos^2 \varphi (1 - \cos^2 \alpha) + \sin^2 \alpha \sin^2 \varphi = \\ = \cos^2 \varphi \sin^2 \alpha + \sin^2 \alpha \sin^2 \varphi,$$

and this expression yields $\sin^2 \alpha$.

805. Expanding the expression $\cos(\alpha + \beta)$, we get:

$$\sin^2 \alpha + \sin^2 \beta + 2 \sin\alpha \sin\beta \cos\alpha \cos\beta - 2\sin^2 \alpha \sin^2 \beta$$

Leave the third term unchanged, group all the rest of the terms and carry out the following transformations:

$$(\sin^2 \alpha - \sin^2 \alpha \sin^2 \beta) + (\sin^2 \beta - \sin^2 \alpha \sin^2 \beta) = \\ = \sin^2 \alpha \cos^2 \beta + \cos^2 \alpha \sin^2 \beta$$

Now the given expression takes the form

$$(\sin\alpha\cos\beta)^2 + (\cos\alpha\sin\beta)^2 + 2\sin\alpha\sin\beta\cos\alpha\cos\beta = \\ = (\sin\alpha\cos\beta + \cos\alpha\sin\beta)^2 = \sin^2(\alpha + \beta)$$

Answer: $\sin^2(\alpha + \beta)$.

806. Transform the sum of the first three terms in the following way:

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = \frac{1-\cos 2\alpha}{2} + \frac{1-\cos 2\beta}{2} + \sin^2 \gamma$$

Since by hypothesis $\gamma = \pi - (\alpha + \beta)$, we have

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 1 - \frac{1}{2}(\cos 2\alpha + \cos 2\beta) + \sin^2(\alpha + \beta) = \\ = 1 - \cos(\alpha + \beta)\cos(\alpha - \beta) + [1 - \cos^2(\alpha + \beta)]$$

or

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2-\cos(\alpha+\beta)\,[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

But the expression in square brackets is equal to $2\cos\alpha\cos\beta$, and since $\alpha+\beta=\pi-\gamma$, we obtain

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2+2\cos\gamma\cos\alpha\cos\beta$$

whence the required relation follows immediately.

807. Represent the left member in the form

$$\cot A\cot B+(\cot A+\cot B)\cot C$$

The expression in parentheses is equal to $\frac{\sin(A+B)}{\sin A\sin B}$, and the factor $\cot C$, on replacing C by an equal expression $\pi-(A+B)$, takes the form $-\cot(A+B)$. Hence, the given expression is equal to

$$\cot A\cot B-\frac{\cos(A+B)}{\sin A\sin B}$$

Using the formula for cosine of a sum, transform it as follows:

$$\cot A\cot B-\left(\frac{\cos A\cos B}{\sin A\sin B}-\frac{\sin A\sin B}{\sin A\sin B}\right)=\cot A\cot B-(\cot A\cot B-1)=1$$

808. Replace the factors $\cos\frac{\pi}{5}$ and $\cos\frac{2\pi}{5}$ by $\frac{\sin\frac{2\pi}{5}}{2\sin\frac{\pi}{5}}$ and $\frac{\sin\frac{4\pi}{5}}{2\sin\frac{2\pi}{5}}$, respectively. Then the left member takes the form $\sin\frac{4\pi}{5}:4\sin\frac{\pi}{5}$. And since $\sin\frac{4\pi}{5}=\sin\left(\pi-\frac{\pi}{5}\right)=\sin\frac{\pi}{5}$, the left member becomes $\frac{1}{4}$.

809. Transform the left member using the formula for a sum of cosines. We get $2\cos\frac{2\pi}{5}\cos\frac{\pi}{5}$. Then proceed as in the preceding problem.

810. Since $1+\cos\alpha=2\cos^2\frac{\alpha}{2}$, the given expression takes the form $2\cos^2\frac{\alpha}{2}+\cos\frac{\alpha}{2}$ or $2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+\frac{1}{2}\right)$. Write $\cos 60^\circ$ instead of $\frac{1}{2}$; we get $2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+\cos 60^\circ\right)$.

Answer: $4\cos\frac{\alpha}{2}\cos\left(\frac{\alpha}{4}+30^\circ\right)\cos\left(\frac{\alpha}{4}-30^\circ\right)$.

811. Transforming the given expression as in the preceding problem, we get $2\cos\alpha\left(\cos\alpha-\frac{\sqrt{2}}{2}\right)$. Instead of $\frac{\sqrt{2}}{2}$ write $\cos 45^\circ$.

Answer: $4\cos\alpha\sin\frac{45^\circ+\alpha}{2}\sin\frac{45^\circ-\alpha}{2}$.

812. Rewrite the given expression in the form $\cos^2(\alpha+\beta)-\sin^2(\alpha-\beta)$, the latter expression is reduced to a form convenient for taking logarithms in the same way as in Problem 656.
Answer: $\cos 2\alpha \cos 2\beta$.
813. Group the terms as follows:

$$(1+\cos\alpha)+(\tan\alpha+\sin\alpha)$$

and take $\tan\alpha$ outside the brackets in the second group. We get $(1+\cos\alpha)\times\times(1+\tan\alpha)$. Instead of $1+\tan\alpha$ write

$$\tan 45^\circ+\tan\alpha=\frac{\sin(45^\circ+\alpha)}{\cos 45^\circ\cos\alpha}$$

Answer: $\dfrac{2\sqrt{2}\cos^2\dfrac{\alpha}{2}\sin(45^\circ+\alpha)}{\cos\alpha}$.

814. Using the formula $1-\cos\alpha=2\sin^2\dfrac{\alpha}{2}$ and $\sin\alpha=2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}$, we get in the numerator $2\sin^2\dfrac{\alpha}{2}+2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}=2\sin\dfrac{\alpha}{2}\left(\sin\dfrac{\alpha}{2}+\cos\dfrac{\alpha}{2}\right)$. The expression in parentheses is equal to $\sin\dfrac{\alpha}{2}+\sin\left(90^\circ-\dfrac{\alpha}{2}\right)$. Making use of the formula for a sum of sines, reduce it to the form $\sqrt{2}\cos\left(45^\circ-\dfrac{\alpha}{2}\right)$.

Answer: $2\sqrt{2}\cos\left(45^\circ-\dfrac{\alpha}{2}\right)$.

815. The given expression is equal to $\dfrac{\cos\alpha-\sin\alpha+1}{\cos\alpha}$. The numerator is transformed to $2\sqrt{2}\cos\dfrac{\alpha}{2}\sin\left(45^\circ-\dfrac{\alpha}{2}\right)$ (see the preceding problem). The fraction can be still simplified by representing the denominator in the form

$$\sin(90^\circ-\alpha)=2\sin\left(45^\circ-\frac{\alpha}{2}\right)\cos\left(45^\circ-\frac{\alpha}{2}\right)$$

Answer: $\dfrac{\sqrt{2}\cos\dfrac{\alpha}{2}}{\cos\left(45^\circ-\dfrac{\alpha}{2}\right)}$.

816. Since $\cos\alpha-\cos 3\alpha=2\sin 2\alpha\sin\alpha$, we have

$$2\sin 2\alpha\sin\alpha+\sin 2\alpha=2\sin 2\alpha\left(\sin\alpha+\frac{1}{2}\right)=2\sin 2\alpha\,(\sin\alpha+\sin 30^\circ)$$

Answer: $4\sin 2\alpha\sin\left(\dfrac{\alpha}{2}+15^\circ\right)\cos\left(\dfrac{\alpha}{2}-15^\circ\right)$.
817. The given expression is equal to

$$\frac{\tan\alpha+1}{1-\tan\alpha}+\frac{\tan\alpha-1}{1+\tan\alpha}=\frac{4\tan\alpha}{1-\tan^2\alpha},\quad\text{i.e.}\quad 2\,\frac{2\tan\alpha}{1-\tan^2\alpha}$$

Answer: $2\tan 2\alpha$.

818. Replacing $\sin 2\beta$ by $2\sin\beta\cos\beta$ and reducing by $2\sin\beta$, we get $\frac{1-\cos\beta}{1+\cos\beta}$; applying the formula

$$\tan\frac{\beta}{2}=\pm\sqrt{\frac{1-\cos\beta}{1+\cos\beta}}$$

we obtain $\tan^2\frac{\beta}{2}$.

Answer: $\tan^2\frac{\beta}{2}$.

819. Transforming the sum $\cos\alpha+\sin\alpha$ in the numerator $\sqrt{2}-(\cos\alpha+\sin\alpha)$ and the difference $\sin\alpha-\cos\alpha$ in the denominator in the same way as in Problem 814, we get

$$\frac{\sqrt{2}\,[1-\cos(\alpha-45^\circ)]}{\sqrt{2}\sin(\alpha-45^\circ)}=\frac{2\sin\frac{\alpha-45^\circ}{2}}{2\sin\frac{\alpha-45^\circ}{2}\cos\frac{\alpha-45^\circ}{2}}$$

Answer: $\tan\frac{\alpha-45^\circ}{2}$.

820. Transform the sum of the last two terms:

$$\cot 2\alpha+\csc 2\alpha=\frac{\cos 2\alpha+1}{\sin 2\alpha}=\frac{2\cos^2\alpha}{2\sin\alpha\cos\alpha}=\cot\alpha$$

Answer: $2\cot\alpha$.

821. Replace $\cos 2\alpha$ by $\cos^2\alpha-\sin^2\alpha$, and $\sin 2\alpha$ by $2\sin\alpha\cos\alpha$.

Answer: 1.

822. Replace $2\sin^2\alpha-1$ by $-\cos 2\alpha$ and represent the given expression in the form $2\left(\frac{\sqrt{3}}{2}\sin 2\alpha-\frac{1}{2}\cos 2\alpha\right)$. Write $\cos 30^\circ$ instead of $\frac{\sqrt{3}}{2}$ and $\sin 30^\circ$ instead of $\frac{1}{2}$.

Answer: $2\sin(2\alpha-30^\circ)$.

823. The numerator is equal to

$$\frac{\cos 2\alpha\cos\alpha+\sin 2\alpha\sin\alpha}{\cos 2\alpha\cos\alpha}=\frac{\cos(2\alpha-\alpha)}{\cos 2\alpha\cos\alpha}=\frac{1}{\cos 2\alpha}$$

The denominator is equal to

$$\frac{\sin^2\alpha+\cos^2\alpha}{\sin\alpha\cos\alpha}=\frac{1}{\frac{1}{2}\sin 2\alpha}$$

Answer: $\frac{1}{2}\tan 2\alpha$.

824. The given expression is equal to

$$2+\frac{2}{\sin 4\alpha}=\frac{2}{\sin 4\alpha}(1+\sin 4\alpha)$$

(see the preceding problem). The expression in parentheses is equal to

$$1+\cos(90^\circ-4\alpha)=2\cos^2\frac{90^\circ-4\alpha}{2}$$

Answer: $\dfrac{4\cos^2(45^\circ-2\alpha)}{\sin 4\alpha}$.

825. The last addend is equal to $\cos^2 x$, thus the given expression can be written in the form $(\tan x-1)(1-\sin x)+\cos^2 x$. Replacing $\cos^2 x$ by $1-\sin^2 x$ and taking $1-\sin x$ outside the brackets, we obtain

$$(1-\sin x)[(\tan x-1)+1+\sin x]=$$
$$=(1-\sin x)(\tan x+\sin x)=$$
$$=(1-\sin x)\tan x(1+\cos x)$$

The first factor is transformed in the same way as in the preceding problem.

Answer: $4\tan x\cos^2\dfrac{x}{2}\sin^2\left(45^\circ-\dfrac{x}{2}\right)$.

826. The numerator and denominator of the fraction are equal to

$$(1+\cos 2\alpha)+(\cos\alpha+\cos 3\alpha)=2\cos^2\alpha+2\cos 2\alpha\cos\alpha$$

and

$$\cos\alpha+\cos 2\alpha, \text{ respectively.}$$

Answer: $2\cos\alpha$.

827. The given expression is equal to

$$(1-\sin^2\beta)-\sin^2\alpha\cos^2\alpha-\cos^4\alpha=\cos^2\beta-\cos^2\alpha(\sin^2\alpha+\cos^2\alpha)$$

We get the expression $\cos^2\beta-\cos^2\alpha$, which is transformed as in the solution of Problem 656.

Answer: $\sin(\alpha+\beta)\sin(\alpha-\beta)$.

828. Reduce the given expression to a common denominator $\cos x\cos y\cos z$. The numerator will be

$$\sin x\cos y\cos z+\sin y\cos z\cos x+$$
$$+\sin z\cos x\cos y-\sin[(x+y)+z]$$

The last term is equal to $-\sin(x+y)\cos z-\cos(x+y)\sin z$. The sum of the first two terms and the term $-\sin(x+y)\cos z$ are mutually annihilated, and the numerator takes the form:

$$\sin z\cos x\cos y-\cos(x+y)\sin z=\sin z[\cos x\cos y-\cos(x+y)]$$

Expanding the expression $\cos(x+y)$, we get in the numerator $\sin z\sin x\sin y$.

Answer: $\tan x\tan y\tan z$.

829. The given expression is equal to $2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}+\sin\gamma$. But by hypothesis $\gamma=180^\circ-(\alpha+\beta)$; hence, we get

$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}$$

Take $2\sin\frac{\alpha+\beta}{2}$ outside the brackets $\left(\text{or, which is the same, } 2\sin\frac{180^\circ-\gamma}{2}=\right.$ $\left.=2\cos\frac{\gamma}{2}\right)$. The bracketed expression becomes $\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}$, which is transformed according to the formula for a sum of cosines.

Answer: $4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$.

CHAPTER XII

TRIGONOMETRIC EQUATIONS

830. After simplifications we get $\sin 5x - \sin 3x = 0$. Using the formula for a difference of sines, we have $2\sin x\cos 4x = 0$, and the equation is reduced to the two equations: $\sin x = 0$ and $\cos 4x = 0$. From the first one we have $x = \pi n$ (n is any integer), from the second $4x = 2\pi n \pm \frac{\pi}{2} = \frac{\pi}{2}(4n \pm 1)$, i.e.

$$x=\frac{\pi}{8}(4n\pm 1)$$

The expression $4n \pm 1$ comprises all odd numbers (the numbers $-3, 1, 5, 9, 13$, etc. are yielded by the expression $4n+1$; the numbers $-1, 3, 7, 11, 15$, etc., by the expression $4n-1$). Therefore, instead of $4n \pm 1$ we may write $2n+1$ (or $2n-1$), where n is any integer.

Answer: $x=\pi n$; $x=\frac{\pi}{8}(2n+1)$, where n is any integer.

831. Transform the left member of the equation in the following way:

$$\sin x+\sin 2x+\sin 3x+\sin 4x=(\sin x+\sin 3x)+(\sin 2x+\sin 4x)=$$
$$=2\sin 2x\cos x+2\sin 3x\cos x=2\cos x(\sin 2x+\sin 3x)=$$
$$=4\sin\frac{5x}{2}\cos\frac{x}{2}\cos x$$

The equation takes the form

$$\sin\frac{5x}{2}\cos\frac{x}{2}\cos x=0$$

and reduces to three equations:

$$\sin\frac{5x}{2}=0;\quad \cos\frac{x}{2}=0;\quad \cos x=0$$

Answer: $x=72^\circ n$; $x=180^\circ(2n+1)$; $x=90^\circ(2n+1)$.

832. Perform the following transformations:

$$\cos(x+60^\circ)=\cos[90^\circ-(30^\circ-x)]=\sin(30^\circ-x)$$

and

$$1+\cos 2x=2\cos^2 x$$

The equation takes the form

$$\sin(x+30^\circ)+\sin(30^\circ-x)=2\cos^2 x$$

Apply the formula for the sum of sines; this is the result:

$$\sin 30^\circ \cos x-\cos^2 x=0 \text{ or } \cos x\left(\frac{1}{2}-\cos x\right)=0$$

Answer: $x=90^\circ(2n+1)$; $x=60^\circ(6n\pm 1)$.

833. Transpose all the terms of the equation to the left side and group them in the following way:

$$(\sin x+\sin 3x)-(\cos x+\cos 3x)+(\sin 2x-\cos 2x)=0$$

Transforming the first two groups, we get

$$2\sin 2x\cos x-2\cos 2x\cos x+(\sin 2x-\cos 2x)=0$$

or

$$(2\cos x+1)(\sin 2x-\cos 2x)=0$$

This equation is reduced to the following two:

$$2\cos x+1=0 \text{ and } \sin 2x-\cos 2x=0$$

The first one yields: $\cos x=-\frac{1}{2}$; $x=2\pi n\pm\frac{2}{3}\pi$. Dividing the second equation by $\cos 2x$, we get $\tan 2x=1$, whence $2x=\pi n+\frac{\pi}{4}$.

Answer: $x=\frac{2\pi}{3}(3n\pm 1)$; $x=\frac{\pi}{8}(4n+1)$.

834. Perform the following grouping:

$$(\cos 2x+\cos 6x)-(1+\cos 8x)=0$$

Using the formula $2\cos^2\frac{\alpha}{2}=1+\cos\alpha$ and transforming the sum of cosines, we obtain

$$2\cos 4x\cos 2x-2\cos^2 4x=0$$

Take $2\cos 4x$ outside the brackets and transform the difference of cosines $\cos 2x-\cos 4x$. We obtain the equation

$$\cos 4x\sin 3x\sin x=0$$

It is reduced to the following three:

$$(1)\ \cos 4x=0;\quad (2)\ \sin 3x=0;\quad (3)\ \sin x=0$$

No consideration may be given to the third equation, since all its solutions are covered by the solutions of the equation $\sin 3x=0$. Indeed, if $\sin x=0$, then also $\sin 3x=3\sin x-4\sin^3 x=0$.

Answer: $x=\frac{\pi}{8}(2n+1)$; $x=\frac{\pi n}{3}$.

835. Represent the right member in the form $2\sin\frac{3x}{2}\cos\frac{3x}{2}$ (instead of $\sin 3x$). The equation takes the form

$$2\sin\frac{3x}{2}\sin\frac{x}{2}=2\sin\frac{3x}{2}\cos\frac{3x}{2}$$

or

$$\sin\frac{3x}{2}\left(\sin\frac{x}{2}-\cos\frac{3x}{2}\right)=0$$

Write the bracketed expression in the form

$$\cos\left(\frac{\pi}{2}-\frac{x}{2}\right)-\cos\frac{3x}{2}=2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\sin\left(x-\frac{\pi}{4}\right)$$

Hence, the given equation is reduced to the following three:

$$(1)\ \sin\frac{3x}{2}=0;\quad (2)\ \sin\left(\frac{\pi}{4}+\frac{x}{2}\right)=0;\quad (3)\ \sin\left(x-\frac{\pi}{4}\right)=0$$

Answer: $x=\frac{2\pi n}{3}$; $x=\frac{\pi}{2}(4n-1)$; $x=\frac{\pi}{4}(4n+1)$.

836. The right member is equal to

$$\sin[90^\circ-(x+30^\circ)]=\sin(60^\circ-x)=-\sin(x-60^\circ)$$

The equation takes the form

$$\sin(x-60^\circ)=-\sin(x-60^\circ) \text{ or } \sin(x-60^\circ)=0$$

whence $x-60^\circ=180^\circ n$.

Answer: $x=60^\circ(3n+1)$.

837. Replacing $2\sin^2 x$ by $1-\cos 2x$, reduce the equation to the form $2\sin 3x\cos 2x-\cos 2x=0$. This equation is reduced to the following two: (1) $\cos 2x=0$; (2) $\sin 3x=\frac{1}{2}$. Since $\frac{1}{2}$ is $\sin 30^\circ$, the second equation yields

$$3x=180^\circ n+(-1)^n\,30^\circ$$

Answer: $x=45^\circ(2n+1)$; $x=60^\circ n+(-1)^n\,10^\circ$.

838. Rewrite the right member: $3(\sin x\cos x-\sin^2 x+1)=3(\sin x\cos x+\cos^2 x)=3\cos^2 x(\tan x+1)$. The given equation is reduced to two ones: (1) $\tan x+1=0$; (2) $\sin^2 x-3\cos^2 x=0$. From the second one we get $\tan x=\pm\sqrt{3}$.

Answer: $x=\frac{\pi}{4}(4n-1)$; $x=\frac{\pi}{3}(3n\pm 1)$.

839. We have the equation

$$\cos 4x+2\cos^2 x=0$$

Since $2\cos^2 x=1+\cos 2x$, the left member is equal to

$$(1+\cos 4x)+\cos 2x=2\cos^2 2x+\cos 2x$$

We get the equation

$$\cos 2x(2\cos 2x+1)=0$$

which is reduced to:

$$(1)\ \cos 2x=0 \text{ and } (2)\ 2\cos 2x+1=0$$

The second one yields $2x=360^\circ n\pm 120^\circ$.

Answer: $x=180^\circ n\pm 45^\circ$; $x=180^\circ n\pm 60^\circ$.

840. Multiplying both members of the equation by $\sin x$ and replacing unity in the right member by $\sin^2 x + \cos^2 x$, we get the equation $\sin x \cos x = \cos^2 x$.

Note. Multiplying both members of the equation by $\sin x$, we introduce no extraneous solutions, since $\sin x$ never vanishes at either of the found values of x.

Answer: $x_1 = \frac{\pi}{2}(2n+1)$; $x_2 = \frac{\pi}{4}(4n+1)$.

841. Rewrite the equation in the following way:

$$\sin 3x - \sin\left(\frac{\pi}{2} - 2x\right) = 0$$

It is reduced to the following two equations:

$$(1)\ \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) = 0 \quad \text{and} \quad (2)\ \sin\left(\frac{5x}{2} - \frac{\pi}{4}\right) = 0$$

The first one yields $\frac{x}{2} + \frac{\pi}{4} = \frac{\pi}{2}(2n+1)$, whence $x = \frac{\pi}{2}(4n+1)$. The second gives $x = \frac{\pi}{10}(4n+1)$.

Answer: $x = \frac{\pi}{2}(4n+1)$; $x = \frac{\pi}{10}(4n+1)$.

842. Add $2\sin^2\frac{x}{3}\cos^2\frac{x}{3}$ to both members of the equation; then in the left member we have

$$\sin^4\frac{x}{3} + 2\sin^2\frac{x}{3}\cos^2\frac{x}{3} + \cos^4\frac{x}{3} = \left(\sin^2\frac{x}{3} + \cos^2\frac{x}{3}\right)^2 = 1$$

and the equation takes the form

$$1 = \frac{5}{8} + 2\sin^2\frac{x}{3}\cos^2\frac{x}{3} \quad \text{or} \quad 2\sin^2\frac{x}{3}\cos^2\frac{x}{3} = \frac{3}{8}$$

Multiply both members of the equations by 2 and apply the double-angle formula for sine. We obtain $\sin^2\frac{2x}{3} = \frac{3}{4}$, whence $\sin\frac{2x}{3} = \pm\frac{\sqrt{3}}{2}$.

Answer: $x = \frac{\pi}{2}(3n \pm 1)$.

843. Represent the equation in the form $3\tan^2 x - (1 + \tan^2 x) = 1$, whence $\tan x = \pm 1$.

Answer: $x = 45^\circ (2n+1)$.

844. Replace $1 + \cos 4x$ by $2\cos^2 2x$.

Answer: $x = \frac{\pi n}{2} + \frac{\pi}{4}$; $x = \frac{\pi n}{4} + (-1)^n \frac{\pi}{24}$.

845. Add $2\sin^2 x \cos^2 x$ to both members of the equation. We get $(\sin^2 x + \cos^2 x)^2 = \cos 4x + 2\sin^2 x \cos^2 x$ or $1 - \cos 4x = \frac{1}{2}\sin^2 2x$.

Answer: $x = \frac{\pi}{2} n$.

846. Replace $\sin 2x$ by $2\sin x\cos x$ and divide all the terms by $\cos^2 x$. It is obvious that no roots are lost. Indeed, if $\cos x = 0$, then $\sin x = \pm 1$, but these values do not satisfy the given equation. We get

$$3 - \tan^2 x - 2\tan x = 0$$

whence

$$\tan x = 1 \text{ and } \tan x = -3$$

Answer: $x = \pi n + \frac{\pi}{4}$; $x = \pi n - \arctan 3$.

847. Write $\sin^2 x + \cos^2 x$ instead of unity and, dividing both members by $\cos^2 x$ (see the solution of the preceding problem), we get

$$\tan^2 x + \sqrt{3}\tan x = 0$$

whence

$$\tan x = 0 \text{ and } \tan x = -\sqrt{3}$$

Answer: $x = \pi n$; $x = \frac{\pi}{3}(3n - 1)$.

848. Replace 2 by $2\sin^2 x + 2\cos^2 x$, and then the equation is solved as the preceding one.

Answer: $x = \pi n + \frac{\pi}{4}$; $x = \pi n - \arctan \frac{7}{4}$.

849. *Answer*: $x = \frac{\pi}{4}(4n + 1)$; $x = \pi n + \arctan \frac{3}{2}$.

850. Replace $\sqrt{3}$ by $\cot 30°$ (we introduce an "auxiliary angle" 30°). Then the given equation becomes

$$\sin x + \frac{\cos 30°}{\sin 30°}\cos x = 1$$

or

$$\sin x \sin 30° + \cos x \cos 30° = \sin 30°$$

or

$$\cos(x - 30°) = \frac{1}{2}$$

Hence, $x - 30° = 360°n \pm 60°$.

Answer: $x = 360°n + 90° = 90°(4n + 1)$; $x = 360°n - 30° = 30°(12n - 1)$.

851. The left member can be represented in the form of a product: $\sqrt{2}\cos(x - 45°)$. Then we get the equation $\cos(x - 45°) = \frac{1}{\sqrt{2}}$; it yields $x - 45° = 360°n - 45°$ and $x - 45° = 360°n + 45°$, i.e. $x = 360°n$ and $x = 360°n + 90°$, or $x = 90° \cdot 4n$ and $x = 90°(4n + 1)$.

Alternate method. Squaring both members of the equation, we get

$$\sin^2 x + 2\sin x\cos x + \cos^2 x = 1$$

or $\sin 2x = 0$. This equation has the solutions $x = 90°n$, but among them there are extraneous ones (compare with the preceding result).

Extraneous solutions have resulted from squaring both members, whereby we introduced one more equation (in addition to the given one): $\sin x + \cos x = -1$ (which also yields $\sin 2x = 0$). To reject extraneous roots we have to accomplish a check. At $n = 0$ we have $x = 0^\circ$ and the given equation is satisfied. It is also satisfied at $n = 4, 8, 12$ and, in general, at $n = 4k$ (i.e. at $x = 90^\circ \cdot 4k = 360^\circ k$). At $n = 1$ we have $x = 90^\circ$; the given equation is satisfied once again. It is also satisfied at $n = 5, 9, 13$ and, in general, at $4k + 1$ (i.e. at $x = 90^\circ (4k + 1) = 90^\circ + 360^\circ k$). But at $n = 2, 6, 10$ (in general, at $n = 4k + 2$), the same as at $n = 3, 7, 11$ (in general at $n = 4k + 3$) the given equation is not satisfied (instead, the equation $\sin x + \cos x = -1$ is satisfied).

Answer: $x = 90^\circ \cdot 4n$; $x = 90^\circ (4n + 1)$.

852. Transform the right member:

$$1 + \sin 2x = \sin^2 x + \cos^2 x + 2 \sin x \cos x = (\sin x + \cos x)^2,$$

now the equation takes the form

$$\sin x + \cos x = (\sin x + \cos x)^2$$

or

$$(\sin x + \cos x)(\sin x + \cos x - 1) = 0$$

The latter is reduced to two equations:

$$(1)\ \sin x + \cos x = 0$$

$$(2)\ \sin x + \cos x - 1 = 0$$

Solving the first one, we find $x = \frac{\pi}{4}(4n - 1)$. The second one is solved in the preceding problem.

Answer: $x = \frac{\pi}{4}(4n - 1)$; $x = \frac{\pi}{2}(4n + 1)$; $x = \frac{\pi}{2} \cdot 4n$.

853. Solved in the same way as Problem 851.

Answer: $x = 15^\circ (8n + 1)$.

854. Using the formula

$$\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

we get

$$\frac{1}{2}[\cos(x - 7x) - \cos(x + 7x)] = \frac{1}{2}[\cos(3x - 5x) - \cos(3x + 5x)]$$

or, after simplifications, $\cos 6x - \cos 2x = 0$. This equation is reduced to the following two: $\sin 4x = 0$; $\sin 2x = 0$, all the roots of the second equation being among the roots of the first one.

Answer: $x = \frac{\pi n}{4}$.

855. Apply to both members of the equation the formula

$$\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

Answer: $x = \frac{\pi n}{2}$; $x = \frac{\pi}{8}(2n + 1)$.

856. We have

$$4 \sin x \sin 2x \sin 3x = \sin 2(2x)$$

or

$$\sin 2x (2 \sin x \sin 3x - \cos 2x) = 0$$

Replace $2 \sin x \sin 3x$ by $\cos 2x - \cos 4x$ (see Problem 854) to get the following equation

$$\sin 2x (\cos 2x - \cos 4x - \cos 2x) = 0 \text{ or } \sin 2x \cos 4x = 0.$$

Answer: $x = \frac{\pi n}{2}$; $x = \frac{\pi}{8}(2n+1)$.

857. Replace $\sin^2 x$ by $1 - \cos^2 x$; we get

$$5 \cos^2 x + 4 \cos x - 3 = 0$$

whence $\cos x = \frac{\sqrt{19} - 2}{5}$. The other root $\cos x = -\frac{\sqrt{19} + 2}{5}$ is not suitable, since its absolute value is more than unity.

Answer: $x = 2\pi n \pm \arccos \frac{\sqrt{19} - 2}{5}$.

858. Using the formula $\cos 2\alpha$ and expressing cosine through sine, we get $10 \sin^2 x + 4 \sin x - 5 = 0$.

Answer: $x = \pi n + (-1)^n \arcsin \frac{-2 \pm \sqrt{54}}{10}$.

859. Applying the formula for tangent of a sum, we get

$$\tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}$$

and reduce the given equation * to $\tan^2 x - 4 \tan x + 1 = 0$.

Answer: $x = \pi n + \arctan (2 \pm \sqrt{3})$.

860. Since $\tan^2 \frac{\alpha}{2} = \frac{1 - \cos x}{1 + \cos x}$ and $\sec x = \frac{1}{\cos x}$, we have the equation

$$\frac{8(1 - \cos x)}{1 + \cos x} = 1 + \frac{1}{\cos x}$$

which is reduced to the form

$$9 \cos^2 x - 6 \cos x + 1 = 0 \text{ or } (3 \cos x - 1)^2 = 0$$

Answer: $x = 2\pi n \pm \arccos \frac{1}{3}$.

861. The left member is equal to

$$\frac{\cos\left(\frac{\pi}{2} - x\right)}{1 + \cos x} = \frac{\sin x}{1 + \cos x} = \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} = \tan \frac{x}{2}$$

---

* When getting rid of the denominator one should be careful not to introduce extraneous solutions, but we do not conduct analysis in the next three problems (since they have no extraneous solutions). Beginning with Problem 865 much attention is paid to such an analysis. See also Problem 867.

The right member is equal to $\sec^2\frac{x}{2}-1=\tan^2\frac{x}{2}$. We obtain

$$\tan\frac{x}{2}=\tan^2\frac{x}{2}$$

Answer: $x=2\pi n$; $x=\frac{\pi}{2}(4n+1)$.

862. Since

$$\cos(\pi-x)=-\cos x \quad\text{and}\quad \sin\frac{\pi+x}{2}=\cos\frac{x}{2}$$

we have

$$1+\cos x+\cos\frac{x}{2}=0 \quad\text{or}\quad 2\cos^2\frac{x}{2}+\cos\frac{x}{2}=0$$

Answer: $x=\pi(2n+1)$; $x=\frac{4\pi}{3}(3n\pm1)$.

863. Applying the reduction formulas

$$\sin\left(\frac{3\pi}{2}-x\right)=-\cos x \quad\text{and}\quad \tan\left(\frac{\pi}{2}-\frac{x}{2}\right)=\cot\frac{x}{2}$$

we get the equation

$$2(1+\cos x)-\sqrt{3}\cot\frac{x}{2}=0$$

Let us make use of the formula

$$\cot\frac{x}{2}=\frac{1+\cos x}{\sin x};$$

then the obtained equation is reduced to the following two equations:

$$(1)\ 1+\cos x=0 \quad\text{and}\quad (2)\ \sin x=\frac{\sqrt{3}}{2}$$

Answer: $x=\pi(2n+1)$; $x=\pi n+(-1)^n\frac{\pi}{3}$.

864. Replacing $\cos^2 x$ by $1-\sin^2 x$, after simplifications we get $3\sin x+\cos x=0$ or $\tan x=-\frac{1}{3}$,

Answer: $x=\pi n-\arctan\frac{1}{3}$.

865. The left member is equal to

$$\frac{\cos x}{\sin x}+\frac{\sin x}{1+\cos x}=\frac{\cos x+\cos^2 x+\sin^2 x}{\sin x(1+\cos x)}=\frac{1+\cos x}{\sin x(1+\cos x)}$$

Reduce the fraction by $1+\cos x$, assuming that $1+\cos x\neq0$. We get the equation $\frac{1}{\sin x}=2$, i.e. $\sin x=\frac{1}{2}$ (at this value of $\sin x$ the quantity $\cos x$ is not equal to -1).

Answer: $x=\pi n+(-1)^n\frac{\pi}{6}$.

866. According to the reduction formulas

$$\cot(x-\pi)=-\cot(\pi-x)=\cot x$$

The given equation may be rewritten as

$$2\cot x-(\cos x+\sin x)\left(\frac{1}{\sin x}-\frac{1}{\cos x}\right)=4$$

On reducing the left member to a common denominator this equation takes the form

$$\frac{1}{\sin x\cos x}=4$$

whence $\sin x\cos x=\frac{1}{4}$ or $\sin 2x=\frac{1}{2}$.

Answer: $x=\frac{\pi}{2}n+(-1)^n\frac{\pi}{12}$.

867. The right member is equal to

$$\frac{\frac{1}{\cos x}-\cos x}{2\sin x}=\frac{1-\cos^2 x}{2\sin x\cos x}=\frac{\sin^2 x}{2\sin x\cos x}$$

Reduce the fraction by $\sin x$. It is assumed that $\sin x\neq 0$, should we obtain such a solution, for which $\sin x=0$, it would not be suitable. The given equation (after applying some reduction formulas to its left member) takes the form

$$\sin x+\tan x=\frac{1}{2}\tan x \quad\text{or}\quad \sin x+\frac{1}{2}\tan x=0$$

This equation may be represented in the form

$$\sin x\left(1+\frac{1}{2\cos x}\right)=0$$

and it is reduced to two equations

$$\sin x=0 \quad\text{and}\quad 1+\frac{1}{2\cos x}=0$$

But the first equation yields extraneous solutions, since we reduced the fraction by $\sin x$ before. To get a better understanding substitute $\sin x=0$ into the right member; then instead of $\cos x$ we have to substitute 1 or -1. In both cases we get the indeterminate form $\frac{0}{0}$.

Answer: $x=\frac{2\pi}{3}(3n\pm 1)$.

868. The left member is equal to

$$\frac{1-\tan\frac{x}{2}}{1-\frac{1}{\tan\frac{x}{2}}}=-\frac{\tan\frac{x}{2}\left(1-\tan\frac{x}{2}\right)}{1-\tan\frac{x}{2}}$$

Reducing by $1-\tan\frac{x}{2}$ (we assume that $1-\tan\frac{x}{2}\neq 0$, see the solution of the preceding problem), we get $-\tan\frac{x}{2}$, and the equation takes the form

$$-\tan\frac{x}{2}=2\sin\frac{x}{2} \text{ or } \sin\frac{x}{2}\left(\sec\frac{x}{2}+2\right)=0$$

It is reduced to two equations:

$$(1)\ \sin\frac{x}{2}=0 \text{ and } (2)\ \cos\frac{x}{2}=-\frac{1}{2}$$

From the second equation we find $\frac{x}{2}=360°n\pm 120°$ and get the solution $x=720°n\pm 240°$. The first equation yields only extraneous solutions ($x=360°n$), though for another reason than in the preceding problem. Namely, the quantity $\cot\frac{x}{2}$, entering the given equation, loses sense ("becomes equal to infinity") at $x=360°n$; hence, the whole left member of the equation has no (direct) meaning.

Answer: $x=240°(3n\pm 1)$.

869. Using reduction formulas, we obtain the equation $\sin x-\tan x=$ $=\sec x-\cos x$ or $\sin x-\frac{\sin x}{\cos x}=\frac{1}{\cos x}-\cos x$. Multiply both members of the equation by $\cos x$ (or, which is the same, reduce it to a common denominator and then reject it). It is assumed that $\cos x\neq 0$, since if $\cos x=0$, then the expressions $\frac{\sin x}{\cos x}$ and $\frac{1}{\cos x}$ lose their meaning ("become infinitely great"). We get the equation

$$\cos x\sin x-\sin x=\sin^2 x$$

which is equivalent to the following two:

$$(1)\ \sin x=0;\quad (2)\ \cos x-\sin x=1$$

The second one may be rewritten as $\sqrt{2}\cdot\cos(45°+x)=1$ (cf. Problem 851), whence $x=360°n$ and $x=360°n-90°$. The solution $x=360°n$ is found among the solutions of the first equation ($x=180°n$), and the solution $x=360°n-90°$ is an extraneous one, since we have $\cos(360°n-90°)=0$.

Answer: $x=180°n$.

870. Use the formulas: $\sec^2 x-\tan^2 x=1$ and $\cos 2x=\cos^2 x-\sin^2 x$. We get the equation

$$1-\tan x=\frac{\cos^2 x-\sin^2 x}{\cos^2 x}$$

which is reduced to the form $\tan^2 x-\tan x=0$.

Answer: $x=\pi n$; $x=\frac{\pi}{4}(4n+1)$.

871. Rewrite the equation in the form

$$\frac{\sin^3 x(\sin x+\cos x)}{\sin x}+\frac{\cos^3 x(\sin x+\cos x)}{\cos x}=\cos^2 x-\sin^2 x$$

Assuming that $\sin x \neq 0$ and $\cos x \neq 0$, reduce the fractions, transpose all the terms to the left side and take $\sin x+\cos x$ outside the brackets. We obtain

$$(\sin x+\cos x)(\sin^2 x+\cos^2 x-\cos x+\sin x)=0$$

Replace $\sin^2 x+\cos^2 x$ by 1. The equation is reduced to

$$(1)\ \sin x+\cos x=0 \quad \text{and} \quad (2)\ \cos x-\sin x=1$$

The first equation yields $x=\frac{\pi}{4}(4n-1)$; the second (see Problem 869) has two solutions $x=2\pi n$ and $x=\frac{\pi}{2}(4n-1)$. Both of them are extraneous, since at $x=2\pi n$ we have $\sin x=0$, and at $x=\frac{\pi}{2}(4n-1)$ we have $\cos x=0$.

Answer: $x=\frac{\pi}{4}(4n-1)$.

872. Use the triple-angle formulas:

$$\sin 3x=3\sin x-4\sin^3 x,\ \cos 3x=4\cos^3 x-3\cos x\ ^*$$

The left member is transformed to the form

$$3\sin x\cos x(\cos^2 x-\sin^2 x)=\frac{3}{2}\sin 2x\cos 2x=\frac{3}{4}\sin 4x$$

and the given equation takes the form $\sin 4x=\frac{1}{2}$.

Answer: $x=\frac{\pi n}{4}+(-1)^n\frac{\pi}{24}$.

873. Rewrite the given equation:

$$\tan 2x=\tan 3x-\tan x$$

and divide both members of the equality by $1+\tan x\tan 3x$ to apply to the right member the formula for tangent of a difference of two angles. We get

$$\frac{\tan 2x}{1+\tan x\tan 3x}=\tan(3x-x),$$

whence

$$\tan 2x=\tan 2x(1+\tan x\tan 3x)$$

or

$$\tan x\tan 2x\tan 3x=0$$

Consider the following three equations separately:

$$(1)\ \tan 3x=0;\quad (2)\ \tan 2x=0;\quad (3)\ \tan x=0$$

The solution of the first is $x=\frac{\pi n}{3}$. The third equation yields nothing new since all its solutions ($x=\pi m$) are found among the solutions of the first equation (at $n = 3m$ we have $\frac{\pi n}{3} = \pi m$). The second equation yields $x = \frac{\pi n}{2}$. At even n these solutions again yield nothing new (at $n = 2k$ we have $\frac{\pi n}{2} = \pi k$); with odd n ($n = 2n' + 1$) they are not the solutions of the given equation. Indeed, the quantities $\tan x$ and $\tan 3x$, entering the equation, lose their meaning ("become infinitely great") at $x = \frac{\pi}{2}(2n' + 1)$. Therefore, the second equation should be rejected.

---

* If they are not familiar to the reader, it is easy to reduce them using the formulas for sine and cosine of a sum of two angles: 2α and α and then the formulas for sine and cosine of 2α.

Answer: $x = \frac{\pi n}{3}$.

874. Applying the formula for cosine of a difference, reduce the right member to the form $\sqrt{2}\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)$. Therefore, express the left member through the argument $\frac{x}{2}$. We have

$$(1 + \cos x) + \sin x = 2\cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\cos\frac{x}{2}\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)$$

Transposing all the terms to the left side we get the equation

$$\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)\left(2\cos\frac{x}{2} - \sqrt{2}\right) = 0$$

equivalent to the following two equations: one yields $x = 360°n - 90°$, the other $x = 720°n \pm 90°$. In the latter expression the double sign may be replaced by the plus sign, since all the quantities $720°n - 90°$ are among the quantities $360°n - 90°$ (if in the expression $360°n - 90°$ we take only even n i.e. if we put $n = 2n'$, we get $720°n' - 90°$).

Answer: $x = 360°n - 90°$; $x = 720°n + 90°$.

875. Rewrite the given equation: $\sin^2 2x = \sin 3x + \sin x$; hence, $\sin^2 2x = 2\sin 2x\cos x$. Transposing all the terms to the left side we get

$$\sin 2x(\sin 2x - 2\cos x) = 0 \text{ or } 2\sin 2x\cos x(\sin x - 1) = 0$$

The equation is reduced to:

$$(1)\ \sin 2x = 0; \quad (2)\ \cos x = 0; \quad (3)\ \sin x = 1$$

Equations (2) and (3) are of no interest, since all their solutions are among the solutions of the first one. (We have $\sin 2x = 2\sin x\cos x = 2\sin x\sqrt{1 - \sin^2 x}$, so that if $\cos x = 0$, or if $\sin x = 1$, then $\sin 2x = 0$.)

Answer: $x = 90°n$.

876. The left member is equal to $2\cos^2 x - 3\cos x$. The right one loses its meaning at $x = \frac{\pi}{2}n$, since $\cot 2x$ "becomes infinitely great". Therefore, we consider that $x \neq \frac{\pi}{2}n$. The denominator of the right member is equal to

$$\frac{\cos 2x}{\sin 2x} - \frac{\cos x}{\sin x} = \frac{(2\cos^2 x - 1) - 2\cos^2 x}{2\sin x \cdot \cos x} = \frac{-1}{2\sin x \cdot \cos x},$$

thus, the right member is equal to

$$-\csc(\pi-x)\cdot 2\sin x\cdot\cos x=-2\csc x\cdot\sin x\cdot\cos x$$

The product $\csc x\cdot\sin x$ $\left(\text{i.e., } \frac{\sin x}{\sin x}\right)$ can be replaced by unity, since the values of x, at which the fraction $\frac{\sin x}{\sin x}$ would turn into the indeterminate form $\frac{0}{0}$, are rejected. We get the equation

$$2\cos^2 x-3\cos x=-2\cos x \quad\text{or}\quad \cos x\left(\cos x-\frac{1}{2}\right)=0$$

whence $\cos x=0$ or $\cos x=\frac{1}{2}$. In the first case we obtain the values $x=\frac{\pi}{2}(2k+1)$, which were rejected above.

Answer: $x=\frac{\pi}{3}(6n\pm 1)$.

877. The left member is equal to

$$(\cos x+\sin x)^2+1=2+2\cos x\sin x,$$

the right member is equal to $\frac{2\sin^2 x}{\tan^2 x}=2\cos^2 x$, assuming that $\sin x\neq 0$. The equation takes the form

$$2(1-\cos^2 x)+2\cos x\sin x=0 \quad\text{or}\quad \sin^2 x+\sin x\cos x=0$$

It is equivalent to the two equations: $\sin x+\cos x=0$ and $\sin x=0$, but at $\sin x=0$ the right member has no (direct) meaning.

Answer: $x=-\frac{\pi}{4}+\pi n$.

878. The right member is equal to

$$2\sin\left(\frac{\pi}{4}+x\right)\cos\left(\frac{\pi}{4}+x\right)=\sin\left(\frac{\pi}{2}+2x\right)=\cos 2x=\cos^2 x-\sin^2 x$$

Then proceed as in the preceding problem.

Answer: $x=\frac{\pi}{4}(4n-1)$; $x=\pi n$.

879. The left member is equal to $2-\sin 3x$, the right one to

$$1-2\cos\left(\frac{\pi}{4}-\frac{3x}{2}\right)\sin\left(\frac{\pi}{4}-\frac{3x}{2}\right)=1-\sin\left(\frac{\pi}{2}-3x\right)=1-\cos 3x$$

The equation takes the form

$$\cos 3x-\sin 3x+1=0$$

Solve it using the (first) method of Problem 851, transforming $\cos 3x-\sin 3x$ to $\sqrt{2}\sin\left(\frac{\pi}{4}-3x\right)$. We obtain

$$\sin\left(\frac{\pi}{4}-3x\right)=-\frac{1}{\sqrt{2}}, \text{ i.e. } \sin\left(3x-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

Consequently,

$$3x-\frac{\pi}{4}=(-1)^n\frac{\pi}{4}+\pi n,\ \text{i.e. } 3x=\frac{\pi}{4}[1+(-1)^n]+\pi n$$

At even n the expression in square brackets is equal to 2, and at odd n it is equal to zero. Therefore, if we put $n=2n'$ (n' is an integer), we get $3x=\frac{\pi}{2}+2\pi n'$, and if we put $n=2n'+1$, we get $3x=\pi(2n'+1)$.

Alternate solution. Besides the alternate method indicated in Problem 851 (which introduces extraneous roots), we can use here (as also in Problem 851) the following method. Getting, as above, the equation $\cos 3x-\sin 3x+1=0$, use the formulas

$$1+\cos 3x=2\cos^2\frac{3x}{2}\quad\text{and}\quad \sin 3x=2\sin\frac{3x}{2}\cos\frac{3x}{2}$$

We get an equation which is reduced to the following two: one $\left(\cos\frac{3x}{2}=0\right)$ yields $\frac{3x}{2}=\frac{\pi}{2}(2n+1)$, i.e. $3x=\pi(2n+1)$. The other $\left(\cos\frac{3x}{2}-\sin\frac{3x}{2}=0\right)$ yields $\frac{3x}{2}=\frac{\pi}{4}+\pi n$, i.e. $3x=\frac{\pi}{2}+2\pi n$.

Answer: $x=\frac{\pi}{3}(2n+1)$; $x=\frac{\pi}{6}(4n+1)$.

880. Represent $1+\sin 2x$ in the form

$$(\cos^2 x+\sin^2 x)+2\sin x\cos x=(\cos x+\sin x)^2$$

and replace $\tan x$ by $\frac{\sin x}{\cos x}$. Reduce all the terms to a common denominator ($\cos x$) and then get rid of it, assuming that $\cos x\neq 0$. We get the equation

$$(\cos x-\sin x)(\cos x+\sin x)^2-(\cos x+\sin x)=0$$

which yields two equations: the first one

$$\cos x+\sin x=0$$

has the solution

$$x=\frac{\pi}{4}(4n-1),$$

the second

$$\cos^2 x-\sin^2 x-1=0,\ \text{or}\ \cos 2x-1=0$$

has the solution

$$x=\pi n$$

Answer: $x=\frac{\pi}{4}(4n-1)$; $x=\pi n$.

881. Represent $1-\sin 2x$ in the form $(\cos x-\sin x)^2$, and $\cos 2x$ in the form $(\cos x+\sin x)(\cos x-\sin x)$. Reduce the fraction by $\cos x-\sin x$, assuming that this quantity is non-zero. We get the equation

$$\cos x+\sin x=\frac{\cos x+\sin x}{\cos x-\sin x}$$

Getting rid of the denominator (under the same assumption), we obtain

$$(\cos x+\sin x)(\cos x-\sin x)-(\cos x+\sin x)=0$$

or

$$(\cos x-\sin x-1)(\cos x+\sin x)=0$$

Solving the equation $\cos x+\sin x=0$, we find $x=\pi n-\frac{\pi}{4}$. The equation $\cos x-\sin x-1=0$ can be solved in the following way (see Problem 879). Represent it in the form $\sqrt{2}\cdot\sin\left(\frac{\pi}{4}-x\right)=1$, i.e., $\sin\left(x-\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$. Hence, $x-\frac{\pi}{4}=(-1)^n\left(-\frac{\pi}{4}\right)+\pi n$. With even $n=2m$ we have $x=\pi n=$ $=2\pi m$. With odd n $(=2m-1)$ we have $x=\frac{\pi}{2}+\pi n=\frac{\pi}{2}(4m-1)$.

Apply the other method from Problem 879.

Answer: $x=\frac{\pi}{4}(4n-1)$; $x=2\pi n$; $x=\frac{\pi}{2}(4n-1)$.

882. The right member is equal to $\cos 2x$, and the left one to $(\cos x+\sin x)^2(\cos x-\sin x)=$

$$=(\cos x+\sin x)(\cos^2 x-\sin^2 x)=(\cos x+\sin x)\cos 2x$$

Answer: $x=\frac{\pi}{4}(2n+1)$; $x=2\pi n$; $x=\frac{\pi}{2}(4n+1)$.

883. The left member is equal to

$$\frac{1}{4}-\frac{4\sin^2 x\cos^2 x}{4\cos^2 x}=\frac{1}{4}-\sin^2 x$$

(assuming that $\cos x\neq 0$). To the right member apply the formula $\sin\alpha\sin\beta=\frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$. We get $\frac{1}{2}(\cos 60^\circ-\cos 2x)=$ $=\frac{1}{2}\left[\frac{1}{2}-(1-2\sin^2 x)\right]=\frac{-1+4\sin^2 x}{4}$. Now the equation takes the form

$$\frac{1}{4}-\sin^2 x=-\left(\frac{1}{4}-\sin^2 x\right)$$

whence $\sin^2 x=\frac{1}{4}$, i.e. $\sin x=\frac{1}{2}$ or $\sin x=-\frac{1}{2}$. The two solutions $x=$ $=180^\circ n+(-1)^n 30^\circ$ and $x=180^\circ n-(-1)^n 30^\circ$ can be represented by one formula: $x=180^\circ n\pm 30^\circ$.

Answer: $x=30^\circ(6n\pm 1)$.

884. The left member is equal to $\sin 60^\circ\cos x$; the right one to

$$\tan x\cos^4 x+\cot x\sin^4 x=\sin x\cos^3 x+\cos x\sin^3 x$$

(assuming that $\cos x\neq 0$ and $\sin x\neq 0$). This expression is equal to

$$\sin x\cos x(\cos^2 x+\sin^2 x)=\sin x\cos x$$

The equation is reduced to the form $\cos x(\sin 60^\circ-\sin x)=0$.

It is reduced to two equations, one of which ($\cos x = 0$) yields an extraneous root.

Answer: $x = 180°n + (-1)^n\, 60°$.

885. Using the formula $\tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$, we get the left member $\sec^2 x - 1 = \tan^2 x$ (reducing by $\sin x$ we assume that $\sin x \neq 0$). The left member is equal to

$$\frac{\sin(x-30°)+\sin(x+30°)}{\cos x} = \frac{2\sin x\cos 30°}{\cos x} = \sqrt{3}\tan x.$$

The equation is reduced to the form $\tan x(\tan x - \sqrt{3}) = 0$ and is equivalent to two equations, one of which, namely $\tan x = 0$, yields extraneous solutions (since if $\tan x = 0$, then also $\sin x = 0$).

Answer: $x = 60°\ (3n+1)$; $x = 2\pi n$.

886. The expression $\tan\frac{x}{2} + \cot\frac{x}{2}$ is transformed to

$$\frac{1}{\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{2}{\sin x}$$

We get the equation

$$\sqrt{2}\sin x = \frac{1}{\sqrt{2}\sin x}$$

Answer: $x = \frac{\pi}{4}(2n+1)$.

887. The left member is equal to $2\sqrt{2}\cdot\frac{\sqrt{2}}{2}(\sin x + \cos x)$; the right one to $\frac{2\cos^2 x}{1+\sin x} = \frac{2(1-\sin^2 x)}{1+\sin x} = 2(1-\sin x)$. We get the equation

$$2(\sin x + \cos x) = 2(1-\sin x) \quad \text{or} \quad (1-\cos x) - 2\sin x = 0$$

or

$$2\sin^2\frac{x}{2} - 4\sin\frac{x}{2}\cos\frac{x}{2} = 0.$$

Answer: $x = 2\pi n$; $x = 2(\pi n + \arctan 2)$.

888. The fraction in the left member is equal to

$$\frac{2}{\sqrt{3}}(\sin 2x - \cos 2x\tan x)\cos^2 x =$$

$$= \frac{2}{\sqrt{3}}(\sin 2x\cos x - \cos 2x\sin x)\cos x = \frac{2}{\sqrt{3}}\sin x\cos x$$

The right member is equal to

$$(\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) = \cos^2 x - \sin^2 x$$

(double angles are of no use here). Write the equation in the form:

$$(1-\cos^2 x)+\sin^2 x-\frac{2}{\sqrt{3}}\sin x\cos x=0$$

or

$$2\sin^2 x-\frac{2}{\sqrt{3}}\sin x\cos x=0$$

Answer: $x = 180°n$; $x = 180°n + 30°$.

889. The left member is equal to $3\sin x - 4\sin^3 x$, and the right one to $4\sin x\,(1 - 2\sin^2 x)$. We get the equation

$$\sin x\,(4\sin^2 x - 1) = 0$$

Answer: $x = 180°n$; $x = 180°n \pm 30°$.

890. The right member is equal to

$$\sin x+\cos x\cot\frac{x}{2}=\frac{\sin x\sin\frac{x}{2}+\cos x\cos\frac{x}{2}}{\sin\frac{x}{2}}$$

The numerator of this expression is equal to $\cos\left(x-\frac{x}{2}\right)=\cos\frac{x}{2}$, thus, we get $\cot\frac{x}{2}$ in the right member. The left member is equal to

$$\frac{1+\cos x}{\cos x}=\frac{2\cos^2\frac{x}{2}}{\cos x}$$

The equation takes the form

$$\frac{2\cos^2\frac{x}{2}}{\cos x}-\cot\frac{x}{2}=0$$

Taking $\cot\frac{x}{2}$ outside the brackets, we obtain

$$\cot\frac{x}{2}\left(\frac{2\cos\frac{x}{2}\sin\frac{x}{2}}{\cos x}-1\right)=0,\ \text{i.e.,}\ \cot\frac{x}{2}(\tan x-1)=0$$

Answer: $x=\pi n+\frac{\pi}{4}$; $x=2\pi n+\pi$.

891. The denominator of the fraction is equal to

$$\frac{\sin 2x\cos x-\cos 2x\sin x}{\cos x\cos 2x}=\frac{\sin x}{\cos x\cos 2x}=\frac{\tan x}{\cos 2x}$$

The whole fraction is equal to $\sin 2x$. The equation takes the form

$$\sin 2x - 2\sin(45° + x)\cos(45° + x) = 0$$

or

$$\sin 2x - \cos 2x = 0$$

whence $\tan 2x = 1$.

Answer: $x = 90°n + 22°30'$.

892. We have

$$\tan (x - 45°) \tan (x + 45°) = \tan (x - 45°) \cot (45° - x) = -1$$

it is assumed that $x \neq 45° (2n + 1)$, since otherwise one of the factors vanishes, and the other becomes infinitely great. The denominator of the right member is transformed to

$$-\frac{\cos x}{\sin \frac{x}{2} \cos \frac{x}{2}} = -\frac{2 \cos x}{\sin x} = -2 \cot x$$

it is assumed that $x \neq 180°n$, since then either $\tan \frac{x}{2}$, or $\cot \frac{x}{2}$ loses its meaning (becomes infinitely great). We get the equation

$$-\tan x = -\frac{4 \cos^2 x}{2 \cot x}$$

which (assuming that $x \neq 180°n$) is reduced to the form $\tan x \cos 2x = 0$. The last equation has the solution $x = 180°n$ and $x = 45° (2n + 1)$, but they disagree with the above assumptions.

Answer: The equation has no solutions.

893. The right member is equal to $-\tan x$ (see the preceding problem). Let us represent the left member in the form:

$$\frac{1}{2} [\tan (x + 45°) - \cot (x + 45°)] =$$

$$= \frac{\sin^2 (x + 45°) - \cos^2 (x + 45°)}{2 \sin (x + 45°) \cos (x + 45°)} = -\cot (2x + 90°)$$

We get the equation

$$\tan 2x = -\tan x$$

It can be written in the form

$$\tan 2x = \tan (-x)$$

wherefrom we conclude that the angles $2x$ and $-x$ differ by $180°n$, and from the equation $2x = -x + 180°n$ we find $x = 60°n$.

Answer: $x = 60°n$.

894. The left member is equal to

$$\frac{\sin 2x}{\cos (x + \alpha) \cos (x - \alpha)} = \frac{\sin 2x}{\frac{1}{2} (\cos 2\alpha + \cos 2x)}$$

We get the equation

$$\frac{2 \sin 2x}{\cos 2\alpha + \cos 2x} = 2 \cot x$$

Using the formulas $\sin 2x=2\sin x\cos x$ and $\cot x=\frac{\cos x}{\sin x}$, reduce the equation to the form

$$\cos x\,(2\sin^2 x-\cos 2x-\cos 2\alpha)=0$$

The equation

$$2\sin^2 x-\cos 2x-\cos 2\alpha=0$$

with $2\sin^2 x$ replaced by $1-\cos 2x$ yields

$$2\cos 2x=1-\cos 2\alpha$$

whence

$$\cos 2x=\sin^2\alpha$$

Answer: $x=\frac{\pi}{2}(2n+1)$; $x=\pi n\pm\frac{1}{2}\arccos(\sin^2\alpha)$.

895. The left member is equal to $1-\sin x$; the denominator of the right member, to

$$\tan\frac{x}{2}-\tan\left(\frac{\pi}{2}+\frac{x}{2}\right)=\tan\frac{x}{2}+\cot\frac{x}{2}$$

This expression is reduced to the form $\frac{2}{\sin x}$. We get the equation $1-\sin x=\sin x$.

Answer: $x=180^\circ n+(-1)^n 30^\circ$.

896. The left member is equal to $\tan x$. The right one (cf. Problem 894), to $1+2\tan x$.

Answer: $x=45^\circ(4n-1)$.

897. We have

$$\sin^4 x=(\sin^2 x)^2=\left(\frac{1-\cos 2x}{2}\right)^2;$$

analogously,

$$\sin^4\left(x+\frac{\pi}{4}\right)=\left[\frac{1-\cos\left(2x+\frac{\pi}{2}\right)}{2}\right]^2=\left(\frac{1+\sin 2x}{2}\right)^2$$

The equation takes the form $1-\cos 2x+\sin 2x=0$, or

$$2\sin^2 x+2\sin x\cos x=0$$

Answer: $x=\pi n$; $x=\pi n-\frac{\pi}{4}$.

897a. Represent the equation (see the preceding problem) in the form

$$\left(\frac{1-\cos 2x}{2}\right)^2+\left(\frac{1+\sin 2x}{2}\right)^2+\left(\frac{1-\sin 2x}{2}\right)^2=\frac{9}{8}$$

After algebraic transformations we get

$$3-2\cos 2x+\cos^2 2x+2\sin^2 2x=\frac{9}{2}$$

Replacing $\sin^2 2x$ by $1-\cos^2 2x$, we obtain the equation

$$\cos^2 2x+2\cos 2x-\frac{1}{2}=0$$

It yields $\cos 2x = -1 + \frac{\sqrt{6}}{2}$ $\left(\cos 2x = -1 - \frac{\sqrt{6}}{2}\right.$ is rejected, since its absolute value exceeds unity$\left.\right)$.

Answer: $x = \pi n \pm \frac{1}{2} \arccos\left(-1 + \frac{\sqrt{6}}{2}\right)$.

898. Represent the left member of the equation in the form $\frac{1}{2}(\cos x + \cos y)$. Solving the system, we find $\cos x = \frac{1}{2}$; $\cos y = \frac{1}{2}$.

Answer: $x = 2\pi k \pm \frac{\pi}{3}$; $y = 2\pi l \pm \frac{\pi}{3}$.

899. Since

$$\sin x \sin y = \frac{\cos(x-y) - \cos(x+y)}{2}$$

the second equation can be written in the following way:

$$\cos(x-y) - \cos(x+y) = 2m$$

But $x + y = \alpha$; consequently,

$$\cos(x-y) = 2m + \cos\alpha$$

whence

$$x - y = 2\pi n \pm \arccos(2m + \cos\alpha)$$

and the given system is reduced to the following two systems:

$$\begin{cases} x + y = \alpha \\ x - y = 2\pi n + \arccos(2m + \cos\alpha) \end{cases}$$

and

$$\begin{cases} x + y = \alpha \\ x - y = 2\pi n - \arccos(2m + \cos\alpha) \end{cases}$$

Answer:

$$\begin{cases} x_1 = \pi n + \frac{\alpha}{2} + \frac{1}{2}\arccos(2m + \cos\alpha) \\ y_1 = -\pi n + \frac{\alpha}{2} - \frac{1}{2}\arccos(2m + \cos\alpha) \end{cases}$$

$$\begin{cases} x_2 = \pi n + \frac{\alpha}{2} - \frac{1}{2}\arccos(2m + \cos\alpha) \\ y_2 = -\pi n + \frac{\alpha}{2} + \frac{1}{2}\arccos(2m + \cos\alpha) \end{cases}$$

900. Using the formula

$$\tan\alpha + \tan\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$$

write the second equation in the following way: $\frac{\sin(x+y)}{\cos x \cos y} = m$. Replacing $\cos x \cos y$ by $\frac{\cos(x+y)+\cos(x-y)}{2}$ and $x+y$ by α, we get

$$\frac{2\sin\alpha}{\cos\alpha+\cos(x-y)} = m$$

or

$$\cos(x-y) = \frac{2\sin\alpha}{m} - \cos\alpha$$

Hence, we have either

$$x-y = 2\pi n + \arccos\left(\frac{2\sin\alpha}{m} - \cos\alpha\right)$$

or

$$y-x = 2\pi n + \arccos\left(\frac{2\sin\alpha}{m} - \cos\alpha\right)$$

Either of these equations should be solved together with the equation $x+y=\alpha$. By the way, of the two systems obtained one differs from the other only in that the unknowns change their roles, therefore it is sufficient to solve one of the systems.

Answer: $x_1\,(=y_2) = \pi n + \frac{\alpha}{2} + \frac{1}{2}\arccos\left(\frac{2\sin\alpha}{m} - \cos\alpha\right)$

$y_1\,(=x_2) = -\pi n + \frac{\alpha}{2} - \frac{1}{2}\arccos\left(\frac{\sin\alpha}{m} - \cos\alpha\right)$

901. Solved as the preceding problem.

Answer:

$x_1 = \frac{\pi}{4}(4n+1)$	$x_2 = -\pi n$
$y_1 = -\pi n$	$y_2 = \frac{\pi}{4}(4n+1)$

902. Since $1 = 2^0$ and $4 = 16^{\frac{1}{2}}$, the given system may be rewritten as

$$\begin{cases} \sin x + \cos y = 0 \\ \sin^2 x + \cos^2 y = \frac{1}{2} \end{cases}$$

whence

$$(1)\ \sin x = \frac{1}{2},\quad \cos y = -\frac{1}{2} \quad\text{and}\quad (2)\ \sin x = -\frac{1}{2},\quad \cos y = \frac{1}{2}$$

Answer: $x_1 = 180°n + (-1)^n\,30°,\ y_1 = 360°n \pm 120°$

$x_2 = 180°n - (-1)^n\,30°,\ y_2 = 360°n \pm 60°$

903. The second equation can be written in the form

$$\frac{\sin x \sin y}{\cos x \cos y} = \frac{1}{3}$$

where, by virtue of the first equation, $\sin x \sin y = \frac{1}{4\sqrt{2}}$. We get the system of equations:

$$\cos x \cos y = \frac{3}{4\sqrt{2}}, \quad \sin x \sin y = \frac{1}{4\sqrt{2}}$$

Adding and subtracting them, we get

$$\cos (x-y) = \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos (x+y) = \frac{1}{2\sqrt{2}}$$

whence

$$x+y = 2\pi m \pm \arccos \frac{1}{2\sqrt{2}}, \quad x-y = 2\pi k \pm \frac{\pi}{4}$$

where m and k are arbitrary integers. In each of these equations we can take any sign.

Note. The numbers $m+k$ and $m-k$ are also integers, but not completely arbitrary (if one of them is even, the other is also even; and if one of them is odd, the other is also odd).

Answer: (1) $x = \pi n + \frac{1}{2} \arccos \frac{1}{2\sqrt{2}} + \frac{\pi}{8}$

$y = \pi t + \frac{1}{2} \arccos \frac{1}{2\sqrt{2}} - \frac{\pi}{8}$

(2) $x = \pi n + \frac{1}{2} \arccos \frac{1}{2\sqrt{2}} - \frac{\pi}{8}$

$y = \pi t + \frac{1}{2} \arccos \frac{1}{2\sqrt{2}} + \frac{\pi}{8}$

(3) $x = \pi n - \frac{1}{2} \arccos \frac{1}{2\sqrt{2}} + \frac{\pi}{8}$

$y = \pi t - \frac{1}{2} \arccos \frac{1}{2\sqrt{2}} - \frac{\pi}{8}$

(4) $x = \pi n - \frac{1}{2} \arccos \frac{1}{2\sqrt{2}} - \frac{\pi}{8}$

$y = \pi t - \frac{1}{2} \arccos \frac{1}{2\sqrt{2}} + \frac{\pi}{8}$

where $n = m + k$; $t = m - k$ (m and k are any integers).

904. Square both members of each of the given equations and then add them by terms. We get

$$1 = 4 \sin^2 y + \frac{1}{4} \cos^2 y \quad \text{or} \quad 1 = 4(1 - \cos^2 y) + \frac{1}{4} \cos^2 y$$

whence $\cos^2 y = \frac{4}{5}$ and $\sin^2 y = \frac{1}{5}$. In each of the expressions $\cos y = \pm \frac{2}{\sqrt{5}}$ and $\sin y = \pm \frac{1}{\sqrt{5}}$ either sign may be taken (thus, in the interval between 0 and 360° the angle y can have four values). Substituting these values into the given equations, we find that the angles x and y satisfy one of the following four relationships:

$$(1)\ \cos x=\frac{1}{\sqrt{5}},\quad \sin x=\frac{2}{\sqrt{5}},\quad \cos y=\frac{2}{\sqrt{5}},\quad \sin y=\frac{1}{\sqrt{5}}$$

$$(2)\ \cos x=\frac{1}{\sqrt{5}},\quad \sin x=-\frac{2}{\sqrt{5}},\quad \cos y=\frac{2}{\sqrt{5}},\quad \sin y=-\frac{1}{\sqrt{5}}$$

$$(3)\ \cos x=-\frac{1}{\sqrt{5}},\quad \sin x=\frac{2}{\sqrt{5}},\quad \cos y=-\frac{2}{\sqrt{5}},\quad \sin y=\frac{1}{\sqrt{5}}$$

$$(4)\ \cos x=-\frac{1}{\sqrt{5}},\quad \sin x=-\frac{2}{\sqrt{5}},\quad \cos y=-\frac{2}{\sqrt{5}},\quad \sin y=-\frac{1}{\sqrt{5}}$$

Consider the first of them. If we take separately the equality $\cos x=\frac{1}{\sqrt{5}}$, then it yields $x=2\pi n\pm\arccos\frac{1}{\sqrt{5}}$. But (by the definition of the principal value of arccos) the angle $\varphi=\arccos\frac{1}{\sqrt{5}}$ belongs to the first or second quadrants, where the function of sine is always positive. Hence, the plus sign should be retained. Indeed, from the equality $x=2\pi n\pm\varphi$ it follows that $\sin x=\pm\sin\varphi=\pm\frac{2}{\sqrt{5}}$. But in the first relationship $\sin x=\frac{2}{\sqrt{5}}$ $\left(\text{but not} -\frac{2}{\sqrt{5}}\right)$. The same with the angle y, thus in case of relationship (1) we get

$$x=2\pi n+\arccos\frac{1}{\sqrt{5}},\quad y=2\pi n_1+\arccos\frac{2}{\sqrt{5}}$$

where n and n_1 are any integers. Reasoning in the same way, we find that for the second relationship

$$x=2\pi n-\arccos\frac{1}{\sqrt{5}},\quad y=2\pi n_1-\arccos\frac{2}{\sqrt{5}}$$

The third and fourth relationships are considered analogously.

Answer: $x=2\pi n\pm\arccos\left(\pm\frac{1}{\sqrt{5}}\right)$

$y=2\pi n_1\pm\arccos\left(\pm\frac{2}{\sqrt{5}}\right)$

where the signs in parentheses are the same for x and for y and the signs before arccos are also the same.

CHAPTER XIII

INVERSE TRIGONOMETRIC FUNCTIONS

905. We have

$$\arcsin\left(-\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3}, \ \operatorname{arccot}(-1)=\frac{3\pi}{4}$$

$$\arccos\frac{1}{\sqrt{2}}=\frac{\pi}{4}, \quad \arccos(-1)=\pi$$

Answer: $\frac{5\pi}{6}$.

906. The angle $\varphi = \arccos x$ is found in the interval between 0 and 180° (by the definition of the principal value of arccos). Hence, $\sin\varphi$ is positive (or equal to zero). We have $\cos\varphi = x$, whence $\sin\varphi = +\sqrt{1-x^2}$ (the radical is taken only with the plus sign). Consequently,

$$\tan\varphi=\frac{\sqrt{1-x^2}}{x},$$

i.e.

$$\tan(\arccos x)=\frac{\sqrt{1-x^2}}{x}$$

which completes the proof.

907. See the solution of the preceding problem.

908. Let us put $\operatorname{arccot}\left(-\frac{3}{4}\right)=\varphi$, thus $\cot\varphi=-\frac{3}{4}$. The angle φ is in the interval between 90° and 180° (since the principal value of arccot is contained between 0 and 180°). It is required to find $\sin\frac{\varphi}{2}$. Let us use the formula

$$\sin\frac{\varphi}{2}=\pm\sqrt{\frac{1-\cos\varphi}{2}}$$

where out of the two signs only the plus sign is taken (since the angle $\frac{\varphi}{2}$ belongs to the first quadrant). First we have to find $\cos\varphi$ using the formula

$$\cos\alpha=\frac{\cot\alpha}{\pm\sqrt{1+\cot^2\alpha}}$$

we get

$$\cos\varphi=\frac{-\frac{3}{4}}{\sqrt{1+\frac{9}{16}}}=-\frac{3}{5}$$

(the radical is taken with the plus sign only, since φ belongs to the second quadrant). Now we find

$$\sin\frac{\varphi}{2}=\sqrt{\frac{1-\cos\varphi}{2}}=\sqrt{\frac{1+\frac{3}{5}}{2}}=\frac{2}{\sqrt{5}}$$

Answer: $\sin\left[\frac{1}{2}\operatorname{arccot}\left(-\frac{3}{4}\right)\right]=\frac{2}{\sqrt{5}}$.

909. Put $\arcsin\left(-\frac{2\sqrt{2}}{3}\right)=\varphi$, hence $\sin\varphi=-\frac{2\sqrt{2}}{3}$. The angle φ is in the interval between $-90°$ and $0°$ (since the principal value of arcsin is contained between $-90°$ and $+90°$). It is required to find $\sin\frac{\varphi}{2}$. This value is negative. Therefore in the formula

$$\sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}}$$

only the minus sign is to be retained. We get

$$\sin\frac{\varphi}{2}=-\sqrt{\frac{1-\cos\varphi}{2}}$$

where

$$\cos\varphi=\sqrt{1-\left(-\frac{2\sqrt{2}}{3}\right)^2}=\frac{1}{3}$$

we take the radical only with the plus sign!).

Answer: $\sin\left[\frac{1}{2}\arcsin\left(\frac{-2\sqrt{2}}{3}\right)\right]=-\frac{1}{\sqrt{3}}$.

910. The angle $\varphi=\arccos\left(-\frac{4}{7}\right)$ is contained between 90° and 180° (see the solutions of the two previous problems). Hence $\cot\frac{\varphi}{2}$ is positive, and

$$\cot\frac{\varphi}{2}=\sqrt{\frac{1+\cos\varphi}{1-\cos\varphi}}$$

(the radical is taken only with the plus sign). Substitute $\cos\varphi=-\frac{4}{7}$ into this expression.

Answer: $\cot\left[\frac{1}{2}\arccos\left(-\frac{4}{7}\right)\right]=\frac{\sqrt{33}}{11}$.

911. Since

$$\arctan\frac{\sqrt{3}}{3}=\frac{\pi}{6}$$

and

$$\arcsin\frac{\sqrt{3}}{2}=\frac{\pi}{3}$$

we have

$$\tan\left(5\cdot\frac{\pi}{6}-\frac{1}{4}\cdot\frac{\pi}{3}\right)=\tan\frac{3\pi}{4}=-1$$

Answer: -1.

912. We have

$$\arctan \sqrt{3}=\frac{\pi}{3}$$

and

$$\arccos\frac{1}{2}=\frac{\pi}{3}$$

Then proceed as in the preceding problem.

Answer: $-\frac{\sqrt{3}}{2}$.

913. *Answer:* $\frac{1}{2}$.

914. Let us put

$$\arctan(3+2\sqrt{2})=\alpha \tag{1}$$

$$\arctan\frac{\sqrt{2}}{2}=\beta \tag{2}$$

It is required to prove that

$$\alpha-\beta=\frac{\pi}{4} \tag{3}$$

Find $\tan(\alpha-\beta)$:

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta};$$

with the aid of (1) and (2) we get

$$\tan(\alpha-\beta)=\frac{(3+2\sqrt{2})-\frac{\sqrt{2}}{2}}{1+(3+2\sqrt{2})\frac{\sqrt{2}}{2}}=1 \tag{4}$$

On the other hand, it is obvious from (1) and (2) that either of the angles α and β lies between 0 and $\frac{\pi}{2}$, and $\alpha>\beta$ $\left(\text{since } 3+2\sqrt{2}>\frac{\sqrt{2}}{2}\right)$; consequently, the angle $\alpha-\beta$ a priori lies between 0 and $\frac{\pi}{2}$, hence, from (4) we get $\alpha-\beta=\frac{\pi}{4}$, which completes the proof.

Note. To prove that the angle $\alpha-\beta$ is just equal to $\frac{\pi}{4}$, i.e. to 45° (but not to 225° or to −135° and so on) we can make use of the corresponding tables to find directly the angles α and β. Here we may confine ourselves to rough approximations (for instance, taking into account only degrees). Thus, putting $\sqrt{2}\approx 1.4$, we find $\alpha\approx\arctan 5.8$, which corresponds to about 80° (the error a priori does not exceed $\frac{1^\circ}{2}\Big)$. In the same way we find $\beta \approx \arctan 0.7$, which corresponds to about 35° $\left(\text{the error is a priori less than } \frac{1^\circ}{2}\right)$. Consequently, $\alpha-\beta$ does not differ from 45° by more than 1°, and, hence, is exactly equal to 45°.

915. Put

$$\arccos \sqrt{\frac{2}{3}}=\alpha$$

$$\arccos \frac{\sqrt{6}+1}{2\sqrt{3}}=\beta^*$$

so that $\cos\alpha=\sqrt{\frac{2}{3}}$ and $\cos\beta=\frac{\sqrt{6}+1}{2\sqrt{3}}$. Either of the angles α and β belongs to the first quadrant **. It is required to prove, that $\alpha-\beta=\frac{\pi}{6}$.

Find $\sin(\alpha-\beta)$, but first compute

$$\sin\alpha=\sqrt{1-\cos^2\alpha}$$

and

$$\sin\beta=\sqrt{1-\cos^2\beta}$$

(each radical is taken only with the plus sign, since α and β belong to the first quadrant). We find

$$\sin\alpha=\frac{1}{\sqrt{3}} \quad \text{and} \quad \sin\beta=\sqrt{\frac{5-2\sqrt{6}}{12}}$$

hence,

$$\sin(\alpha-\beta)=\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{6}+1}{2\sqrt{3}}-\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{5-2\sqrt{6}}{12}}=$$

$$=\frac{\sqrt{6}+1}{6}-\frac{\sqrt{2}}{6}\cdot\sqrt{5-2\sqrt{6}}$$

Let us prove that the found irrational expression is equal to $\frac{1}{2}$. To this end transform the "double irrationality" $\sqrt{5-2\sqrt{6}}=\sqrt{5-\sqrt{24}}$. It can be performed with the aid of the formula

$$\sqrt{A-\sqrt{B}}=\sqrt{\frac{A+\sqrt{A^2-B}}{2}}-\sqrt{\frac{A-\sqrt{A^2-B}}{2}}$$

---

* This problem may be solved without introducing auxiliary quantities α and β using the method mentioned in the note to the preceding problem.

** The principal value of arccos lies between 0 and π.

(at $A=5$, $B=24$); we obtain

$$\sqrt{\frac{5+1}{2}}-\sqrt{\frac{5-1}{2}}=\sqrt{3}-\sqrt{2}$$

But it is simpler to represent the radicand $5-2\sqrt{6}$ in the form $3+2-2\sqrt{2}\cdot\sqrt{3}=(\sqrt{3}-\sqrt{2})^2$, and then we have

$$\sqrt{5-2\sqrt{6}}=\sqrt{(\sqrt{3}-\sqrt{2})^2}=\sqrt{3}-\sqrt{2}\,*$$

Since either of the angles α and β lies within the interval between 0 and $\frac{\pi}{2}$ the angle $\alpha-\beta$ undoubtedly lies within the interval from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, then from the equality $\sin(\alpha-\beta)=\frac{1}{2}$ it follows that $\alpha-\beta=\frac{\pi}{6}$, which completes the proof **.

916. Let (see the two previous problems)

$$\arcsin\frac{4}{5}=\alpha,\ \arcsin\frac{5}{13}=\beta,\ \arcsin\frac{16}{65}=\gamma$$

Then

$$\sin\alpha=\frac{4}{5},\qquad \cos\alpha=\frac{3}{5}$$

$$\sin\beta=\frac{5}{13},\qquad \cos\beta=\frac{12}{13}$$

$$\sin\gamma=\frac{16}{65},\qquad \cos\gamma=\frac{63}{65}$$

Hence

$$\sin(\alpha+\beta)=\frac{4}{5}\cdot\frac{12}{13}+\frac{3}{5}\cdot\frac{5}{13}=\frac{63}{65}\ \text{and}\ \cos(\alpha+\beta)=\frac{16}{65}$$

Both angles α and β belong to the first quadrant; therefore the angle $\alpha+\beta$ lies between 0° and 180°, and since $\cos(\alpha+\beta)$ is positive, $\alpha+\beta$ belongs to the first quadrant. Furthermore, $\cos(\alpha+\beta)=\sin\gamma$ and $\sin(\alpha+\beta)=\sin\left(\frac{\pi}{2}-\gamma\right)$.

---

* The number $\sqrt{3}-\sqrt{2}$ is positive.

** If $\cos(\alpha-\beta)$ is computed instead of $\sin(\alpha-\beta)$, we would find $\cos(\alpha-\beta)=\frac{\sqrt{3}}{2}$; between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ we would have two values of $\alpha-\beta$, namely, $-\frac{\pi}{6}$ and $+\frac{\pi}{6}$; therefore we would have first to establish that $\alpha>\beta$, i.e. that $\cos\alpha<\cos\beta$.

Therefore, $\alpha+\beta$ and $\frac{\pi}{2}-\gamma$ may differ only by $2\pi n$, and since $\frac{\pi}{2}-\gamma$ also belongs to the first quadrant, we have $n=0$. Consequently, $\alpha+\beta=\frac{\pi}{2}-\gamma$, i.e. $\alpha+\beta+\gamma=\frac{\pi}{2}$ which completes the proof.

917. We have $\arccos\frac{1}{2}=\frac{\pi}{3}$; let us denote $\arccos\left(-\frac{1}{7}\right)$ by β, so that $\cos\beta=-\frac{1}{7}$. The angle β is contained between $\frac{\pi}{2}$ and π (see the three previous problems). Therefore

$$\sin\beta=+\sqrt{1-\left(\frac{1}{7}\right)^2}\ \left[\text{but not } -\sqrt{1-\left(\frac{1}{7}\right)^2}\right],$$

i.e.

$$\sin\beta=\frac{4}{7}\sqrt{3}$$

We find

$$\cos\left(\frac{\pi}{3}+\beta\right)=\cos\frac{\pi}{3}\cos\beta-\sin\frac{\pi}{3}\sin\beta=$$

$$=\frac{1}{2}\cdot\left(-\frac{1}{7}\right)-\frac{\sqrt{3}}{2}\cdot\frac{4}{7}\sqrt{3}=-\frac{13}{14}$$

To prove the validity of the given identity we have to make sure that the angle $\frac{\pi}{3}+\beta$ belongs to the second quadrant $\left[\text{since the angle }\arccos\left(-\frac{13}{14}\right)\text{ in the right member lies in the second quadrant}\right]$. The angle $\beta=\arccos\left(-\frac{1}{7}\right)$ is contained between $\frac{\pi}{2}$ and π; consequently, the angle $\frac{\pi}{3}+\beta$ lies between $\frac{5\pi}{6}$ and $\frac{4\pi}{3}$. But it does not, however, follow from this estimate that the angle $\frac{\pi}{3}+\beta$ belongs to the second quadrant $\left(\text{since the angle }\frac{4\pi}{3}\text{ is already found in the third quadrant}\right)$. But taken into account that $-\frac{1}{7}>-\frac{1}{2}$ and that, consequently, $\arccos\left(-\frac{1}{7}\right)<\arccos\left(-\frac{1}{2}\right)$, i.e. $\arccos\left(-\frac{1}{7}\right)<\frac{2\pi}{3}$, it follows that $\frac{\pi}{3}+\arccos\left(-\frac{1}{7}\right)<\pi$. And since this angle is more than $\frac{5\pi}{6}$, it lies in the second quadrant. Hence, the given identity is proved.

Note. The fact that the angle $\frac{\pi}{3}+\beta$ belongs to the second (and not to the third) quadrant can also be demonstrated in a different way: we have

$$\sin\left(\frac{\pi}{3}+\beta\right)=\sin\frac{\pi}{3}\cos\beta+\cos\frac{\pi}{3}\sin\beta=$$
$$=\frac{\sqrt{3}}{2}\cdot\left(-\frac{1}{7}\right)+\frac{1}{2}\cdot\frac{4}{7}\sqrt{3}=\frac{3}{14}\sqrt{3}$$

Since this number is positive, the angle $\frac{\pi}{3}+\beta$ belongs to the second quadrant.

918. Put

$$\arctan\frac{1}{5}=\alpha \quad \text{and} \quad \arctan\frac{1}{4}=\beta$$

whence

$$\tan\alpha=\frac{1}{5} \quad \text{and} \quad \tan\beta=\frac{1}{4}$$

Compute

$$\tan(2\alpha+\beta)=\frac{\tan 2\alpha+\tan\beta}{1-\tan 2\alpha\tan\beta}$$

First find

$$\tan 2\alpha=\frac{2\cdot\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}=\frac{5}{12}$$

and then

$$\tan(2\alpha+\beta)=\frac{\frac{5}{12}+\frac{1}{4}}{1-\frac{5}{12}\cdot\frac{1}{4}}=\frac{32}{43}$$

The angles $\alpha=\arctan\frac{1}{5}$ and $\beta=\arctan\frac{1}{4}$ belong to the first quadrant, but it does not yet follow from this fact that the angle $2\alpha+\beta$ belongs to the first (and not to the third) quadrant. But if we take into consideration that either of the angles α and β is less than $\frac{\pi}{4}$ (since their tangents are less than unity), it proves that $2\alpha+\beta$ is less than $\frac{3\pi}{4}$, and since, furthermore, $\tan(2\alpha+\beta)=\frac{32}{43}$ is positive, $2\alpha+\beta$ lies in the first quadrant, i.e. $2\alpha+\beta=\arctan\frac{32}{43}$, which completes the proof.

Note. Instead of proving that the angle $2\alpha+\beta$ remains within the limits of the first quadrant, we can find this angle approximately with the aid of the corresponding tables (see the note to Problem 914). We get: $\alpha=\arctan\frac{1}{5}\approx 11^\circ$, $\beta=\arctan\frac{1}{4}\approx 14^\circ$, hence $2\alpha+\beta\approx 36^\circ$.

919. Put

$$\arctan\frac{1}{3}=\alpha,\ \arctan\frac{1}{5}=\beta,\ \arctan\frac{1}{7}=\gamma,\ \arctan\frac{1}{8}=\delta$$

and first find

$$\tan(\alpha+\beta)=\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\cdot\frac{1}{5}}=\frac{4}{7},$$

then

$$\tan[(\alpha+\beta)+\gamma]=\frac{\frac{4}{7}+\frac{1}{7}}{1-\frac{4}{7}\cdot\frac{1}{7}}=\frac{7}{9}$$

and, finally,

$$\tan[(\alpha+\beta+\gamma)+\delta]=\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\cdot\frac{1}{8}}=1$$

As in the preceding problem, prove that the angle $\alpha+\beta+\gamma+\delta$ lies in the first quadrant. Consequently, $\alpha+\beta+\gamma+\delta=\frac{\pi}{4}$.

920. We have $\arctan(x^2-3x-3)=\frac{\pi}{4}$, whence $x^2-3x-3=\tan\frac{\pi}{4}$, i.e. $x^2-3x-3=1$. Hence, $x_1=4$; $x_2=-1$.
Answer: $x_1=4$; $x_2=-1$.

Note. If instead of the equation $\arctan(x^2-3x-3)=\frac{\pi}{4}$ we would have the equation $\arctan(x^2-3x-3)=-\frac{3\pi}{4}$, then the latter would have no solution, since the principal value of arctan cannot be equal to $-\frac{3\pi}{4}$. If no attention is paid to this circumstance, we can obtain the same equation $x^2-3x-3=1$, but the roots of the last equation are not suitable.

921. We have

$$\arcsin(x^2-6x+8.5)=\frac{\pi}{6}$$

whence $x^2-6x+8.5=0.5$.
Answer: $x_1=4$; $x_2=2$.

922. Taking tangents of both members of the equation and remembering that $\tan(\arctan\alpha)=\alpha$, we get

$$\frac{(x+2)-(x+1)}{1+(x+2)(x+1)}=1$$

whence $x_1=-1$; $x_2=-2$. Check these roots. If $x=-1$, then

$$\arctan(x+2)=\arctan 1=\frac{\pi}{4}$$

and

$$\arctan(x+1) = \arctan 0 = 0$$

thus the given equation is not satisfied. We prove in the same way that the second root is also not suitable.

Answer: $x_1 = -1$; $x_2 = -2$.

Note. Why such a check is necessary is clear from the following example. Consider the equation

$$\arctan(x+2) - \arctan(x+1) = -\frac{3\pi}{4}$$

which differs from the given one only by the value of the constant term. It is impossible to state beforehand that it has no solutions (cf. Problem 920). If, say, $\arctan(x+2)$ is equal to $-\frac{\pi}{3}$, and $\arctan(x+1)$ to $\frac{5\pi}{12}$ (these values can be principal values of arctangent), then the left member would be equal to $-\frac{3\pi}{4}$. Taking tangents of both members of the equation under consideration, we again get the equation

$$\frac{(x+2)-(x+1)}{1+(x+2)(x+1)} = 1$$

but now neither of the roots $x_1 = -1$, $x_2 = -2$ is suitable. See also the Note to Problem 925.

923. Take tangent of both members of the equation. First find (see the preceding problem):

$$\tan\left(2\arctan\frac{1}{2}\right) = \frac{2\cdot\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} = \frac{4}{3}$$

and then we obtain

$$\frac{\frac{4}{3}-x}{1+\frac{4x}{3}} = 1$$

The root of this equation is $x = \frac{1}{7}$; it should be checked (see the Note to the preceding problem). Substituting $x = \frac{1}{7}$ into the left member of the equation, we get $2\arctan\frac{1}{2} - \arctan\frac{1}{7}$. The angle $\alpha = \arctan\frac{1}{2}$ lies between 0 and $\frac{\pi}{4}$ $\left(\text{since } \tan\alpha = \frac{1}{2} < 1\right)$. The angle $\beta = \arctan\frac{1}{7}$ lies within the same range. The angle 2α belongs to the first quadrant, and the angle $2\alpha - \beta$ lies between $-\frac{\pi}{4}$ and $\frac{\pi}{2}$. But $\tan(2\alpha - \beta) = 1$, hence, $2\alpha - \beta = \pi n + \frac{\pi}{4}$. But only at $n = 0$ the angle $2\alpha - \beta$ turns out to lie within the found boundaries. Consequently, the given equation is satisfied.

Answer: $x = \frac{1}{7}$.

924. Let us take sines of both members of the equation. We can put

$$\arcsin\frac{2}{3\sqrt{x}}=\alpha \text{ and } \arcsin\sqrt{1-x}=\beta$$

(see the solution of Problem 915), but we can do without them making use of the formulas $\sin(\arcsin x) = x$ (which immediately follows from the definition of arcsine) and $\cos(\arcsin x) = \sqrt{1-x^2}$*. Consequently, sine of the left member is equal to

$$\frac{2}{3\sqrt{x}}\cdot\sqrt{1-(\sqrt{1-x})^2}-\sqrt{1-\left(\frac{2}{3\sqrt{x}}\right)^2}\cdot\sqrt{1-x}$$

and the given equation takes the form

$$\frac{2}{3\sqrt{x}}\sqrt{x}-\frac{\sqrt{9x-4}}{3\sqrt{x}}\sqrt{1-x}=\frac{1}{3}$$

Solving it, we find $x=\frac{2}{3}$. This root should be checked (see the Note to Problem 922), i.e. we have to prove the identity

$$\arcsin\sqrt{\frac{2}{3}}-\arcsin\sqrt{\frac{1}{3}}=\arcsin\frac{1}{3}$$

It is proved in the same way as in Problem 917.

Answer: $x=\frac{2}{3}$.

925. Taking tangents of both members of the equation, we get

$$\frac{\frac{a}{b}-\frac{a-b}{a+b}}{1+\frac{a}{b}\cdot\frac{a-b}{a+b}}=x$$

whence $x = 1$. This value should be checked (see the Note to Problem 922). Substituting $x = 1$ into the given equation, we get

$$\arctan\frac{a}{b}-\arctan\frac{a-b}{a+b}=45^\circ \tag{1}$$

Introduce the following notation

$$\arctan\frac{a}{b}=\varphi \tag{2}$$

Here the angle φ (the principal value of arctangent) lies between -90° and 90°

$$-90^\circ<\varphi<90^\circ. \tag{3}$$

---

* This formula is deduced in the following way. Put $\arcsin x = \alpha$. Then $\sin\alpha = x$ and $\cos\alpha = \sqrt{1-x^2}$. The radical is taken only with the plus sign since the angle $\alpha = \arcsin x$ lies between -90° and $+90^\circ$ (the principal value of arcsine!). Substituting $\arcsin x$ for α, we get the required formula.

Using this notation, we have

$$\arctan\frac{a-b}{a+b}=\arctan\frac{b\tan\varphi-b}{b\tan\varphi+b}=\arctan\tan(\varphi-45^\circ) \tag{4}$$

and so we have to check the equality

$$\varphi-\arctan\tan(\varphi-45^\circ)=45^\circ \tag{5}$$

This equality is true if and only if

$$\arctan\tan(\varphi-45^\circ)=\varphi-45^\circ \tag{6}$$

And the equality (6) holds true when the angle $\varphi-45^\circ$ (the principal value of arctangent) lies within the following interval:

$$-90^\circ<\varphi-45^\circ<90^\circ, \tag{7}$$

i.e. when

$$-45^\circ<\varphi<135^\circ \tag{8}$$

Taking into consideration (3), we get a more narrow interval for the angle

$$-45^\circ<\varphi<90^\circ \tag{9}$$

From (2) and (9) we find

$$\frac{a}{b}=\tan\varphi>\tan(-45^\circ),$$

i.e.

$$\frac{a}{b}>-1 \tag{10}$$

Conversely, for $\frac{a}{b}>-1$ the angle φ satisfies inequality (9). Consequently, the given equation has a solution ($x=1$) for $\frac{a}{b}>-1$. For $\frac{a}{b}<-1$ there is no solution.

For example, at $a=-\sqrt{3}$, $b=1$ we have

$$\arctan\frac{a}{b}=\arctan(-\sqrt{3})=-60^\circ$$

$$\arctan\frac{a-b}{a+b}=\arctan\frac{-\sqrt{3}-1}{-\sqrt{3}+1}\approx\arctan 3.732=75^\circ$$

thus, the left member of the given equation is equal to -135°, and the right one at $x=1$ is equal to 45°.

Answer: $x=1$ for $\frac{a}{b}>-1$; the equation has no solution for $\frac{a}{b}<-1$.

926. Let us take cosines of both members of the equation. We get $\sqrt{1-9x^2}=4x$ (see Problem 924). This equation has only one root $x=\frac{1}{5}$. Let us check it. The angle $\alpha=\arcsin 3x=\arcsin\frac{3}{5}$ belongs to the first quadrant; the angle $\beta = \arccos 4x = \arccos \frac{4}{5}$ also belongs to the first quadrant. Here $\sin\alpha = \frac{3}{5}$; hence, $\cos\alpha = \frac{4}{5}$. On the other hand, $\cos\beta = \frac{4}{5}$; hence, $\alpha = \beta$.

Answer: $x = \frac{1}{5}$.

927. Let us take the sines of both members of the equation. We get (see Problem 924) $2x\sqrt{1-x^2} = \frac{10x}{13}$, whence $x_1 = 0$, $x_2 = +\frac{12}{13}$, $x_3 = -\frac{12}{13}$. Check these roots.

Answer: $x = 0$.

928. From the first equation we find

$$\tan(x+y) = \frac{2a}{1-a^2},$$

i.e.

$$\frac{\tan x + \tan y}{1-\tan x \tan y} = \frac{2a}{1-a^2}$$

Taking into consideration the second equation, we get

$$\frac{\tan x + \tan y}{1-a^2} = \frac{2a}{1-a^2}$$

or

$$\tan x + \tan y = 2a$$

From the system of equations

$$\tan x + \tan y = 2a$$
$$\tan x \tan y = a^2$$

we find $\tan x = a$; $\tan y = a$. Hence, it follows that $x = 180°n + \arctan a$, $y = 180°m + \arctan a$, where n and m are integers. But only one of them may be taken arbitrarily, since according to the first equation the quantity $x + y$ must be contained between $-90°$ and $+90°$ (as the principal value of arctangent).

To identify suitable values of n and m substitute the found expressions into the first equation. We get

$$180°(n+m) + 2\arctan a = \arctan\frac{2a}{1-a^2} \qquad \text{(A)}$$

Since by hypothesis $|a| < 1$, the angle $\arctan a$ lies between $-45°$ and $+45°$, i.e. $2\arctan a$ lies between $-90°$ and $+90°$. The angle $\arctan\frac{2a}{1-a^2}$ (the principal value of arctangent) lies within the same range. Consequently, these two angles differ by less than 180°. Therefore, the equality (A) holds true only at $n + m = 0$.

Answer: $x = 180°n + \arctan a$, $y = -180°n + \arctan a$.

TO THE READER

Mir Publishers welcome your comments on the content, translation, and design of the book.

We would also be pleased to receive any suggestions you care to make about our future publications.

Our address is:
USSR, 129820,
Moscow, I-110, GSP,
Pervy Rizhsky Pereulok, 2,
Mir Publishers

Printed in the Union of Soviet Socialist Republics

UDC 511+512+513+514(075.4)=20

A $\frac{0222-233}{041(01)-74}$